PACIFIC JOURNAL OF MATHEMATICS
Vol. 103, No. 2, 1982

ON THE LEAST NUMBER OF FIXED POINTS FOR INFINITE COMPLEXES

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Let K be a connected infinite and locally finite simplicial complex. The main theorem of this paper is the following: let L be a two-dimensionally connected infinite subcomplex of K, whose boundary $\dot{L}$ in K consists of vertices only, and $f\colon |K|\to|K|$ be a map. Then there exists a map $F\colon |K|\to|K|$, that has the following properties: (1) $F\simeq f$ rel $|\overline{K-L}|$; and, (2) F has no fixed point on $|L|-|\dot{L}|$.

The main theorem implies that if an infinite and locally finite complex K is two dimensionally connected, then the least number of fixed points of any mapping class from $|K|$ to itself is null. At the same time, the main theorem also enables us to compute the least number $m(K)$ of the fixed points of the identity mapping class of $|K|$ by means of the following result: $m(K)$ is equal to the least number $n(K)$ of the fixed points of the good displacements of the welding set $\dot{M}(K)$ of K, where $\dot{M}(K)$ is the set of the boundary vertices of all these maximal two-dimensionally connected and finite subcomplexes of K.

In this paper, an infinite complex means a complex whose simplices are countable infinite. On the other hand, a locally finite complex means a complex K satisfying the following two conditions: For each simplex σ of K, $\mathrm{St}_K(\sigma)$ consists of number of finite simplices and $|\mathrm{St}_K(\sigma)|$ is an open subset of $|K|$. The second condition means the topology of $|K|$ is the weak topology. If x is a point of $|K|$, then it belongs to just one simplex of K which is called the carrier of x and is denoted by $\mathrm{Tr}_K(x)$. A complex K is called two-dimensionally connected if for any two maximal simplices σ and τ of K, there are simplices of K

$$\sigma=\sigma_0, \sigma_1, \cdots, \sigma_{n-1}, \sigma_n=\tau$$

such that σ_{i-1} and σ_i, $i=1,\cdots,n$, have a common face of dimension greater than zero.

Suppose that M is a subset of $|K|$ and that $f\colon M\to|K|$ is a map such that $\overline{\mathrm{Tr}_K(x)}\cap\overline{\mathrm{Tr}_K[f(x)]}\neq\phi$ for any $x\in M$, then we say that f satisfies $S(K)$ on M. The following Lemma 1 is the generalization of Lemma 2.3 and Lemma 1.3 of [6].

LEMMA 1. *Let K be a locally finite complex and τ the common face of its maximal simplices σ_1 and σ_2, where the dimension of τ*

is greater than zero. Suppose we are given points $A \in \sigma_1$, $B \in \tau$ *and a map* $f\colon |K| \to |K|$ *such that* A *is an isolated fixed point of* f *and it is the only fixed point of* f *on* $[A, B]$. *Then we can find a map* $F\colon |K| \to |K|$ *and* $\delta > 0$ *such that*:

$$F \cong f \text{ rel} [|K| - U([A, B], \delta)]$$

and F *on* $U([A, B], \delta)$ *has only one fixed point* C *belonging to* σ_2. *If* f *satisfies* $S(K)$ *on* $[A, B]$ *then* F *satisfies* $S(K)$ *on* $\bar{U}([A, B], \delta)$.

LEMMA 2. *Let* K *be a locally finite complex and* $f\colon |K| \to |K|$ *be a map. Then there is a map* F, $F \cong f\colon |K| \to |K|$ *such that each fixed point of* F *is isolated and lies in a maximal simplex of* K.

Proof. We can find a simplicial approximation $q\colon R \to K$ to f, where R is a barycentric subdivision of a subdivision H of the complex K. We first prove that q has a maximum of one fixed point on the closure of each simplex of R as follows. If σ^n is a simplex of R and x_1, x_2 are two fixed points of q such that the open segment $(x_1, x_2) \subset \sigma^n$ belongs to σ^n, then the straight line $y = tx_1 + (1 - t)x_2$ intersects $\dot{\sigma}^n$ at two points y_1 and y_2, which are fixed points of q. Because x_i is a fixed point of the simplicial map q, then $|\mathrm{Tr}_R(x_i)| \subset |\mathrm{Tr}_H(x_i)|$, so the dimension of $\mathrm{Tr}_H(x_i)$ is n. The dimension of the carrier of (x_1, x_2) in H is n. Similarly, we have $|\mathrm{Tr}_R(y_i)| \subset |\mathrm{Tr}_H(y_i)|$, so the dimension of $\mathrm{Tr}_H(y_i)$ is equal to the dimension of $\mathrm{Tr}_R(y_i)$ and less than n, for $i = 1, 2$. Since R is the barycentric subdivision of H, σ^n has a face σ^{n-1}, such that all the points of $\bar{\sigma}^n$ which have the carrier in H of dimension less than n belong to $\bar{\sigma}^{n-1}$. This fact implies that $y_1, y_2 \in \bar{\sigma}^{n-1}$, which is a contradiction, because then we would have $(x_1, x_2) \subset \bar{\sigma}^{n-1}$.

Next we denote all the fixed points of q as $x_1, x_2, \cdots$, so:

$$|\mathrm{St}_R[\mathrm{Tr}_R(x_i)]| \cap |\mathrm{St}_R[\mathrm{Tr}_R(x_j)]| = \phi, \text{ for } i \neq j .$$

We choose $\delta_i > 0$, $i = 1, 2, \cdots$, such that:

$$\bar{U}(x_i, \delta_i) \subset |\mathrm{St}_R[\mathrm{Tr}_R(x_i)]|, \; i = 1, 2, \cdots ,$$

then:

$$\bar{U}(x_i, \delta_i) \cap \bar{U}(x_j, \delta_j) = \phi, \; i \neq j .$$

From [1] (Kapitel 14) we can find the maps $g_i\colon \bar{U}(x_i, \delta_i) \to |K|$ with $\varepsilon_i = \sup\{\rho[q(x), g_i(x)] \mid x \in \bar{U}(x_i, \delta_i)\}$, where ρ is the metric of $|K|$, with ε_i sufficiently small so that the following three conditions are satisfied:

(1) $\overline{\mathrm{Tr}_K[g_i(x)]} \cap \overline{\mathrm{Tr}_K[q(x)]} \neq \phi$, for all $x \in \bar{U}(x_i, \delta_i)$;

(2) each fixed point of g_i is isolated and lies in a maximal simplex of R as well as in $\bar{U}(x_i, \delta_i/2)$; and,

(3) $\alpha[q(x), g_i(x), (2 - 2\rho(x, x_i)/\delta_i)t] \neq x$, for all $0 \leqq t \leqq 1$ when $\delta_i/2 \leqq \rho(x, x_i) \leqq \delta_i$.

Using the short homotopy α of Lemma 1.1 of [6] we define

$$f_t(x) = \begin{cases} q(x), \ x \in |K| - \bigcup_i U(x_i, \delta_i) , \\ \alpha[q(x), \ g_i(x), \ (2 - 2\rho(x, x_i)/\delta_i)t], \ \delta_i/2 \leqq \rho(x, x_i) \leqq \delta_i , \\ \alpha[q(x), \ g_i(x), \ t], \ 0 \leqq \rho(x, x_i) \leqq \delta_i/2 , \end{cases}$$

so f_t is a homotopy between q and f_1. Finally, let $F = f_1$, then each fixed point of F is isolated and lies in a maximal simplex of K.

LEMMA 3. *Assume that K is a locally finite complex, M is a subcomplex consisting of vertices only, and that $g = M \to |K|$ is a map satisfying $S(K)$ on M. Then there is a map $F_1: |K| \to |K|$ that has the following properties*:

(1) *F_1 satisfies $S(K)$ on $|K|$;*

(2) *$F_1(x) = g(x)$, for all $x \in M$; and,*

(3) *each fixed point of F_1 on $|K| - M$ is isolated and lies in a maximal simplex of K.*

Proof. In the proof of Lemma 2, let $f = 1$; thus we can choose ε_i to be sufficiently small to ensure that $F(x)$ satisfies $S(K)$ on $|K|$. Since M consists of vertices of K, then

$$\overline{\mathrm{Tr}_K(x)} \cap \overline{\mathrm{Tr}_K[g(x)]} \cap \overline{\mathrm{Tr}_K[F(x)]} \neq \phi ,$$

for all $x \in M$. Writing $M = \{y_1, y_2, \cdots\}$, we can find $\eta_i > 0$, that have the following properties:

$$\bar{U}(y_i, \eta_i) \cap \bar{U}(y_j, \eta_j) = \phi, \ i \neq j ;$$
$$\bar{U}(y_i, \eta_i) \subset \mathrm{St}_K(y_i), \ F[\bar{U}(y_i, \eta_i)] \subset \mathrm{St}_K(y_i) ;$$
$$F[\bar{U}(y_i, \eta_i)] \cap \bar{U}(y_i, \eta_i) = \phi, \ i = 1, 2, \cdots .$$

We choose a path $P_i = [F(y_i), A_i, y_i, B_i, g(y_i)]$ in $\mathrm{St}_K(y_i)$, parametrized by length, such that points A and B belong to the maximal simplices of K. Defining the map $F_1: |K| \to |K|$ as:

$$F_1(x) = \begin{cases} F(x), \ x \in |K| - \bigcup_i U(y_i, \eta_i) ; \\ F\left[\left(\frac{2\rho(x, y_i)}{\eta_i} - 1\right)x + \left(2 - \frac{2\rho(x, y_i)}{\eta_i}\right)y_i\right], \\ \qquad \eta_i/2 \leqq \rho(x, y_i) \leqq \eta_i ; \\ P_i\left(1 - \frac{2\rho(x, y_i)}{\eta_i}\right), \ 0 \leqq \rho(x, y_i) \leqq \eta_i/2 , \end{cases}$$

F_1 satisfies the conditions of this lemma.

THEOREM 1. *Assume that K is an infinite and locally finite complex and that L is a two-dimensionally connected infinite subcomplex which has the boundary $\dot{L}$ consisting of some vertices of K. Assume that $f: |K| \to |K|$ is a map and that each fixed point of f on $|L| - |\dot{L}|$ is isolated and lies in a maximal simplex of L. Then there exists a map $F: |K| \to |K|$ which has the following two properties:*

(a) $F \cong f \operatorname{rel} |\overline{K - L}|$; *and,*

(b) *F has no fixed points on $|L| - |\dot{L}|$.*

If f satisfies $S(K)$ on $|K|$ then F also satisfies $S(K)$ on $|K|$.

Proof. The basic method of constructing F from f is to push a fixed point of f further away on L. First we choose the route of pushing the fixed point of f. We construct a one-dimensional complex R such that there exists a one-to-one correspondence g from all the maximal simplices of L to all the vertices of R, where two vertices $g(\sigma_1)$ and $g(\sigma_2)$ constitute a one-dimensional simplex in R if, and only if, σ_1 and σ_2 have a common face of dimension greater than zero. Then R is a connected, infinite and locally finite complex. We choose a tree S in R which is a simply connected subcomplex of R and contains all the vertices of R.

We now construct a function N on the simplices of S by inductive definition. In complex S, if a vertex τ^0 is a face of a single one-dimensional simplex τ^1 only, then we define $N(\tau^0) = 1$ and $N(\tau^1) = 1$. Evidently, $S - N^{-1}(1)$ is a subcomplex of S. In complex $S - \bigcup_{r=1}^{i-1} N^{-1}(r)$, if a vertex τ^0 is a face of a single one-dimensional simplex τ^1 only, then we define $N(\tau^0) = i$ and $N(\tau^1) = i$. Evidently, $S - \bigcup_{r=1}^{i} N^{-1}(r)$ is a subcomplex of S. Let $T = S - \bigcup_{r=1}^{\infty} N^{-1}(r)$. If T is nonempty, then T is a subcomplex of S and we define $N(\tau) = 0$ for all $\tau \in T$. As a result, function N has the following properties (1) and (2):

(1) $S - \bigcup_{r=1}^{i} N^{-1}(r)$ is simply connected, for $i = 1, 2, \cdots$.

(2) if τ^0 is a vertex of $S - T$, then there exists another vertex σ^0 of S such that we have either $N(\sigma^0) > N(\tau^0)$ or $N(\sigma^0) = 0$, where τ^0 and σ^0 constitute a one-dimensional simplex of S.

(3) If T is nonempty, from (1) we know that T is a simply connected and infinite subcomplex of S. (See Fig. 1). In this case, we pick a vertex A in T and construct a function V on all the vertices of T as follows: For a vertex τ^0, $V(\tau^0)$ is defined to be the least number of edges from A to τ^0 in T. In this case the property (4) is similar to property (2):

(4) if τ^0 is a vertex of T, then there exists another vertex

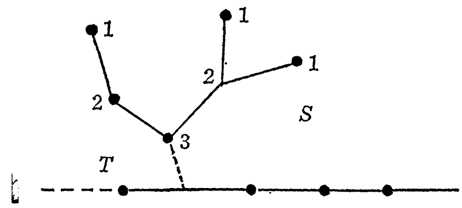

FIGURE 1

σ^0 of T such that $V(\sigma^0) > V(\tau^0)$, where τ^0 and σ^0 constitute a one-dimensional simplex of T.

Based on the Lemma 1 and property (2), we can move the fixed points of f from $g^{-1}N^{-1}(1)$ to $\{g^{-1}N^{-1}(r)/r = 0 \text{ or } r > 1\}$, and subsequently move the fixed points of f from $g^{-1}N^{-1}(i)$ to $\{g^{-1}N^{-1}(r)/r = 0 \text{ or } r > i\}$, and so on, thereby moving all the fixed points of f to $\{g^{-1}N^{-1}(0)\}$. Further, based on the Lemma 1 and property (4), we can move the fixed points of f from $g^{-1}V^{-1}(1)$ to $\{g^{-1}V^{-1}(r)/r > 1\}$, and subsequently move the fixed points of f from $g^{-1}V^{-1}(i)$ to $\{g^{-1}V^{-1}(r)/r > i\}$ and so on. Finally, we get a map F such that $F \cong f \operatorname{rel} \overline{|K - L|}$ and F has no fixed points on $|L| - |\dot{L}|$.

From the Theorem 1 we deduce:

THEOREM 2. *Suppose K is an infinite and locally finite two-dimensionally connected complex, then the least number of the fixed points of any mapping class from $|K|$ to itself is zero.*

DEFINITION 1. Let K be a locally finite complex and M_i, $i = 1, 2, \cdots$, be all its maximal two-dimensionally connected finite subcomplexes, thus the boundary $\dot{M}_i$ consists of some vertices of K. Denote $\dot{M}(K) = \bigcup_i \dot{M}_i$, $\dot{M}(K)$ is called the welding set of K. A *good displacement* is a map $g: \dot{M}(K) \to |K|$ such that:

(1) $g(a) \in |\mathrm{St}_K(a)|$, for all $a \in \dot{M}(K)$; and,

(2) if g has no fixed points in $\dot{M}_i$, then the number of points in $\dot{M}_i$ whose images under g are outside $|M_i|$ is exactly $\chi(M_i)$.

THEOREM 3. *Let K be a locally finite complex, then the least number $m(K)$ of fixed points of the identity mapping class is equal to the least number of fixed points $n(K)$ of all the good displace-*

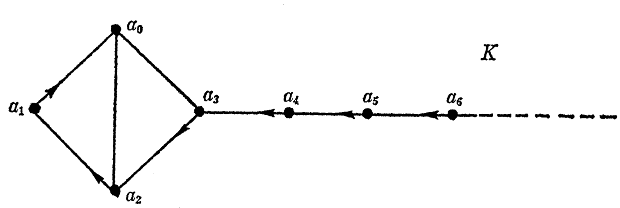

FIGURE 2

ments.

In Lemma 4 we shall prove $m(K) \leqq n(K)$ and in Lemma 5 we shall prove $m(K) \geqq n(K)$.

EXAMPLE 1. In Fig. 2, the welding set $\dot{M}(K)$ of K is $\{a_0, a_1, a_2, \cdots\}$ and the arrows represent a good displacement which has the least fixed points. From Theorem 3 we have $m(K) = 1$. Replacing each 1-dimensional closed simplex $\tau_i = a_j a_k$ of K by a 2-dimensionally connected complex M_i, such that $\dot{M}_i = \{a_j, a_k\}$, we get a complex K_1 with $\dot{M}(K_1) = \dot{M}(K)$. If each M_i is an n-dimensional closed simplex, then $m(K_1) = 1$ results from Theorem 3; if for each M_i, either $\chi(M_i) > 2$ or $\chi(M_i) < 0$, then $m(K_1) = \infty$ from Theorem 3.

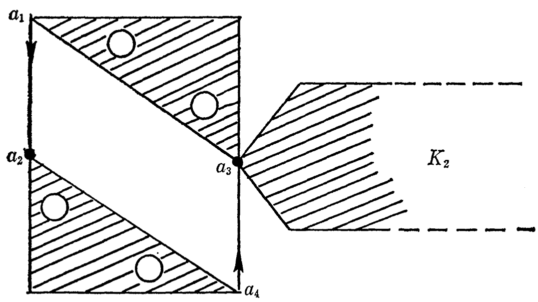

FIGURE 3

EXAMPLE 2. In Fig. 3, the welding set $\dot{M}(K_2)$ of K_2 is $\{a_1, a_2, a_3, a_4\}$, and the arrows represent a good displacement which has least fixed points. From Theorem 3 we have $m(K_2) = 2$.

LEMMA 4. *If g is a good displacement of K, there will be a map G: $|K| \to |K|$ such that:*

(1) *$G(x) = g(x)$, for all $x \in \dot{M}(K)$;*

(2) *G satisfies $S(K)$ on $|K|$; and,*

(3) *G has no fixed points on $|K| - \dot{M}(K)$.*

Proof. Applying Lemma 3, we get a map F_1: $|K| \to |K|$ that has the following three properties:

(1) F_1 satisfies $S(K)$ on $|K|$;

(2) $F_1(x) = g(x)$, for all $x \in \dot{M}(K)$; and,

(3) each fixed point of F_1 on $|K| - \dot{M}(K)$ is isolated and lies in a maximal simplex of K.

From Theorem 1, there exists a map F, such that, F satisfies $S(K)$ on $|K|$, $F \cong F_1: |K| \to |K|$ rel $\bigcup_i M_i$, and F in $|K - \bigcup_i M_i|$ has no fixed points.

Since g is a good displacement, if g has no fixed points on $\dot{M}_i$, the fixed point index of F in M_i is zero, (see Appendix). From Lemma 1, we may move all the fixed points of F on $|M_i - \dot{M}_i|$ to any single point and then cancel this fixed point (see page 123 of [2]). If the map g in $\dot{M}_i$ has a fixed point A, then applying Lemma 1 as many times as necessary we may move all the fixed points of F on $|M_i| - |\dot{M}_i|$ to A and finally get the map G.

In order to prove $n(K) \leqq m(K)$, we introduce the concept of fixed point classes on an open subset.

DEFINITION 2. Assume that U is an open subset of the polyhedron $|K|$ of a locally finite complex K where $\bar{U}$ is compact. Assume that a map f: $\bar{U} \to |K|$ has no fixed point on $\dot{U}$. Fixed points a and b of f in U are said to belong to the same *fixed point class* if there is a path $P(t)$ on U such that $P(0) = a$, $P(1) = b$, and $f[P(t)] \cong P(t)$ rel $\{a, b\}$ on $|K|$.

We may define the index of fixed point classes. The fixed point class with a nonzero index is called an essential fixed point class. The number of essential fixed point classes of f on U is finite.

DEFINITION 3. Suppose that a homotopy f_t: $\bar{U} \to |K|$, $0 \leqq t \leqq 1$, has no fixed points on $\dot{U}$, $f_0(a) = a$, $f_1(b) = b$ and that $P(t)$ is a path on U connecting a and b such that

$$f_t[P(t)] \cong P(t) \operatorname{rel} \{a, b\} \text{ on } |K| .$$

Thus we say there is a *homotopy correspondence* between the fixed point class of f_0 on U which contains a and the fixed point class of f_1 on U which contains b. This homotopy correspondence is a one-to-one correspondence between all the essential fixed point classes of f_0 and all the essential fixed point classes of f_1. The corresponding classes have the same index.

LEMMA 5. *Suppose that K is a locally finite complex and that $1 \cong f$: $|K| \to |K|$. Then there exists a good displacement g such that the number of fixed points of g is not greater than the number of fixed points of f.*

Proof.

(1) If f has fixed points on $|M_s| - \dot{M}_s$ for some M_s of K, we arbitrarily assign a point in $\dot{M}_s$. The set of the assigned points and the fixed points of f on $\dot{M}(K)$ are denoted by $\{b_1, b_2, \cdots\}$, then the number of points in $\{b_1, b_2, \cdots\}$ is not greater than the number

of fixed points of f. We write

$$\{c_1, c_2, \cdots\} = \dot{M}(K) - \{b_1, b_2, \cdots\} .$$

Let $f_t: 1 \cong f: |K| \to |K|$, then $f_t(c_i)$ is a path from c_i to $f(c_i)$. Based on $f_t(c_i)$, we can construct a path $Q_i(t) = \alpha_1^i \cdot \alpha_2^i \cdots \alpha_h^i \cdot \beta^i$ that has the following four properties:

(a) for $j = 1, 2, \cdots, h$, there are points b_j^i, $c_j^i \in \mathrm{St}_K(c_i)$ and polygonal arcs θ_j^i from b_j^i to c_j^i not containing c_i (see Fig. 4) such that

$$\alpha_j^i = [c_i, b_j^i] \cdot \theta_j^i \cdot [c_j^i, c_i],\ j = 1, 2, \cdots, h ;$$

(b) $\beta^i = [c_i, b_{h+1}^i] \cdot \theta_{h+1}^i$;
where $b_{h+1}^i \in \mathrm{St}_K(c_i)$ and θ_{h+1}^i is a polygonal arc from b_{h+1}^i to $f(c_i)$ not containing c_i;

(c) $\alpha_1^i \cdot \alpha_2^i \cdots \alpha_r^i \not\cong 1$, $r = 1, \cdots, h$;

(d) $f_t(c_i) \cong Q_i(t) \operatorname{rel} \{c_i, f(c_i)\}$, $1 = 1, 2, \cdots$.

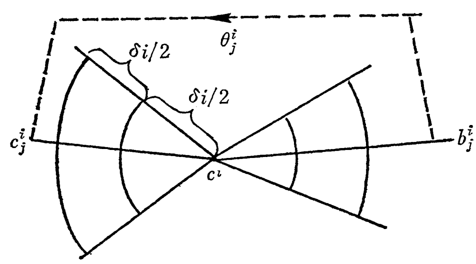

FIGURE 4

From the homotopy extension theorem, there is another homotopy $f_t: 1 \cong f: |K| \to |K|$ with $f_t(c_i) = Q_i(t)$, $i = 1, 2, \cdots$.

(2) For each c_i, we choose a sufficiently small $\delta_i > 0$ such that:

(a) $\bar{U}(c_i, \delta_i) \subset \mathrm{St}_K(c_i)$;

(b) $\bar{U}(c_i, \delta_i) \cap \bar{U}(c_j, \delta_j) = \phi$, $i \neq j$;

(c) $\bar{U}(c_i, \delta_i) \cap \theta_j^i = \phi$, $j = 1, \cdots, h+1$;

(d) $b_j^i \in |K| - \bigcup_i U(c_i, \delta_i)$, $j = 1, \cdots, h+1$,
$c_j^i \in |K| - \bigcup_i U(c_i, \delta_i)$, $j = 1, \cdots, h$.

We define a map $F: |K| \to |K|$ by

$$F(x) = \begin{cases} x, \quad x \in |K| - \bigcup_i U(c_i, \delta_i) , \\ \left[\dfrac{2\rho(x, c_i)}{\delta_i} - 1\right]x + \left[2 - \dfrac{2\rho(x, c_i)}{\delta_i}\right]c_i , \\ \qquad \delta_i/2 \leqq \rho(x, c_i) \leqq \delta_i , \\ Q_i\left[1 - \dfrac{2\rho(x, c_i)}{\delta_i}\right],\ 0 \leqq \rho(x, c_i) \leqq \delta_i/2, \end{cases}$$

thus

$$F \cong f \operatorname{rel}\{c_1, c_2, \cdots\}.$$

(3) The fixed point set of F on $|K|$ is $N_1 \cup N_2$, where

(a) $N_1 = |K| - \bigcup_i U(c_i, \delta_i)$;

(b) $N_2 = \bigcup_i \{d_1^i, d_2^i, \cdots, d_{h+1}^i, e_1^i, e_2^i, \cdots, e_h^i\}$, where $d_j^i \in (c_i, b_j^i)$, $j = 1, 2, \cdots, h+1$, and $e_j^i \in (c_j^i, c_i)$, $j = 1, \cdots, h$; moreover,

(c) $\delta_i/2 > \rho(c_i, d_1^i) > \rho(c_i, e_1^i) > \rho(c_i, d_2^i) > \rho(c_i, e_2^i) \cdots \rho(c_i, d_{h+1}^i) > 0$.

(4) If $\dot{M}_s \subset \{c_1, c_2, \cdots\}$, then F on $\dot{M}_s$ has no fixed points, and we can discuss the fixed point classes of F on $|M_s| - \dot{M}_s$.

(a) If $d_1^i \in |M_s|$, then d_1^i and $N_1 \cap |M_s|$ belong to the same fixed point class, the reason being $b_1^i \in N_1 \cap |M_s|$, and $F([b_1^i, d_1^i]) \cdot [d_1^i, b_1^i] = [b_1^i, c_i] \cdot [c_i, d_1^i] \cdot [d_1^i \cdot b_1^i] \cong 1$. Excluding these $(\bigcup_i d_1^i) \cap |M_s|$, each fixed point of $N_2 \cap |M_s|$ does not belong to the same fixed point class as $N_1 \cap |M_s|$. This fact will be proved in (b) and (c).

(b) Suppose $b(t)$ is a path from b_r^i to d_r^i $(r > 1)$ in $|M_s| - \dot{M}_s$, then there exists a loop β based at b_r^i such that $\beta \subset |M_s| \cap N_1$ and $b(t) \cong \beta \cdot [b_r^i, d_r^i] \operatorname{rel}\{b_r^i, d_r^i\}$. Hence,

$$\begin{aligned} F(b(t)) \cdot b(t)^{-1} &\cong F(\beta \cdot [b_r^i, d_r^i]) \cdot [d_r^i, b_r^i] \cdot \beta^{-1} \\ &= \beta \cdot F([b_r^i, d_r^i]) \cdot [d_r^i, b_r^i] \cdot \beta^{-1} \\ &\cong \beta \cdot [b_r^i, c_i] \cdot \alpha_1^i \cdot \alpha_2^i \cdots \alpha_{r-1}^i \cdot [c_i, d_r^i] \cdot [d_r^i, b_r^i] \cdot \beta^{-1} \\ &\cong \beta \cdot [b_r^i, c_i] \cdot \alpha_1^i \cdot \alpha_2^i \cdots \alpha_{r-1}^i \cdot [c_i, b_r^i] \cdot \beta^{-1} \not\cong 1 \end{aligned}$$

on $|K|$; because we required that $\alpha_1^i \cdot \alpha_2^i \cdots \alpha_{r-1}^i \not\cong 1$.

(c) Similarly, suppose $b(t)$ is a path from c_r^i to $e_r^i (r \geqq 1)$ in $|M_s| - \dot{M}_s$, then there exists a loop β of c_r^i such that $\beta \subset |M_s| \cap N_1$ and $b(t) \cong \beta \cdot [c_r^i, e_r^i] \operatorname{rel}\{c_r^i, d_r^i\}$. Hence,

$$b(t) \not\cong F(b(t)) \text{ on } |K|.$$

(d) Since f in $|M_s|$ has no fixed points, the index of each fixed point class of F on $|M_s| - \dot{M}_s$ is zero, in particular the index of the fixed point class containing $|M_s| \cap N_1$ is zero.

(5) We define a map g: $\dot{M}(K) \to |K|$ as follows:

$$g(c_i) = b_1^i, \; i = 1, 2, \cdots;$$

and,

$$g(b_j) = b_j, \; j = 1, 2, \cdots.$$

Consider the fixed point class of F on $|M_s| - \dot{M}_s$. Since the index of the fixed point class containing $|M_s| \cap N_1$ is zero, then there are exactly $\chi(M_s)$ points in $\dot{M}_s$, whose images under g are outside $|M_s|$ (see Appendix), so g is a good displacement.

APPENDIX. The proof of Lemma 2 of [7] (it was published previously in Chinese).

LEMMA. *Assume that K is a locally finite complex and M is a maximal two-dimensionally connected finite subcomplex. Assume that $g: \dot{M} \to |K|$ is a map such that*

(1) $g(a) \in |\mathrm{St}_K(a)|$, *for all* $a \in \dot{M}$;

(2) $g(a) \neq a$, *for all* $a \in \dot{M}$, *and g maps χ_g points of $\dot{M}$ outside of* $|M|$; *and*

(3) $[a, g(a)] \cap [b, g(b)] = \varnothing$, *for any* $a, b \in \dot{M}$.

If a map F: $|M| \to |K|$ has the following two properties:

(i) $F(a) = g(a)$, *for all* $a \in \dot{M}$; *and*

(ii) *F satisfies $S(K)$ on* $|M|$,

then $J(F, |M| - \dot{M})$ the index of fixed points of F on $|M| - \dot{M}$, equals $\chi(M) - \chi_g$.

Proof. We denote the points of $\dot{M}$ by a_j, $j = 1, \cdots, r$. Assume that $g(a_j) \notin |M|$ for $j = 1, 2, \cdots, \chi_g$ and $g(a_j) \in |M|$ for $j = \chi_g + 1, \chi_g + 2, \cdots, r$. First choose b_j, $j = 1, \cdots, \chi_g$, so that $g(a_j) \in (a_j, b_j)$, $[a_j, b_j[\subset \mathrm{St}_K(a_j) \subset |K|$ and that any two segments of $\{[a_j, b_j] \mid j = 1, \cdots, \chi_g\}$ are disjoint (from property 3). Let K' denote the complex composed of M and $[a_j, b_j]$, $j = 1, \cdots, \chi_g$. Let g' be the map g considered as a map from $\dot{M}$ to $|K'|$. Applying Lemma 3 to g' and K', we know there exists a map $G_1: |K'| \to |K'|$ such that $G_1(a_j) = g'(a_j) = g(a_j)$, $j = 1, \cdots, r$, G_1 satisfies $S(K')$ on $|K'|$. Define a map G_2: $|K'| \to |K'|$ as follows:

$$G_2(x) = \begin{cases} G_1(x), & x \in |M|; \\ g(a_j), & x \in [a_j, b_j],\ j = 1, 2, \cdots, \chi_g\,. \end{cases}$$

Since G_2 is homotopic to the identity map,

$$J(G_2, K') = \chi(K') = \chi(M)$$

by "Axiom 4" on page 52 of [2]. Since G_2 on $[a_j, b_j]$, $j = 1, \cdots, \chi_g$, only has one fixed point $g(a_j)$ of index $+1$, we obtain

$$J(G_2, |M| - \dot{M}) + \chi_g = \chi(M)$$

i.e.,

$$J(G_2, |M| - \dot{M}) = \chi(M) - \chi_g\,.$$

Now, denote the inclusion map of $|K'|$ into $|K|$ by I, and let $G_3 = IG_2: |M| \to |K|$. We have $J(G_3, |M| - \dot{M}) = \chi(M) - \chi_g$. Finally, recall the map F: $|M| \to |K|$ assumed in this lemma. Since it has the two properties listed, the map $\alpha(x, F(x), t)$ ([2], pages 124–126), for $x \in |M|$, $0 \leqq t \leqq 1$, is a homotopy equivalence between the identify mapping I and F. Since G_1 satisfies $S(K')$ on $|K'|$, it also satisfies $S(K)$ on $|K'|$. Moreover, G_3 satisfies $S(K)$ on $|M|$. So $\alpha(x, G_3(x), t)$,

$x \in |M|$, $0 \leqq t \leqq 1$, is a homotopy equivalence from I to G_3. Furthermore $\alpha(x, F(x), t) = \alpha(x, G_3(x), t)$ when $x \in \dot{M}$, $0 \leqq t \leqq 1$. Consequently, employing the homotopy extension theorem on $|M|$, $F \cong G_3 \operatorname{rel} \dot{M}$. Thus we get the conclusion of this lemma: $J(F, |M| - \dot{M}) = J(G_3; |M| - \dot{M}) = \chi(M) - \chi_g$.

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Received August 7, 1980 and in revised form February 4, 1981.

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