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TOPOLOGICAL TRANSVERSALITY II. APPLICATIONS TO THE NEUMANN PROBLEM FOR $y'' = f(t, y, y')$

A. GRANAS, R. B. GUENTHER AND J. W. LEE

In this paper the Neumann problem for the nonlinear equation $y'' = f(t, y, y')$ is studied. A priori bounds are derived and the results of Granas, Guenther and Lee, are invoked to obtain existence theorems. The existence theorems are in many cases quite different from those of the Dirichlet problem, e.g. it is possible to obtain general existence theorems where $f(t, y, y')$ can grow very rapidly in the y' variable.

1. Introduction. Several existence theorems are established for the Neumann problem,

$$\text{(N)} \qquad \begin{aligned} &y'' = f(t, y, y'), \qquad 0 \leq t \leq 1, \\ &y'(0) = r, \qquad y'(1) = s, \end{aligned}$$

where f: $[0, 1] \times R \times R \to R$ is continuous and r, s are given numbers. Here $R = (-\infty, \infty)$. The Neumann problem is homogeneous if $r = s = 0$; otherwise, it is inhomogeneous. Our results include analogues for the Neumann problem of well-known results for the Dirichlet problem as well as some surprising results which set the existence theory for the nonlinear Neumann problem apart from that of the other standard boundary value problems. Also, an unexpectedly sharp distinction in the behavior of the homogeneous and inhomogeneous Neumann problems emerges. In the comparisons made below, we restrict our remarks to the Dirichlet and Neumann problems for $y'' = f(t, y, y')$; however, the assertions made for the Dirichlet problem hold under virtually the same hypotheses when any set of standard Sturm-Liouville boundary conditions replace the Dirichlet conditions, provided these conditions do not include a pure Neumann condition at $t = 0$ and/or $t = 1$. Thus, the comparisons indicate the special position of the Neumann problem among Sturm-Liouville problems. Precise formulations of the results for Sturm-Liouville problems can be found in [4] and [5].

We say that a function $f(t, y, p)$ defined on $[0, 1] \times R \times R$ satisfies the *Bernstein growth condition* if there are nonnegative functions $A(t, y)$ and $B(t, y)$ defined on $[0, 1] \times R$ and bounded on compact sets such that $|f(t, y, p)| \leq A(t, y)p^2 + B(t, y)$. The following result for the Dirichlet problem is well known; see, for instance, [4].

THEOREM 1.1. *Let f be continuous on* $[0, 1] \times R \times R$ *and assume*:

(i) *There is a constant* $M \geq 0$ *such that*,

$$yf(t, y, 0) \geq 0 \quad \text{for all } |y| \geq M;$$

(ii) *f satisfies the Bernstein growth condition. Then the Dirichlet problem*,

$$\text{(D)} \qquad \begin{aligned} &y'' = f(t, y, y'), \qquad 0 \leq t \leq 1, \\ &y(0) = r, \qquad y(1) = s, \end{aligned}$$

has at least one solution for any choice of r and s.

Theorem 1.1 extends the following basic theorem proved by S. Bernstein in **[1]**.

THEOREM 1.2. *Let f be continuously differentiable on* $[0, 1] \times R \times R$ *and assume*:

(i)′ *There is a constant* $k > 0$ *such that*

$$f_y(t, y, p) \geq k, \quad \text{on } [0, 1] \times R \times R;$$

(ii) *f satisfies the Bernstein growth condition. Then the Dirichlet problem* (D) *has a unique solution for any choice of r and s.*

The analogue of Theorem 1.1 for the homogeneous Neumann problem is known to be true; see **[4]**. Surprisingly, the analogue of Theorem 1.1 does not hold for inhomogeneous boundary data. A simple counterexample is,

$$y'' = y'^2, \qquad y'(0) = 1, \qquad y'(1) = s,$$

where s is arbitrary. The hypotheses of Theorem 1.1 hold; however, by direct integration any solution to $y'' = y'^2$ and $y'(0) = 1$ satisfies $y = y(0) - \log|1 - t|$. This shows that the Neumann problem has no solution on $0 \leq t \leq 1$. In contrast to this, Theorem 1.2 below establishes the analogue to Bernstein's Theorem 1.2 for the inhomogeneous Neumann problem.

The Neumann and Dirichlet problems differ in another important respect. Examples, going back at least to S. Bernstein, show that conditions (i) and (ii) of Theorem 1.1 cannot be relaxed substantially and still imply the solvability of the Dirichlet problem, unless rather strong additional assumptions are made about the explicit structure of the nonlinearity f. (In this regard see **[5]**.) On the other hand, we establish existence theorems below for the inhomogeneous Neumann problem under fairly arbitrary rates of growth of $f(t, y, p)$ in its p variable. In fact, in Theorem 3.3 below the principal growth rate restriction on f is that $|f(t, y, p)|$

should not be too small when $|p| \to \infty$ and (t, y) is restricted to a compact set!

In this introduction as well as in what follows, we often assume that f satisfies the Bernstein growth condition. In each such case, we could slightly extend the generality of our presentation by replacing the Bernstein condition by an appropriate Nagumo condition. Since the statements of our results are less technical when the Bernstein condition is used rather than a Nagumo condition and the gain in generality is not great, we omit these more general formulations. The interested reader can easily supply them.

2. A general existence principle. In this section we set some standard notation and state, for reference purposes, a general existence theorem whose proof is based on the topological transversality theorem of Granas; see **[2]**, **[4]**, or **[6]** for details.

The Banach space of continuous functions on [0, 1] will de denoted by C or C^0 and,

$$|u|_0 = \max\{|u(t)|: 0 \leq t \leq 1\}$$

defines the norm of a continuous function u. Likewise, $C^k = C^k[0, 1]$ denotes the Banach space of functions u which have a continuous kth derivative on [0, 1] and,

$$|u|_k = \max\{|u|_0, |u'|_0, \dots, |u^{(k)}|_0\}$$

defines the norm on C^k.

Let $\mathfrak{B}$ denote the set of functions u which satisfy the linear inhomogeneous boundary conditions $W_i(u) = r_i$, $i = 1, 2$, where,

$$W_i(u) = a_{i1}u(0) + a_{i2}u'(0) + b_{i1}u(1) + b_{i2}u'(1),$$

a_{i1}, a_{i2}, b_{i1}, b_{i2}, and r_i are given numbers. In this context we let $\mathfrak{B}_0$ be the set of functions satisfying the corresponding homogeneous boundary conditions $W_i(u) = 0$ for $i = 1, 2$, and $C^k_{\mathfrak{B}}$ be the set of functions in C^k which also satisfy the boundary conditions $\mathfrak{B}$.

Let $b(t)$ and $c(t)$ be continuous functions defined on [0, 1], and for u in C^2 define $Lu = u'' + b(t)u' + c(t)u$, a second order linear differential operator. In this context we have:

THEOREM 2.1. *Assume g: $[0, 1] \times R \times R \to R$ is continuous, L: $C^2_{\mathfrak{B}_0} \to C$ is one-to-one (where $\mathfrak{B}$, $\mathfrak{B}_0$, and L are as above) and that there is a constant $M < \infty$ such that $|y|_2 < M$ for each solution y to the boundary value problem,*

$$\begin{aligned} &Ly = \lambda g(t, y, y'), \qquad 0 \leq t \leq 1, \\ &y \in \mathfrak{B}, \end{aligned}$$

for each λ *in* $[0, 1]$. *Then the boundary value problem,*

$$Ly = g(t, y, y'), \qquad 0 \leq t \leq 1,$$
$$y \in \mathfrak{B},$$

has at least one solution in $C^2_{\mathfrak{B}}$.

The following lemmas, see [4] and [5], are often used to establish the a priori bound M in Theorem 2.1.

LEMMA 2.2. *Suppose there is a constant* $M_0 \geq 0$ *such that,*

$$yg(t, y, 0) > 0 \quad \text{for } |y| > M_0.$$

If $y(t)$ *is a solution to the differential equation* $y'' = g(t, y, y')$ *and* $|y(t)|$ *does not achieve its maximum at* $t = 0$ *or* $t = 1$, *then,*

$$|y(t)| \leq M_0 \quad \text{for } t \text{ in } [0, 1].$$

If in addition $y(t)$ *satisfies homogeneous Neumann boundary conditions and* $|y(t)|$ *achieves its maximum at* $t_0 = 0$ *or* 1, *then* $|y(t_0)| \leq M_0$.

LEMMA 2.3. *Let* $M_0 \geq 0$ *be fixed and* $y(t)$ *be a solution to* $y'' = g(t, y, y')$ *whose derivative vanishes at least once in* $[0, 1]$ *and for which* $|y|_0 \leq M_0$. *Suppose there are constants* $A, B \geq 0$ *such that* $|g(t, y, p)| \leq Ap^2 + B$ *for all* (t, y, p) *in* $[0, 1] \times [-M_0, M_0] \times R$. *Then there is a constant* M_1 *depending only on* M_0, A, *and* B *such that* $|y'(t)| \leq M_1$ *for* t *in* $[0, 1]$.

3. The nonlinear Neumann problem. Let f: $[0, 1] \times R \times R \to R$ be continuous and consider the Neumann problem,

$$\text{(N)} \qquad \begin{aligned} y'' &= f(t, y, y'), \qquad 0 \leq t \leq 1, \\ y'(0) &= r, \qquad y'(1) = s, \end{aligned}$$

where r, s are given constants. Our first result is the analogue of Bernstein's Theorem 1.2 for the Neumann problem.

THEOREM 3.1. *Assume* $f(t, y, p)$ *is continuously differentiable with respect to* y *for all* (t, y, p) *in* $[0, 1] \times R \times R$ *and:*

(i) *There is a constant* $k > 0$ *such that*

$$f_y(t, y, p) \geq k,$$

for (t, y, p) *in* $[0, 1] \times R \times I$ *where* $I = [\min(r, s), \max(r, s)]$;

(ii) f *satisfies the Bernstein growth condition. Then the Neumann problem* (N) *has a solution for all choices of* r *and* s.

Proof. To prove existence, consider the family of Neumann problems,

$$(\mathrm{N})_\lambda \qquad \begin{aligned} &y'' = \lambda[f(t, y, y') - y] + y \\ &y'(0) = r, \qquad y'(1) = s, \end{aligned}$$

for λ in [0, 1]. This family of problems can be written as

$$\begin{aligned} &Ly = \lambda g(t, y, y'), \\ &y \in \mathfrak{B}, \end{aligned}$$

where $Ly = y'' - y$, $g(t, y, y') = f(t, y, y') - y$, and $\mathfrak{B}$ stands for the inhomogeneous Neumann conditions. It is easily checked that L: $C^2_{\mathfrak{B}_0} \to C$ is one-to-one in this case. So the existence of a solution to $(\mathrm{N})_1$, that is to (N), follows if the a priori bound required in Theorem 2.1 can be established. Thus, to show that (N) has a solution it suffices to prove that there is a constant $M < \infty$ such that $|y|_2 < M$ for each solution y to $(\mathrm{N})_\lambda$ for λ in [0, 1].

Suppose y is a solution to $(\mathrm{N})_\lambda$ for some λ in [0, 1] and define $u = y - \rho$ where

$$\rho(t) = \frac{s - r}{2} t^2 + rt.$$

Then u satisfies

$$(3.1) \qquad \begin{aligned} &u'' = F(t, u, u'), \\ &u'(0) = 0, \qquad u'(1) = 0, \end{aligned}$$

where

$$F(t, u, u') = \lambda[f(t, u + \rho, u' + \rho') - (u - \rho)] + (u + \rho) - (s - r).$$

Also,

$$\begin{aligned} uF(t, u, 0) &= u\{\lambda f(t, u + \rho, \rho') - \lambda(u + \rho) + (u + \rho) - (s - r)\} \\ &= u\{\lambda[f(t, u + \rho, p') - f(t, \rho, \rho')] + \lambda f(t, \rho, \rho') \\ &\qquad\qquad -\lambda(u + \rho) + (u + \rho) - (s - r)\} \\ &\geq \lambda k u^2 + (1 - \lambda) u^2 - l|u|, \end{aligned}$$

where,

$$l = \max\{|f(t, w, p)| + |w| + |s - r|\},$$

and the maximum is over (t, w, p) in $[0, 1] \times [-|\rho|_{0'} |\rho|_0] \times I$. The inequality

$$uF(t, u, 0) \geq [\lambda k + (1 - \lambda)] u^2 - l|u|$$

implies $uF(t, u, 0) > 0$ for $|u| > l/k'$ where $k' = \min(l, k)$. Then Lemma 2.2 implies $|u(t)| \leq l/k' \equiv M_0$ for any solution $u(t)$ to (3.1). Since $F(t, u, p)$ satisfies a Bernstein growth condition because f does, it follows easily that there are constants A, B independent of λ, such that $|F(t, u, p)| \leq Ap^2 + B$ for all (t, u, p) in $[0, 1] \times [-M_0, M_0] \times R$ and λ in $[0, 1]$. Then Lemma 2.3 implies that there is a constant $M_1 < \infty$ independent of λ such that $|u'|_0 < M_1$ for any solution u to (3.1). Finally the differential equation in (3.1) together with the a priori bounds already found for u and u' imply that $|u''|_0 < M_2$ for a constant $M_2 < \infty$ and independent of λ. Thus, $|u|_2 < M' = \max(M_0, M_1, M_2)$ for any solution $u(t)$ to (3.1), and so $|y|_2 < M' + |\rho|_2 \equiv M$ for any solution $y(t)$ to $(\mathrm{N})_\lambda$. The existence proof is complete.

The next result has an essentially different character because it allows quite general growth of f in its derivative variable, p.

THEOREM 3.2. *Suppose $f(t, y, p)$ is continuously differentiable with respect to its three arguments on $[0, 1] \times R \times R$. Assume:*

(i) *There is a constant $k > 0$ such that $f_y(t, y, p) \geq k$ for all (t, y, p) in $[0, 1] \times R \times I$ where $I = [\min(r, s), \max(r, s)]$;*

(ii) *There is a constant M_1 such that*

$$pf_t(t, y, p) + p^2 f_y(t, y, p) > 0,$$

for all (t, y) in $[0, 1] \times [-\hat{M}_0, \hat{M}_0]$ and $|p| > M_1$ where $\hat{M}_0 = |\rho|_0 + l/k'$ and l, k', $\rho(t)$ are as defined above. Then the Neumann problem (N) *has a solution.*

REMARK. Condition (ii) is often checked by inspection in practice by showing that for (t, y) in a compact set, $pf_t + p^2 f_y > 0$ provided $|p|$ is sufficiently large.

Proof. Just as before the existence proof reduces to establishing an a priori bound M such that $|y|_2 < M$ for all solutions $y(t)$ to $(\mathrm{N})_\lambda$ for λ in $[0, 1]$. In view of (i) it follows, exactly as in the proof of Theorem 3.1, that $|y|_0 < |\rho|_0 + l/k'$.

To get an a priori bound on y' we let $u = y'$ and differentiate the original family of problems $(\mathrm{N})_\lambda$ to obtain,

$$\text{(3.2)} \qquad \begin{aligned} &u'' = \lambda\left[f_t(t, y, u) + f_y(t, y, u)u + f_p(t, y, u)u' - u\right] + u, \\ &u(0) = r, \qquad u(1) = s. \end{aligned}$$

Let $G(t, u, u')$ stand for the right-hand side of the differential equation in (3.2). Then,

$$uG(t,u,0) = \lambda\left[f_t(t, y, u)u + f_y(t, y, u)u^2\right] + (1-\lambda)u^2 > 0$$

if $|u| > M_1$. Lemma 2.2 implies that $|u|_0 < \max(M_1, |r|, |s|)$; that is, $|y'|_0 < \max(M_1, |r|, |s|)$ for every solution y to $(\mathrm{N})_\lambda$. The differential equation in $(\mathrm{N})_\lambda$ now yields an a priori bound for $|y''|_0$ and consequently there is a constant M such that $|y|_2 < M$ for each solution $y(t)$ to $(\mathrm{N})_\lambda$ for λ in $[0, 1]$. This completes the existence proof.

A rather interesting corollary of Theorem 3.2 pertains to autonomous differential equations.

COROLLARY 3.3. *Suppose $f = f(y, p)$ is continuously differentiable on $R \times R$ and that $f_y(y, p) \geq k > 0$ for some constant k and all (y, p) in $R \times R$. Then the Neumann problem,*

$$\begin{aligned} &y'' = f(y, y'), \\ &y'(0) = r, \qquad y'(1) = s, \end{aligned}$$

has a unique solution.

Proof. Existence follows from the theorem and uniqueness from §5 in [4].

The preceding theorems are of primary interest in the case of *inhomogeneous* Neumann conditions where $r^2 + s^2 > 0$ because in the case of *homogeneous* Neumann conditions existence of a solution can be established with virtually no growth rate restrictions on the nonlinearity. Specifically we have:

THEOREM 3.4. *Suppose f: $[0, 1] \times R \times R \to R$ is continuous and:*
(i) *There is a constant $M_0 \geq 0$ so that,*

$$yf(t, y, 0) > 0 \quad \text{for } |y| > M_0;$$

(ii) *There is a constant $M_1 > 0$ so that,*

$$\inf|f(t, y, p)| > 0,$$

where the infimum is calculated over (t, y) in

$$[0, 1] \times [-M_0, M_0] \quad \text{and} \quad |p| \geq M_1.$$

Then the homogeneous Newmann problem (N) *with $r = s = 0$ has at least one solution.*

Proof. Let $\varepsilon = \rho/2(M_0 + 1)$, where $\rho > 0$ is the infimum in (ii) and consider the family of problems,

$$(3.3)\quad \begin{aligned} &y'' - \varepsilon y = \lambda[f(t, y, y') - \varepsilon y] + (1 - \lambda)\phi(t, y, y')y'^2, \\ &y'(0) = 0, \qquad y'(1) = 0, \end{aligned}$$

where $\phi(t, y, y')$ is a bounded continuous function for which,

$$\phi(t, y, p) = \operatorname{sgn} f(t, y, p),$$

for (t, y) in $[0, 1] \times [-M_0, M_0]$ and $|p| \geq M_1$. In the appendix, the construction of such a function is given. To prove existence of a solution to (N) we note that Theorem 2.1 is again applicable with $Ly = y'' - \varepsilon y$ in (3.3). Thus, (3.3) with $\lambda = 1$, i.e. (N), will have a solution provided there is a constant $M < \infty$ such that $|y|_2 < M$ for each solution y to (3.3) for λ in $[0, 1]$.

To establish the required a priori bound note that the differential equation in (3.3) can be expressed as,

$$y'' = \lambda f(t, y, y') + (1 - \lambda)\varepsilon y + (1 - \lambda)\phi(t, y, y')y'^2.$$

Since,

$$y[\lambda f(t, y, 0) + (1 - \lambda)\varepsilon y] = \lambda y f(t, y, 0) + (1 - \lambda)\varepsilon y^2 > 0,$$

for $|y| > M_0$ by (i), Lemma 2.2 implies that $|y|_0 \leq M_0$ for any solution $y(t)$ to (3.3) for λ in $[0, 1]$.

Next let $M_1' = \max(1, M_1, \rho)$ and suppose y is a solution to (3.3). Then $|y|_0 \leq M_0$ as we have seen. Suppose $|y'|_0 > M_1'$. Then, in view of the boundary date, there is a point t_0 in $(0, 1)$ where $|y'(t)|$ achieves its maximum, $y''(t_0) = 0$, and $|y'(t_0)| > M_1'$. The differential equation gives,

$$0 = y_0'' = \lambda f(t_0, y_0, y_0') + (1 - \lambda)\left[\varepsilon y_0 + \phi(t_0, y_0, y_0')y_0'^2\right]$$

where $y_0 = y(t_0)$, etc. Multiply the equation above by $\operatorname{sgn} f_0$, where $f_0 = f(t_0, y_0, y_0')$, and use $\phi(t_0, y_0, y_0') = \operatorname{sgn} f(t_0, y_0, y_0')$ to obtain,

$$(\operatorname{sgn} f_0)(\lambda - 1)\varepsilon y_0 = \lambda|f_0| + (1 - \lambda)y_0'^2.$$

Since,

$$\varepsilon|y_0| \leq [\rho/2(M_0 + 1)]M_0 < \rho/2, \qquad |f_0| \geq \rho, y_0'^2 \geq M_1'^2 \geq M_1' \geq \rho,$$

the preceding equation yields the inequality,

$$\frac{\rho}{2} \geq \min(|f_0|, y_0'^2) \geq \rho,$$

a contradiction. Thus, $|y'|_0 \leq M_1'$ for each solution $y(t)$ to (3.3). The differential equation now gives a bound independent of λ for $|y''|_0$ and so $|y|_2 < M$ for some constant M. The proof is complete.

Theorem 3.4 has as a corollary the Neumann problem analogue of a theorem of Nirenberg [7] for periodic boundary data.

COROLLARY 3.5. *Let* α, β: $[0, 1] \times R \times R \to R$ *be continuous. Assume*:

(i) $\alpha(t, y, p) > 0$;

(ii) $|\beta(t, y, p)| \to \infty$ *and* $\alpha(t, y, p)/\beta(t, y, p) \to 0$ *as* $|p| \to \infty$ *uniformly for* (t, y) *in a compact set in* $[0, 1] \times R$.

(iii) *There is a constant* $M \geq 0$ *such that*

$$\frac{\beta(t, y, 0)}{y\alpha(t, y, 0)} < 1 \quad \textit{for } |y| > M.$$

Then the homogeneous Neumann problem,

$$y'' = y\alpha(t, y, y') - \beta(t, y, y'),$$
$$y'(0) = 0, \qquad y'(1) = 0,$$

has a solution.

The proof is obtained by observing that $f(t, y, p) = y\alpha(t, y, p) - \beta(t, y, p)$ satisfies the hypotheses of Theorem 3.4.

As another corollary of Theorem 3.4, we obtain the following counterpart to Theorem 3.1 for the inhomogeneous Neumann problem. In Theorem 3.1 the monotonicity condition $f_y \geq k > 0$ is coupled with a quadratic growth rate restriction on f with respect to increasing $|p|$. Surprisingly, Theorem 3.6 shows that we can eliminate the quadratic growth restriction entirely provided only that f is suitably bounded away from zero as $|p|$ increases.

THEOREM 3.6. *Suppose* f: $[0, 1] \times R \times R \to R$ *is continuous and*:

(i) *There is a constant* $k > 0$ *such that* $f_y(t, y, p) \geq k$ *for* (t, y, p) *in* $[0, 1] \times R \times I$ *where* $I = [\min(r, s), \max(r, s)]$;

(ii) *There is a constant* $M_1 > 0$ *so that*,

$$\inf|f(t, y, p) + (r - s)| > 0,$$

where the infimum is calculated for (t, y) *in* $[0, 1] \times [-M_0 - |\rho|_0, M_0 + |\rho|_0]$ *and* $|\rho| \geq M_1$. *(Here* $M_0 = l_1/k$ *is a constant defined in the proof below.) Then the Neumann problem* (N) *has a solution.*

REMARK. In interesting special cases (ii) is often confirmed by inspection by showing the $|f(t, y, p) + (s - r)|$ becomes infinite for (t, y) in a compact set as $|\rho| \to \infty$.

Proof. Let $\rho(t) = [(s-r)/2]t^2 + rt$ as usual. Then $y(t)$ solves $y'' = f(t, y, y')$ with $y'(0) = r$ and $y'(1) = s$ if and only if $u = y - \rho$ solves,

$$(3.4) \quad \begin{aligned} &u'' = f(t, u + \rho(t), u' + \rho'(t)) - (s - r) \equiv F(t, u, u') \\ &u'(0) = 0, \qquad u'(1) = 0. \end{aligned}$$

From (i) we get

$$\begin{aligned} uF(t, u, 0) &= uf(t, u + \rho(t), \rho'(t)) + u(r - s) \\ &= u[f(t, u + \rho(t), \rho'(t)) - f(t, \rho(t), p'(t))] \\ &\quad + uf(t, \rho(t), \rho'(t)) + u(r - s) \geq u^2 k - |u| l_1, \end{aligned}$$

where $l_1 = \max | f(t, y, p) + (r - s) |$ and the maximum is over $[0, 1] \times J \times I$, I has the usual meaning and $J = [\min \rho(t), \max \rho(t)]$ with the min and max computer over $[0, 1]$. Consequently, $uF(t, u, 0) > 0$ if $| u | > M_0 \equiv l_1/k$.

Next, if $U = \{(t, u, v)\colon (t, u) \in [0,-] \times [-M_0, M_0], \ | v | \geq M_1 + | \rho' |_0\}$ and V is the set of points (t, y, p) when (t, y) in $[0, 1] \times [-M_0 - | \rho |_0, M_0 + | \rho |_0]$ and $| p | \geq M_1$ simple triangle inequality estimates confirm that

$$\inf_U |F(t, u, v)| \geq \inf_V |f(t, y, p) + (r - s)l| > 0,$$

by (ii). Thus the hypotheses of Theorem 3.4 are satisfied by the homogeneous Neumann problem (3.4). Consequently, (3.4) has a solution $u(t)$ and $y(t) = u(t) + \rho(t)$ solves (N).

We conclude this section with a comment about uniqueness of solutions to the nonlinear Neumann problems above: It is proven in §5 of [4] that the inhomogeneous Neumann problem has a unique solution provided $f(t, y, y')$ is continuously differentiable on $[0, 1] \times R \times R$, $f_y \geq 0$ there, and $f_y(t_0, y, p) > 0$ for a fixed t_0 in $[0, 1]$. In many practical applications of Theorems 3.1, 3.2, and 3.6, condition (i) in each theorem is checked by showing that $f_y \geq k > 0$ on $[0, 1] \times R \times R$; in these cases the solution in unique.

4. Examples and remarks.

EXAMPLE 1. The use of Lemma 2.2 to obtain a priori bounds for the homogeneous Neumann problem uses $r = s = 0$ in an essential way. For instance, the Neumann problem,

$$\begin{aligned} &y'' = a^2 y - 1, \qquad a > 0 \text{ constant}, \\ &y'(0) = 0, \qquad y'(1) = s, \end{aligned}$$

can be embedded naturally in the family of problems,

$$\begin{aligned} y'' - y &= \lambda(a^2 y - 1 - y), \\ y'(0) = 0, &\quad y'(1) = s, \end{aligned} \tag{4.1}$$

for λ in [0, 1]. The differential equation is,

$$y'' = \lambda(a^2 y - 1) + (1 - \lambda)y \equiv g(t, y, y'),$$

and we have $yg(t, y, 0) > 0$ for $|y| > m \equiv 1/\min(a^2, 1)$, for all λ in [0, 1]; however, if the initial datum s is chosen so that $s > mM \sinh M$ where $M = \max(a^2, 1)$, then each solution to (4.1) satisfies $|y|_0 > m$. To see this simply note that the unique solution of (4.1) is,

$$y = \frac{s}{\alpha \sinh \alpha} \cosh \alpha t + \frac{\lambda}{\alpha^2},$$

where $\alpha = \sqrt{\lambda a^2 + (1 - \lambda)}$. The solution is strictly increasing so, for t in [0, 1],

$$y(t) \geq \frac{s}{\alpha \sinh \alpha} + \frac{\lambda}{\alpha^2} \geq \frac{s}{M \sinh M} > m.$$

Thus, no solution to (4.1) satisfies $|y|_0 \leq m$.

EXAMPLE 2. It is possible to construct whole families of problems for which (N) is not solvable. The simple observation to follow shows that the existence of a solution to the Neumann problem depends much more delicately on the relation between the boundary data r, s and the nonlinearity f than is the case for the Dirichlet problem. Indeed if problem (N) has a solution y the mean value theorem implies that $y''(\sigma) = s - r$ for some σ in (0, 1). Thus, $s - r$ must be in the range of f. Consequently, (N) cannot have a solution, regardless of any growth rate or monotonicity assumptions which f may satisfy, if f never take the value $s - r$. For instance, none of the problems,

$$\begin{aligned} y'' &= y'^2 + 1, & y'(0) &= 0, & y'(1) &= 0, \\ y'' &= 1 + ty^2, & y'(0) &= 1, & y'(1) &= 0, \\ y'' &= y'^2, & y'(0) &= 2, & y'(1) &= 1, \end{aligned}$$

has a solution; however, in each case, the corresponding Dirichlet problem is solvable. In fact for the first and third problems this can be seen by direct integration. For the second we note that $\beta = 2$ is an upper solution and $\alpha = t^2 - t$ is a lower solution to the Dirichlet problem. Also, the third Neumann problem shows that the condition $f_y \geq k > 0$ in Theorem 3.1 cannot be relaxed to $yf(t, y, 0) \geq 0$ in contrast to the situation for the Dirichlet problem (Theorem 1.1).

EXAMPLE 3. Care must be taken in reducing a Neumann problem to a Dirichlet problem as was done in the proof of Theorem 3.2. For instance, the problem,

$$y'' = y'^2,$$
$$y'(0) = 2, \qquad y'(1) = 1,$$

has no solution by Example 2. On the other hand, if one tried to construct a solution by setting $y' = u$ one would obtain the problem,

$$u'' = 2uu',$$
$$u(0) = 2, \quad u(1) = 1,$$

for u. This problem does in fact have a unique solution,

$$u = \frac{a(1 - be^{2at})}{1 + be^{2at}},$$

where $a \approx 2.0238$ and $b \approx 0.0051$. However, the function $\int_0^t u(s)\,ds +$ constant does not satisfy the original Neumann problem no matter how the constant is chosen.

EXAMPLE 4. The Neumann problem,

$$y'' = 1 + ty'^2 + y,$$
$$y'(0) = 1, \qquad y'(1) = 0,$$

has a solution by Theorem 3.1. (Compare with problem 2 in Example 2.)

EXAMPLE 5. The Neumann problem for each of the equations,

$$y'' = t(y')^{2n+1} + y + 1, \qquad n \geq 0 \text{ an integer},$$
$$y'' = (\cos t)y + y^n(y')^m, \quad n \geq 1 \text{ odd}, m \geq 0 \text{ even},$$
$$y'' = e^y(1 + |y'|^m), \qquad m \geq 0 \text{ and real},$$

has a solution on [0, 1] for arbitrary initial date r, s by Theorem 3.2.

EXAMPLE 6. The Neumann problem,

$$y'' = y^3 + ty + 1 + y'^n, \qquad n \geq 0, \text{ integer},$$
$$y'(0) = 0, \qquad y'(1) = 0,$$

has a solution by Theorem 3.4, while,

$$y'' = y(ty'^2 + 1) - ((\cos t)y'^3 + y^2y'^4),$$
$$y'(0) = 0, \qquad y'(1) = 0,$$

has a solution by Corollary 3.5.

EXAMPLE 7. The Neumann problem,

$$y'' = y(1 + y'^4),$$
$$y'(0) = r, \qquad y'(1) = s,$$

has a unique solution for all r, s by Corollary 3.3. It is interesting to note that Theorem 3.6 does not apply in this case.

EXAMPLE 8. The homogeneous Dirichlet problem for

$$y'' = 1 + y + ay'^4$$

where $a > 0$ is a constant only has a solution for $a < b$ where b is some constant not greater than $3^8/2^5$; see [5]. In contrast, the homogeneous Neumann problem for this equation has a solution for all $a > 0$ by Theorem 3.4. Additionally, by Theorem 3.6 the inhomogeneous Neumann problem has a solution for all r and s.

EXAMPLE 9. To illustrate the scope of Theorem 3.6 we note that it implies that the inhomogeneous Neumann problem for,

$$y'' = \sum_{k=1}^{m} a_k(t) y'^k + b(t, y),$$

always has a solution provided:

(i) $a_k(t)$ is continuous for $k = 1, \ldots, m$, $a_m(t) \neq 0$ for t in $[0, 1]$;

(ii) $b(t, y)$ is continuously differentiable and $b_y(t, y) \geq k > 0$ for some constant k.

The equation in Example 8 is a special case.

EXAMPLE 10. It is useful to contrast the preceding example with some related work in [5]. Let $f(t, y, p)$ be continuous on $[0, 1] \times R \times R$ and $g(t, y, p) = f(t, y, p) - y$. If for each fixed (t, y), $g(t, y, p) = 0$ has both positive and negative solutions for p, define,

$$r(t, y) = \sup\{p : g(t, y, p) = 0\},$$

and,

$$s(t, y) = \inf\{p : g(t, y, p) = 0\}.$$

It is proved in [5] that the homogeneous Neumann problem for $y'' = f(t, y, y')$ has a solution if:

(a) There is a constant $M \geq 0$ such that

$$yf(t, y, 0) \geq 0 \quad \text{for } |y| > M;$$

(b) The functions $r(t, y)$ and $s(t, y)$ are defined and continuous on $[0, 1] \times [-M, M]$.

This general result implies that the homogeneous Neumann problem for $y'' = \Sigma_{k=0}^{m} a_k(t, y) y'^k$ has a solution provided: The $a_k(t, y)$ are continuous on $[0, 1] \times R$ such that,

(i) for some $M \geq 0, ya_0(t, y) \geq 0$ for $|y| > M$;

(ii) the polynomial in p, $\Sigma_{k=0}^{m} a_k(t, y) p^k$ has both positive and negative roots for each (t, y) in $[0, 1] \times [-M, M]$ and $a_m(t, y) \neq 0$ there.

For instance, it is not difficult to check that the homogeneous Neumann problem for,

$$y'' = (\cos t) y'^6 + 2e^t y y'^5 + y'^3 + y'^2 - (t^2 + 1) y' + y + 1,$$

has a solution. In this case $M = 1$ in (i). Note that Theorem 3.6 does not apply here because $f_y = 2e^t y'^5 + 1$; however, Theorem 3.4 for the homogeneous case does apply.

Appendix

A function such as $\phi(t, y, p)$ used in the proof of Theorem 3.4 can be constructed along the following lines. Let M_0 and M_1 be defined as in Theorem 3.4 and $S = [0, 1] \times [-M_0, M_0]$. By the continuity of f and the definition of M_1 there is a P, $0 \leq P < M_1$ such that $\inf |f(t, y, p)| > 0$ for (t, y) in S and $|p| \geq P$. Let,

$$A = \{(t, y, p) \colon (t, y) \in S, |p| \geq M_1\},$$

$$B = \{(t, y, p) \colon (t, y) \in S, |p| \leq P\},$$

and define $\rho\colon S \times R \to [0, 1]$ by

$$\rho(x) = \frac{d(x, B)}{d(x, A) + d(x, B)},$$

where $d(x, A)$ is the distance of $x = (t, y, p)$ from the set A. Then $\rho = 1$ on A and $\rho = 0$ on B. Define $\phi(t, y, p) = \rho(t, y, p) \operatorname{sgn} f(t, y, p)$ where $\operatorname{sgn} r = +1$ if $r \geq 0$ and $\operatorname{sgn} r = -1$ if $r < 0$. Then ϕ is bounded and continuous on $S \times R$ and equal to $\operatorname{sgn} f(t, y, p)$ for $|p| \geq M_1$. The domain of ϕ can be extended to $[0, 1] \times R \times R$ by setting $\phi(t, y, p) = \phi(t, M_0 \operatorname{sgn} y, p)$ for (t, y, p) with $|y| > M_0$.

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UNIVERSITY OF MONTREAL
MONTREAL, CANADA
AND
OREGON STATE UNIVERSITY
CORVALLIS, OR 97331