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APPLICATIONS OF DIFFERENTIATION OF $\mathfrak{L}_p$-FUNCTIONS TO SEMILATTICES

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Let S be a commutative semigroup with identity 1 such that $x^2 = x$ for each $x \in S$ (i.e. S is a semilattice). Let Γ denote the set of semicharacters equipped with topology of simple convergence and μ be a fixed probability measure on Γ. Those real-valued functions f on S which admit disintegrations of the form $f(x) = \int_\Gamma \rho(x)\, d\mu_f(\rho)$ where either $d\mu_f = f' d\mu$ with $f' \in L_p(\mu)$ $(1 \le p \le \infty)$ or μ_f is singular with respect to μ, are characterized. This extends the previous characterization of Alo and Korvin from the case where p is either 1 or ∞ to all $p \in [1, \infty]$. Applications of this theory to the classical L_p-spaces on the n-cube are also presented. The main applications occur upon specializing to the case where S is a Boolean algebra and the functions on S that are being disintegrated are additive. Not only is the Darst decomposition theorem easily recovered, but also the theory of V^P-spaces of set functions introduced by Bochner and extended by Leader is reproved from the point of view of "differentiation". As a by-product, it is shown that every non-atomic probability measure is in the closed convex hull (topology of simple convergence) of those zero-one-valued additive set functions which are not countably additive; a curious result when applied to Lebesgue measure.

1. Preliminary. For each $x \in S$, the shift operator E_x is defined on the class of all real-valued functions $f \mid S \to \mathbf{R}$ by $(E_x f)(y) = f(xy)$. Observe that $E_x E_y = E_{xy}$ and E_1 is the identity operator which we will also denote by I. We will be interested in certain difference operators of the form $\Delta = E_x \prod_{j=1}^{k}(I - E_{x_j})$ where $x, x_1, \ldots, x_k \in S$ and introduce the notation $\Delta f(yx; \{x_j\}) = (\Delta f)(y)$, at all times distinguishing between the function Δf and its evaluation $(\Delta f)(y)$, at y. It follows that $\Delta f(1)$ is the kth difference of f $(\Delta f(x; x_1, \ldots, x_k))$ as defined in [**6** and **8**]. Recall that a real-valued function f on S is called *completely monotonic* (CM) if $(\Delta f)(1) \ge 0$ for all choices of Δ. The class CM(S) of all completely monotonic functions is the same as the "positive definite functions" discussed in [**12**] and the difference operator Δ can be seen to be the operator "L" defined therein. Let $X = \{x_1, \ldots, x_k\}$ be a finite subset and Λ_X (Λ, when X is understood) denote the set of all σ_X (σ, when X is understood) of zero-one-valued functions on $\{1, 2, \ldots, k\}$ and let Δ_σ denote the difference operator $\prod_{j=1}^{k}(E_{x_j})^{\sigma_j}(I - E_{x_j})^{1-\sigma_j}$, where we adopt the convention that an operator (or member of any semigroup) to the power 0 is the identity even if that member is 0 itself. If f is a real-valued function on S then, following [**6**], we set $\| f \|_X = \sum_{\sigma \in \Lambda_X} | \Delta_\sigma f(1) |$. The triangle inequality implies $\| f \|_X$ is an increasing function of X (ordered by inclusion) and we set $\|f\| = \lim_X \|f\|_X$. Functions in the set $\mathrm{BV}(S) = \{f \mid \|f\| < \infty\}$ are called functions of *bounded variation* (or BV-functions) and are discussed thoroughly in **[6]** and **[9]**. Since $\Sigma_{\sigma\in\Lambda}\Pi_j E_{x_j}^{\sigma_j}(I - E_{x_j})^{1-\sigma_j} = \Pi_j(E_{x_j} + (I - E_{x_j})) = I$ we have $\Sigma_\sigma \Delta_\sigma f = f$. Thus $\mathrm{CM}(S) \subset \mathrm{BV}(S)$ and $\|f\| = f(1)$ if $f \in \mathrm{CM}(S)$.

Let g be a fixed completely monotonic function normalized by the condition $g(1) = 1$. Following **[12]** we define a real-valued function f on S to be *continuous* (*with respect to* g) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\Sigma_{\sigma\in T\subset\Lambda_X} |\Delta_\sigma f(1)| < \varepsilon$ whenever $\Sigma_{\sigma\in T\subset\Lambda_X} \Delta_\sigma g(1) < \delta$. It follows from **[11**, Th. A] see also **[10]** or **[12]** that every continuous function is also of bounded variation. The set of all continuous functions with variation norm $\|\cdot\| = \|\cdot\|_1$ will be denoted by $\mathfrak{L}_1(g)$. A BV-function f will be called *singular* (*with respect to* g) if given $\varepsilon > 0$ there exists a finite subset X of S and $T \subset \Lambda_X$ such that both $\Sigma_{\sigma\in T}|\Delta_\sigma f| < \varepsilon$ and $\Sigma_{\sigma\notin T}\Delta_\sigma g(1) < \varepsilon$. It will follow from Th. 1.1 below that every BV-function f admits a unique decomposition of the form $f_1 + f_2$ with f_1 continuous and f_2 singular. Moreover $\|f\| = \|f_1\| + \|f_2\|$.

For $1 < p < \infty$ and with the understanding that $0/0 = 0$, we set $\|f\|_{(p,X)}^p = \Sigma_{\sigma\in\Lambda_X} |\Delta_\sigma f(1)|^p / [\Delta_\sigma g(1)]^{p-1}$. It follows that $\|f\|_{(p,X)}$ increases with p and X cf. **[10]** and we define $\|f\|_p^p = \lim_X \|f\|_{(p,X)}^p$. For $p = \infty$, we define $\|f\|_\infty = \sup_\Delta |\Delta f(1)|/\Delta g(1)$ and set $\mathfrak{L}_p(g) = \{f \mid \|f\|_p < \infty\}$ for all $1 \le p \le \infty$.

A non-identically zero, real-valued function ρ on S is called a *semicharacter* if $\rho(x)\rho(y) = \rho(xy)$ for all $x, y \in S$. The set Γ of all semicharacters on S will be given the topology of simple convergence. Each $\rho \in \Gamma$ is zero-one-valued and $\Delta\rho(x; \{x_j\}) = \rho(x)\Pi_j(1 - \rho(x_j))$. The space Γ is compact and it follows **[6]**, that the collection $\mathfrak{R}$ of all sets $R_\Delta = \{\rho \in \Gamma \mid (\Delta\rho)(1) = 1\}$ is a basis of open and closed subsets for Γ.

Let $\mathfrak{M}(\Gamma)$ denote the regular Borel measures on Γ and $\mathfrak{M}^+(\Gamma)$ the non-negative members of $\mathfrak{M}$.

THEOREM 1.1. *A real-valued function f on S admits a* (*necessarily unique*) *disintegration of the form $f(x) = \int_\Gamma \rho(x)\, d\mu_f(\rho)$ where*

(i) $\mu_f \in \mathfrak{M}^+(\Gamma)$ *if and only if f is* CM.

(ii) $\mu_f \in \mathfrak{M}(\Gamma)$ *if and only if f is* BV.

(iii) $d\mu_f = f'\, d\mu_g$ *with $f \in L_p(\mu_g, \Gamma)$ if and only if $f \in \mathfrak{L}_p(g)$* $(1 \le p \le \infty)$

(iv) μ_f *is singular with respect to μ_g if and only if f is singular.*

Moreover the spaces $\mathfrak{L}_p(g)$ and $L_p(\mu_g, \Gamma)$ are linearly isometric via $f \to f'$ as also are the spaces of BV-*functions and bounded measures each with variation norm.*

Proof. Direct proofs of (i) and (ii) as well as the last mentioned isometry statement are contained in **[6]** and **[8]**. Direct proofs of the remaining part of the assertion may be found in [**10**]. Another proof of the theorem may be accomplished by appealing to the real algebra $\mathcal{Q}$ generated by the shift operators on S and the set up of [**9** and **11**]. The set τ which generates the positive cone P is taken to be $\{E_x, I - E_x \mid x \in S\}$. The idempotent operation on S allows us to restrict our attention to partitions of unity of the form $\{\Pi_j E_{x_j}^{\sigma_j}(I - E_{x_j})^{1-\sigma_j} \mid \sigma \in \Lambda_X\}$. Reduction of the results in [**11**] to our setting is then accomplished upon identification of the linear functionals on $\mathcal{Q}$ with the functions on S via $F \to f$, where $F(E_x) = f(x)$. Note that this biuniquely identifies the positive multiplicative linear functionals on $\mathcal{Q}$ with Γ.

The possibilities offered by Th. 1.1 of obtaining decompositions of BV-functions from known decompositions of measures are numerous. We define a semicharacter to be a *singularity* of f if there exists $\alpha > 0$ such that $|f| - \alpha\rho \in \mathrm{CM}(S)$. Then ρ is a singularity of a BV-function f if and only if ρ is an atom of μ_f. The decomposition of μ_f into its atomic and non-atomic part applies to give

$$f = f_1 + \sum_{i=1}^{\infty} \alpha_i \rho_i,$$

where f_1 has no singularities.

COROLLARY 1.2. *Every* BV-*function f admits three (unique) decompositions of the form $f = f_1 + f_2$ with $\|f\| = \|f_1\| + \|f_2\|$; each respectively satisfying*

(i) *f_1 has no singularities and f_2 is of the form $\sum_{i=1}^{\infty} \alpha_i \rho_i$*

(ii) *f_1 and $-f_2$ are completely monotonic*

(iii) *f_1 is continuous and f_2 is singular.*

In order to establish an $\mathfrak{L}_p$-inversion formula to recover the density function f' of a representing measure of the form $f' \, d\mu_g$ we consider the linear span of $\{E_x g \mid x \in S\}$ and following [**1**] call each of its members *polygonal functions*. Since $(E_y g)(x) = \int_\Gamma \rho(x)\rho(y) \, d\mu_g(\rho)$, then it follows that the evaluation function $\rho \to \rho(y)$ is the derivative of $E_y g$, so that linearity of the differentiation map $f \to f'$ gives

$$\left(\sum_j a_j E_{x_j} g\right)'(\rho) = \sum_j a_j \rho(x_j). \tag{1.2.1}$$

The Stone-Weierstrass theorem implies that these derivatives are uniformly dense in the continuous functions $C(\Gamma)$ on Γ and hence in $L_p(\mu_g)$ for $1 \leq p < \infty$. In particular the derivatives of polygonal functions of the form

$$f_X = \sum_{\sigma \in \Lambda_X} [\Delta_\sigma f(1)/\Delta_\sigma g(1)] \Delta_\sigma g \tag{1.2.2}$$

are computed to be

$$f'_X = \sum_{\sigma \in \Lambda_X} [\Delta_\sigma f(1)/\Delta_\sigma g(1)] \Delta_\sigma \rho(1). \tag{1.2.3}$$

The following proposition is of interest and will be referred to again in §3.

PROPOSITION 1.3. *If f is polygonal there exists a finite set X such that $f = f_Y$ whenever $Y \supset X$.*

Proof. Suppose $f = \Sigma_j a_j E_{x_j} g$. Set $X = \{x_j\}_j$ then $f = \Sigma_j a_j (\Sigma_{\sigma \in \Lambda_X} \Delta_\sigma E_{x_j} g) = \Sigma_j a_j (\Sigma_{\sigma_j = 1} \Delta_\sigma g) = \Sigma_{\sigma \in \Lambda_X} b_\sigma \Delta_\sigma g$, where the last equality is obtained by reversing the order of summation and setting $b_\sigma = \Sigma_{\sigma_j = 1} a_j$. Thus if X is a finite subset of S which contains X, we get $f = \Sigma_{\sigma_X} (\Sigma_{\sigma_Y} b_{\sigma_X} \Delta_{\sigma_X} \Delta_{\sigma_Y} g)$ or

$$f = \sum_{\sigma_Y} b_{\sigma_Y} \Delta_{\sigma_Y} g, \tag{1.3.1}$$

where $b_{\sigma_Y} = b_{\sigma_X}$ whenever $\Delta_{\sigma_X} \cdot \Delta_{\sigma_Y} = \Delta_{\sigma_Y}$. If we apply Δ_{σ_Y} to both sides of (1.3.1) for a fixed σ_Y and evaluate at 1, then we obtain $\Delta_{\sigma_Y} f(1) = b_{\sigma_Y} \Delta_{\sigma_Y} g(1)$ and the assertion follows.

THEOREM 1.4. (L_p*-inversion*). *If $f \in \mathfrak{L}_p(g)$ for $1 \le p < \infty$ then* $\lim_X \| f' - f'_X \|_p = 0$.

Proof. Again we can appeal to the general algebraic setting of [**9** and **11**] as in the proof of Th. 1.1. However, Prop. 1.3 which is not available in that generality, provides a simpler and more illuminating approach and we refer the reader to [**10**] for the details.

COROLLARY 1.5. *If $1 \le p < \infty$, $1 < q \le \infty$ and $(1/p) + (1/q) = 1$ then $\mathfrak{L}_p^*(g) = \mathfrak{L}_q(g)$ via the pairing*

$$\langle f, h \rangle = \lim_X \sum_{\sigma \in \Lambda_X} \Delta_\sigma f(1) \Delta_\sigma h(1)/\Delta_\sigma g(1).$$

Proof. The Riesz representation theorem and Th. 1.1 imply $\mathfrak{L}_p^*(g) = \mathfrak{L}_q(g)$ via the pairing $\langle f', h' \rangle = \int_\Gamma f'(\rho) h'(\rho)\, d\mu_g(\rho)$. But, $\langle (\Delta g)', h' \rangle = \Delta h(1)$, and since $\langle \cdot\,, h' \rangle$ is continuous, we can apply the inversion theorem to obtain:

$$\langle f', h' \rangle = \lim_X \langle f'_X, h' \rangle = \lim_X \left\langle \sum_{\sigma \in \Lambda_X} [\Delta_\sigma f(1)/\Delta_\sigma g(1)] (\Delta_\sigma g)', h' \right\rangle$$

$$= \lim_X \sum_{\sigma \in \Lambda_X} (\Delta_\sigma f(1) \Delta_\sigma h(1)/\Delta_\sigma g(1)) = \langle f, h \rangle.$$

2. Applications to differentiation and the classical Lebesgue spaces. Let S be the closed interval $[0, 1]$ under the semilattice operation $xy = \min[x, y]$, and set $g(x) = x$. Then $\Delta f(x; \{x_j\}) = f(x) - f(x \max_j\{x_j\})$ and it follows that the completely monotonic functions are just the non-negative, non-decreasing functions on $[0, 1]$. Moreover the definitions of bounded variation and continuity (with respect to g) given in §1 agree with the classical notions with the added restriction to the classical definition of absolute continuity that $f(0) = 0$. The semicharacters are the characteristic functions of the form $1_{(x, 1]}$ and $1_{[x, 1]}(x \in S)$ and the map $\Pi \mid \Gamma \to S$ defined by $\Pi(1_{(x, 1]}) = \Pi(1_{[x,1]}) = x$ is seen to be continuous. Let the representing measure be μ_g and its transformed measure, $(\Pi\mu_g)$, be defined on $[0, 1]$ in the usual way by $(\Pi\mu_g)(A) = \mu_g[\Pi^{-1}(A)]$ for each Borel set A. Since μ_g has no atoms, an examination of $(\Pi\mu_g)$ on the subintervals of $[0, 1]$, shows the transformed measure to be Lebesgue measure. Let Df denote the ordinary derivative of $f \in \mathfrak{L}_p(g)$. Since the evaluation function $\rho \to \rho(x)$ is seen to agree (μ_g-almost everywhere) with the composition $1_{[0, x]} \circ \Pi$, standard integration theory shows $f(x) = \int_0^x (Df)\, dt = \int_\Gamma \rho(x)[(Df) \circ \Pi](\rho)\, d\mu_g(\rho)$, from which we get

$$(2.0.1) \qquad (Df) \circ \Pi = f' \qquad (\mu_g\text{-almost everywhere}).$$

Therefore

$$(2.0.2) \quad \int_\Gamma |f'(\rho)|^p \, d\mu_g(\rho) = \int_0^1 |(Df)(x)|^p \, dx \qquad (1 \leq p < \infty).$$

We summarize the foregoing as follows

PROPOSITION 2.1. *If $1 \leq p < \infty$, then in the above notation, $\mathfrak{L}_1(g)$ is the set of all functions f on $[0, 1]$ which are absolutely continuous in the classical sense and for which $f(0) = 0$. The space $\mathfrak{L}_p(g)$ is isometric to the classical Lebesgue space $L_p(dx)$ via differentiation.*

Let $X = \{x_j\}_j$ be a finite subset of $[0, 1]$ such that $x_j < x_{j+1}, f \in \mathfrak{L}_p(g)$. Then the approximating polygonal function f_X used in the inversion (Th. 1.4) are of the form

$$(2.1.1) \qquad f_X(x) = \sum_j \frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} (x - x_{j-1}) 1_{(x_{j-1}, x_j]}(x).$$

That is, f_X is the linear approximation that interpolates f at each node point x_j. Note that Df_X is the expected step function approximation to Df. If $1 < p < \infty$ then we have

$$(2.1.2) \quad \|f\|_p = \lim_X \sqrt[p]{\sum_j |f(x_j) - f(x_{j-1})|^p / (x_j - x_{j-1})^{p-1}},$$

(2.1.3) $$\|f\|_1 = \lim_X \sum_j |f(x_j) - f(x_{j-1})|$$

and

(2.1.4) $$\|f\|_\infty = \sup\left[|f(x) - f(y)|/|x - y|\right]$$

Condition (2.1.4) shows that $\mathfrak{L}_\infty(g)$ is in fact the space of function satisfying the usual Lipschitz condition with the additional property, $f(0) = 0$.

Observe that Cor. 1.2 (ii) is just the usual decomposition of a BV-function f into its two non-negative, non-decreasing parts. Part (iii) of the Corollary along with (2.0.2) imply $f = f_1 + f_2$ with f_1 absolutely continuous and $Df_2 = 0$ a.e. It is easily verified that a function $h \in \mathrm{CM}(S)$ is continuous at x if and only if both $h - \alpha 1_{(x,1]}$ and $h - \alpha 1_{[x,1]}$ are not CM for any $\alpha > 0$. Using the familiar fact that a BV-function is continuous if and only if its variation is continuous, one can characterize the BV-functions with no singularities as the continuous BV-functions. Thus Cor. 1.2(i) provides the familiar decomposition of a BV-function as

$$f = f_1 + \sum_{j=1}^{\infty} \alpha_j 1_{(x_j,1]} + \sum_{j=1}^{\infty} \alpha_j 1_{[x_j,1]}$$

where f_1 is continuous and BV.

Finally we remark that the above extends to the n-dimensional cube with coordinate-wise operations, and leave the details to the reader.

3. Applications to Boolean algebras and finitely additive set functions. We assume that S admits a second operation $\vee$ under which S is a distributive lattice. Recall [**2**] that a valuation is a real-valued function f on S which satisfies $f(x) + f(y) = f(xy) + f(x \vee y)$ ($\Delta f(x \vee y; x, y) = 0$) for all $x, y \in S$. It is easily verified in [**6**] that every valuation satisfies

(3.0.1) $$\Delta f(x; \{x_j\}) = \Delta f(x; \vee_j x_j).$$

Thus a valuation f is completely monotonic if and only if $f(x) \geq 0$ and $\Delta f(x; y) \geq 0$ for all $x, y \in S$. Moreover the formula for the variation of a valuation reduces to

(3.0.2) $$\|f\| = \lim_X \sum_{\sigma \in \Lambda_X} \left|\Delta f\left(\prod x_j^{\sigma_j}; \vee x_j^{1-\sigma_j}\right)\right| \qquad (x_j \in X)$$

with analogous forms for $\|f\|_p$. While formally different, this definition of variation is seen in [**6**] to be equivalent to that given in Birkhoff [**2**, p. 74].

Let Γ_V denote the set of all $\rho \in \Gamma$ such that ρ is a valuation.

PROPOSITION 3.1. *Γ_V is a closed subset of Γ. Moreover a* BV-*function f is a valuation if and only if its representing measure μ_f is supported by Γ_V.*

Proof. Let $\rho_0 \in \Gamma \setminus \Gamma_V$. Then there exist $x, y \in S$ such that $\Delta\rho_0(x \vee y; x, y) = 1$ so that $R_\Delta = \{\rho \in \Gamma \mid \Delta\rho(x \vee y; x, y) = 1\}$ is an open (and closed) set which contains ρ_0 but does not intersect Γ_V. Thus Γ_V is closed. If f is a BV-valuation then $\mu_f(R_\Delta) = \Delta f(x \vee y; x, y) = 0$. Thus μ_f is supported by Γ_V. Conversely if μ_f is supported by Γ_V, then $\Delta f(x \vee y; x, y) = \int_\Gamma \Delta\rho(x \vee y; x, y)\, d\mu_f(\rho) = 0$ so that f is a valuation.

We further specialize to the case where $S(\cdot, \vee, ')$ is a Boolean algebra of subsets of a set Ω. Now it must be remembered that each $x, y \in S$ is a set. Further $xy = x \cap y$, $0 = \varnothing$, $1 = \Omega$ and $x' = \Omega \setminus x$. The valuations f such that $f(0) = 0$ are exactly the functions with the property that $f(x \vee y) = f(x) + f(y)$ whenever $xy = 0$, i.e. the *additive functions*. Since

$$\Delta f(x; y) = f(xy'), \tag{3.1.1}$$

we see that an additive f is completely monotonic if and only if it is non-negative. If we introduce the notation

$$x_\sigma = \prod_j x_j^{\sigma_j}(x_j')^{1-\sigma_j}, \qquad (\sigma \in \Lambda_X), \tag{3.1.2}$$

then $\{x_\sigma \mid \sigma \in \Lambda_X\}$ is a typical partition of Ω into disjoint subsets by member of S. Applying (3.1.1) and (3.1.2) gives

$$\|f\|_{X \cup X'} = \sum_{\sigma \in \Lambda_X} |f(x_\sigma)|,$$

so that

$$\|f\| = \lim_X \sum_{\sigma \in \Lambda_X} |f(x_\sigma)|. \tag{3.1.3}$$

It follows that the definition of variation given here agrees with conventional usage for a finitely additive set function, cf. **[5]**. Moreover it is clear that the definitions of singularity and continuity introduced in §1 reduce to those given by Darst **[4]** for this special case where Cor. 1.2 (iii) has already been observed. However, even in this case the methods used under the present set up provide a simplified proof because of our access to the Lebesgue decomposition via the map $f \to \mu_f$ cf. **[10]**. The additive members of Γ_V are the characteristic functions of ultrafilters. In fact, the identically 1 function is the only member of Γ_V which is not additive and it is seen to be an isolated point. It follows by the same reasoning used in Prop. 3.1

that μ_f is supported by $\Gamma_V \setminus \{1\}$. In summary we assert

PROPOSITION 3.2. *Let g be additive and non-negative and let f be continuous with respect to g, then*

(3.2.1) *f is additive and* BV

(3.2.2) $\|f\|_p^p = \lim_X \sum_{\sigma \in \Lambda_X} |f(x_\sigma)|^p / [g(x_\sigma)]^{p-1}$ *for* $1 < p < \infty$

(3.2.3) $\|f\|_\infty = \sup_x [|f(x)|/g(x)]$

The polygonal functions f_X are of the form:

(3.2.4) $f_X = \sum_{\sigma \in \Lambda_X} [f(x_\sigma)/g(x_\sigma)] E_{x_\sigma} g$

(3.2.5) *If* $1 < p < \infty$, $(1/p) + (1/q) = 1$ *then the dual* $\mathfrak{L}_p^*(g)$ *of* $\mathfrak{L}_p(g)$ *is* $\mathfrak{L}_q(g)$ *under the pairing* $\langle f, h \rangle = \lim_X \Sigma_{\sigma \in \Lambda_X} f(x_\sigma) h(x_\sigma)/g(x_\sigma)$.

REMARK 3.3 If g is additive, the spaces $\mathfrak{L}_p(g)$ are exactly the V^p spaces. Consequently each V^p-space is isometric to the Lebesgue space $L_p(\mu_g)$ on Γ_A.

Let $L_p(g)$ denote the Lebesgue space $\{F \mid \int_\Omega |F|^p(\omega)\, dg(\omega) < \infty\}$ defined for a not necessarily countable additive g $(1 \leq p < \infty)$, cf. [**5**, III.3]. Recall that $L_p(g)$ need not be complete and observe the distinction between $L_p(g)$ as a possibly incomplete Banach space of functions F on Ω, $L_p(\mu_g)$ as a complete space of functions on Γ_A and $\mathfrak{L}_p(g)$ as a V^p-space of additive set functions on the Boolean algebra S. Following usual conventions we will call a function $F \in L_1(g)$ the *Radon-Nikodým derivative* of an additive $f \mid S \to \mathbf{R}$ if $f(x) = \int_x F(\omega)\, dg(\omega)$. Then f is called the *antiderivative* of F. The derivative is unique when it exists and in such cases will be denoted by df/dg. The following theorem is essentially contained in [**3**]. We offer this alternate proof here in part as an application of §1 and in part as motivation of our point of view concerning differentiation. In particular, it indicates the necessity of seeking the derivative of an $\mathfrak{L}_p(g)$-function on the structure space Γ_A rather than Ω when $L_p(g)$ is incomplete.

THEOREM 3.4. *If* $1 \leq p < \infty$ *then* $L_p(g)$ *is densely embedded in* $\mathfrak{L}_p(g)$ *via antidifferentiation. Thus* $\mathfrak{L}_p(g)$ *(hence* $L_p(\mu_g)$*) represents the completion of* $L_p(g)$*. The latter space is complete if and only if the Radon-Nikodým derivative* df/dg *exists for each* $f \in \mathfrak{L}_p(g)$.

Proof. Let $F \in L_p(g)$ and $T \mid L_p(g) \to \mathfrak{L}_p(g)$ be the antidifferentiation map. We apply Hölder's inequality to get

$$\sum_{\sigma \in \Lambda_X} \frac{|T(F)(x_\sigma)|^p}{[g(x_\sigma)]^{p-1}} = \sum_{\sigma \in \Lambda_X} \frac{|\int_{x_\sigma} F(\omega)\, dg(\omega)|^p}{[g(x_\sigma)]^{p-1}}$$

$$\leq \sum_{\sigma \in \Lambda_X} \int_{x_\sigma} |F|^p(\omega)\, dg(\omega) = \|F\|_p^p \quad \text{for } p > 1.$$

It follows that $\|T(F)\|_p \leq \|F\|_p$ for all $p \geq 1$, with equality holding when F is a simple function and X is sufficiently large. In the latter event one finds $T(F)$ to be a polygonal function of the form

$$T(F)(x) = \sum_{\sigma \in \Lambda_X} \frac{T(F)(x_\sigma)}{g(x_\sigma)} g(xx_\sigma).$$

Thus T is a continuous linear map of $L_p(g)$ into $\mathfrak{L}_p(g)$ which preserves the norm of each simple function. Since the simple functions are known to be dense in $L_p(g)$ [**5**, p. 125], T is a norm preserving linear injection. But since the polygonal functions are dense in $\mathfrak{L}_p(g)$, we can complete the proof of the first assertion by showing that df/dg is a simple function whenever f is polygonal. Clearly, $dE_x g/dg = 1_x$ and it follows that $dE_X/dg = \Sigma_{\sigma \in \Lambda_X}[f(x_\sigma)/g(x_\sigma)]1_{x_\sigma}$, g-almost everywhere. The first assertion follows since Prop. 1.3 implies every polygonal function f is of the form f_X. The remaining assertions are clear.

From the above proof we can assert

COROLLARY 3.5. *If $f \in \mathfrak{L}_p(g)$ for $1 \leq p < \infty$ and the Radon-Nikodým derivative df/dg exists then the Radon-Nikodým net $\Sigma_{\sigma \in \Lambda_X}[f(x_\sigma)/g(x_\sigma)]1_{x_\sigma}$ converges in the $L_p(g)$-metric to df/dg.*

The structure space $\Gamma_V \setminus \{1\}$ which is just the set of characteristic functions of S-ultrafilters on Ω will be denoted by Γ_A. The characteristic function of an ultrafilter which contains a smallest member of S will be called a *principal semicharacter*. A principal semicharacter is then one of the form:

$$\rho_y(x) = \begin{cases} 0 & \text{if } xy = 0 \\ 1 & \text{if } xy = y \end{cases}$$

Since the neighborhood $\{\rho \in \Gamma_A \mid \rho(y) = 1\}$ only contains ρ_y we have

REMARK 3.6. The set Γ_p of principal semicharacters is discrete in Γ_A.

If $x \in S$ with $|f|(x) \neq 0$ and $|f|(y)$ is either 0 or $|f|(x)$ for each $y \in S$ such that $y \subset x$, then x will be called an *atom* for f. If f has no atoms it will be called *non-atomic*. At the other extreme, if every $x \in S$ for which $|f|(x) \neq 0$, contains an atom then f will be called *completely atomic*. It is easily verified that every additive BV-function f on S admits a unique decomposition of the form $f = f_1 + f_2$, where f_1 is completely atomic and f_2 is non-atomic.

THEOREM 3.7. *Let f be additive and* BV. *If f is non-atomic then μ_f is supported by $\Gamma_A \setminus \Gamma_p$. If μ_f is concentrated on Γ_p then f is completely atomic and countably additive.*

Proof. Suppose f is non-atomic and let $\rho_y \in \Gamma_p$ in the above notation. Then $\mu_f(\{\rho_y\}) = \mu_f\{\rho \in \Gamma_A \mid \rho(y) = 1\} = f(y) = 0$, since y is not an atom of f. Thus $\{\rho_y\}$ is a neighborhood of ρ_y with measure zero. Therefore ρ_y is not in the support of μ_f. The second assertion follows because if μ_f is concentrated on Γ_p then f is of the form $\Sigma_{i=1}\alpha_i\rho_{y_i}$, with y_i minimal.

In sharp contrast to the usual weak* approximation of probability measures by point masses we offer

COROLLARY 3.8. *Every non-negative, non-atomic, additive function f with $f(\Omega) = 1$ is in the closed convex hull of $\Gamma_A \setminus \Gamma_p$; principal semicharacters are not.*

Proof. Recall that we have imposed the topology of simple convergence on the finitely additive BV-functions and the w^*-topology on $\mathfrak{M}(\Gamma)$. The set $\mathfrak{M}_1^+(\Gamma_A \setminus \Gamma_p)$ of probability measures on $\Gamma_A \setminus \Gamma_p$ is the closed convex hull of $\{\mu_\rho \mid \rho \in \Gamma_A \setminus \Gamma_p\}$. The first assertion follows since the theorem implies the map $f \to \mu_f$ is an affine homeomorphism of the set of all f satisfying its first hypothesis into $\Gamma_A \setminus \Gamma_p$. Finally suppose $\rho_y \in \Gamma_p$ is in the closed convex hull. Then we can find $\alpha_j > 0$, $\rho_j \in \Gamma_A \setminus \Gamma_p$ for $j = 1, 2, \ldots, n$ with $\Sigma_j\alpha_j = 1$ such that $1 = \Sigma_j\alpha_j\rho_j(y') < \rho_y(y') + \varepsilon = \varepsilon$ for any $\varepsilon > 0$; an obvious contradiction.

The characteristic function ρ_ω of an ultrafilter, each of whose members contain a given singleton $\omega \in \Omega$ will be called *point mass* ρ_ω. Then ρ_ω is principal when and only when $\{\omega\} \in S$. Since every neighborhood of a semicharacter $\rho' \in \Gamma_A$ contains a set of the form $\{\rho \in \Gamma_A \mid \rho(x) = 1\}$ and this set contains ρ_ω for each $\omega \in x$, then it follows that the set Γ_{PM} of point masses is dense in Γ_A. If $\{\omega\} \in S$ for each $\omega \in \Omega$, then Remark 3.6 shows that Γ_{PM} is in fact almost all of Γ_A in the topological sense of category. One can identify Ω with Γ_{PM} via the map $\omega \to \rho_\omega$. It is therefore somewhat surprising that Th. 3.7 implies that the representing measure μ_f is supported by the complement $\Gamma_A \setminus \Gamma_{PM}$ whenever f is non-atomic.

REMARK 3.9. If Ω is compact and metrizable and S is the set of Borel subsets of Ω, then $\rho \in \Gamma_A \setminus \Gamma_{PM}$ if and only if ρ is not countable additive. Thus Cor. 3.8 asserts that Lebesgue measure, for example, is in the closed convex hull of those zero-one-valued set functions which are not countable additive.

Added in Proof. In a recent preprint, D. Plachky forwarded a complete characterization of the closed convex hull of non-principal ultrafilters using entirely different techniques than those contained herein.

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