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AFFINE CURVES OVER AN ALGEBRAICALLY NON-CLOSED FIELD

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In this paper, different $\bar{k}$-completions of a curve over an algebraically non-closed field k are compared. If the curve has k-points at infinity, then it is shown to admit a completion which is canonical. If $k = \mathbf{R}$ this is true also for rational curves.

Introduction. If $(V, \mathcal{O}_V)$ is an affine algebraic curve defined over a field k ($k \neq \bar{k}$) then V is the set of k-points of several non-isomorphic $\bar{k}$-curves which are called completions of V.

In this paper we compare these different completions and we prove (Theorem 3.1.) the existence of completions which, without creating new singularities, are extended as large as possible in the sense that they are not affine open sets of larger ones. These completions are called minimal. If a curve satisfies the condition of having k-points "at infinity" (cf. Theorem 3.5) then these minimal completions turn out to be all isomorphic. We shall say that "this is the canonical" completion of the curve. In the case $k = \mathbf{R}$ we prove that the above condition is also necessary if the genus of the curve is bigger than zero, while rational affine real curves always admit canonical complexification.

The involved techniques are mainly those of A. Tognoli (cf. [**6**]). We emphasize the use of affine representations (rather than that of completions (cf. §0)). First we introduce a suitable partial ordering in the set of isomorphism classes of affine representations; we are then able to show that the subset of affine representations which correspond to completions having no non-rational singularities is sufficiently rich and has minimal (up to isomorphism) elements.

0. We first recall some preliminaries. Throughout this paper k denotes a field and $\bar{k}$ an algebraic closure of it. Let V be an algebraic subset of k^n and let $\mathcal{O}_V$ denote the sheaf of regular functions defined over the open sets of V. It is known that

$$\Gamma_V = \Gamma(V, \mathcal{O}_V) = N_V^{-1}(k[X_1,\dots,X_n]/\mathfrak{I}_V)$$

where $\mathfrak{I}_V = \{P \in k[X_1,\dots,X_n] \mid P_{|V} \equiv 0\}$ and $N_V = \{g \in k[X_1,\dots,X_n]/\mathfrak{I}_V \mid g(x) \neq 0 \text{ for each } x \in V\}$. More generally an affine

k-variety $V = (V, \mathcal{O}_V)$ is a topological space V plus a sheaf $\mathcal{O}_V$ of k-valued functions on V which is isomorphic to an irreducible algebraic subset of some k^n plus its sheaf of regular functions (see **[6]** for general references).

A criterion for a k-algebra to be the ring of global regular functions of some affine variety defined over k is given in **[5]**. Ibidem it is also showed that the global sections functor gives an equivalence of categories between the category of affine k-varieties and the opposite to the full subcategory of "k-algebras" consisting of those objects that satisfy the quoted above criterion.

Given now an affine algebraic k-variety V, each embedding $i: V \to k^n$ gives rise to $j: V \to \bar{k}^n$ ([**6**], Def. 1. p. 28) and we call *completion* of V (complexification if $k = \mathbf{R}$) the algebraic variety $\tilde{V}$ which is the closure of $j(V)$ into $\bar{k}^n$. Note that different embeddings of V in k^n may induce different completions of V. We shall study a completion $\tilde{V}$ of V mainly considering the ring P_V of regular functions from $\tilde{V}$ to k. Observe that $P_V \otimes_k \bar{k}$ is the ring of coordinates of $\tilde{V}$ and that $\Gamma_V \simeq N^{-1}P_V$ (where $N = \{g \in P_V \mid g(x) \neq 0$ for all $x \in V)$. The ring P_V is said to be an affine representation of Γ_V. The affine representations have been characterized as follows ([**5**] Prop. 2.1):

PROPOSITION 0.1. *Let V be an affine k-variety. A k-algebra A is an affine representation of Γ_V if and only if*

(i) *A is of finite type over k,*

(ii) $\bigcap_{\mathfrak{M} \in \mathrm{Specm}_k A} \mathfrak{M} = (0)$ *and* $\mathrm{Specm}_k A \simeq V$,

(iii) $\Gamma_V \simeq N_A^{-1}A$ *where* $N_A = \{a \in A \mid a \notin \mathfrak{M}$ *for each* $\mathfrak{M} \in \mathrm{Specm}_k A\}$.

REMARK 0.2. If A is an affine representation of some Γ_V, then:

(a) for every $\mathfrak{M} \in \mathrm{Specm}_k A$ we have $A_{\mathfrak{M}} \simeq (\Gamma_V)_{\mathfrak{M}^e}$

(b) if $\dim_k V = 1$, then for every $\mathfrak{p} \in \mathrm{Specm}\, A - \mathrm{Specm}_k A$ we have $(A - \mathfrak{p})^{-1}\Gamma_V \simeq \mathrm{Fract}(\Gamma_V)$.

We point out that if $V = (V, \mathcal{O}_V)$ is an affine irreducible curve defined over k, then $K = \mathrm{Fract}(\Gamma_V)$ ($\simeq \mathrm{Fract}\, A$ for each affine representation A) is a field of algebraic functions in one variable over k.

From now on, unless contrarily specified, we shall always consider affine irreducible curves and $\mathcal{V}$ will denote the set of all valuations of K over k.

The terms valuation, divisor, genus of K will be used with the same meaning as in **[1]**. For every $v \in \mathcal{V}$, $(R_v, \mathfrak{M}_v)$ will indicate the corresponding local ring with its maximal ideal and $\deg v$ will indicate the integer $[R_v/\mathfrak{M}_v : k]$.

For each affine representation A of Γ_V, we introduce the following non-negative integers:

$$\Xi(A) = \#\{v \in \mathcal{V} \mid R_v \not\supset A\},$$
$$\Xi_n(A) = \#\{v \in \mathcal{V} \mid R_v \not\supset A \text{ and } \deg v = n\}.$$

REMARKS 0.3. (a) $0 < \Xi(A) < \infty$ (cf. **[4]** Lemma 6.5 Ch. I) and so $\Xi_n(A) = 0$ for large n.

(b) there exists a one-to-one correspondence which to each maximal ideal $\mathfrak{M} \in \operatorname{Specm}_k A - \operatorname{Sing}(\operatorname{Spec} A)$ associates a valuation ring R_v such that $\deg v = 1$, $R_v \supset A$ and $R_v \simeq A_{\mathfrak{M}}$. So, as $\operatorname{Specm}_k A \simeq \operatorname{Specm}_k \Gamma_V = \operatorname{Specm} \Gamma_V$, $\Xi_1(A)$ depends only on V and we shall denote this integer by $\Xi_1(V)$.

1. We remark that there exist affine representations of non-singular k-varieties such that the corresponding completions have singularities as $\bar{k}$-varieties:

EXAMPLE 1.1. The $\mathbf{R}$-algebra

$$A = \mathbf{R}[(X^2+1)(X^2+4),\ X(X^2+1)(X^2+4),\ X^3+7X]$$

is an affine representation of the ring of regular algebraic functions on the real line $\mathbf{A}^1_{\mathbf{R}}$ and $A \otimes_{\mathbf{R}} \mathbf{C}$ is the ring of coordinates of a complex curve having two ordinary nodes.

This induces us to introduce among all the affine representations of a fixed Γ_V the following distinction.

DEFINITION 1.2. An affine representation A of Γ_V is said to be a good affine representation if $\operatorname{Sing}(\operatorname{Spec} A) \subset \operatorname{Specm}_k A$.

It is possible to associate to each affine representation A of Γ_V a good one such that its corresponding completion is a "partial normalization" of the completion associated to A.

THEOREM 1.3. *Let A be an affine representation of Γ_V and let A' be its integral closure in Γ_V. Then A' is a good affine representation of Γ_V.*

Proof. First of all, by using criterion (0.1) we show that A' is an affine representation of Γ_V. Clearly $A' \subset \bar{A}$, where $\bar{A}$ is the integral closure of A in K, then A' is a k-algebra and an A-module of finite type as $\bar{A}$ is so and

A is noetherian. The inclusions $A \hookrightarrow A' \hookrightarrow \Gamma_V$ induce $V = \operatorname{Specm} \Gamma_V = \operatorname{Specm}_k \Gamma_V \to \operatorname{Specm}_k A' \to \operatorname{Specm}_k A$ whose composite is a homeomorphism and $\operatorname{Specm}_k A'$ maps injectively into $\operatorname{Specm}_k A$ (use [5], Th. 2.2.). On the other hand it is easy to check that $\Gamma_V = N_{A'}^{-1}A'$. If now $\mathfrak{a} = \bigcap \mathfrak{M}'(\mathfrak{M}' \in \operatorname{Specm}_k A')$, then $\mathfrak{a}\Gamma_V = (0_{\Gamma_V})$, so $\mathfrak{a} = (0_{A'})$, since Γ_V is a ring of fractions of A'. All this shows our claim. To prove that $\operatorname{Sing}(\operatorname{Spec} A') \subset \operatorname{Specm}_k A'$, let $\mathfrak{p}' \in \operatorname{Specm} A' - \operatorname{Specm}_k A'$ and let $\mathfrak{p} = \mathfrak{p}' \cap A \in \operatorname{Specm} A - \operatorname{Specm}_k A$. Then $(A - \mathfrak{p})^{-1}A'$ is integrally closed in $(A - \mathfrak{p})^{-1}\Gamma_V \simeq K$. Since $(A' - \mathfrak{p}')^{-1}A'$ is a ring of fractions of $(A - \mathfrak{p})^{-1}A'$, it follows that $\mathfrak{p}' \notin \operatorname{Sing}(\operatorname{Spec} A')$.

Proposition 1.4. *An affine representation A of Γ_V is good if and only if A is integrally closed in Γ_V.*

Proof. After (1.3) we only have to show that if $\operatorname{Sing}(\operatorname{Spec} A) \subset \operatorname{Specm}_k A$ then $A = A'$, the integral closure of A in Γ_V. Let $\mathfrak{f}' = \operatorname{Ann}_A(A'/A)$ be the conductor of A' in A (resp. let $\mathfrak{f} = \operatorname{Ann}_A(\bar{A}/A)$ be the conductor of $\bar{A}$ in A). Clearly $A = A'$ iff $\mathfrak{f}' = (1_A)$, therefore we need to prove that every $\mathfrak{p} \in \operatorname{Specm} A$ does not contain $\mathfrak{f}'$. If $\mathfrak{p} \in \operatorname{Specm}_k A$ then $A_{\mathfrak{p}} \simeq (A - \mathfrak{p})^{-1}\Gamma_V \simeq (A - \mathfrak{p})^{-1}A'$ and so $\mathfrak{p} \not\supset \mathfrak{f}'$. On the other hand if $\mathfrak{p} \in \operatorname{Specm} A - \operatorname{Specm}_k A$ then $\mathfrak{p} \not\supset \mathfrak{f}$ by hypothesis, therefore $\mathfrak{p} \not\supset \mathfrak{f}'$ as $\mathfrak{f} \subset \mathfrak{f}'$

Remark 1.5. If V is smooth and A is a good affine representation of Γ_V, then both A and Γ_V are Dedekind domains and it is easy to check that:

(a) $\Gamma_V = \bigcap R_v$ (intersection running throughout the set of R_v's such that $R_v \supset \Gamma_V$ and $\deg v = 1$).

(b) $A \cong \Gamma_V \cap (\bigcap R_v)$ (the second intersection now running through the set of R_v's such that $R_v \supset A$ and $\deg v > 1$ (cf. Rem. 0.3 b).

2. We introduce now a partial ordering $\prec$ on the set of (isomorphism classes of) affine representations of a given Γ_V.

Definition 2.1. If A and B are affine representations of a given Γ_V we say that A precedes B ($A \prec B$) if there exists an isomorphism $\alpha\colon N_A^{-1}A \to N_B^{-1}B$ such that its restriction $\alpha|_A$ maps A into B. Plainly $\prec$ is reflexive and transitive, so it remains to show that it is antisymmetric, i.e. if $A \prec B$ and $B \prec A$ then $A \simeq B$. We shall prove this first for good affine representations, then for all ones.

Lemma 2.2. *If A and B are affine representations of a given Γ_V such that A is good and $A \prec B$ via $\alpha\colon A \to B$, then α is a flat ring homomorphism.*

Proof. The induced embedding $\bar{\alpha}$: $\bar{A} \to \bar{B}$ is flat since $\bar{A}$ is Dedekind and $\bar{B}$ is an overring of $\bar{A}$. It is enough to show that for every $\mathfrak{p} \in \operatorname{Spec} B$, $B_{\mathfrak{p}}$ is flat over $A_{\mathfrak{q}}$ where $\mathfrak{q} = \mathfrak{p} \cap A$. For this:

(i) if $\mathfrak{p} \in \operatorname{Sing}(\operatorname{Spec} B)$ then $\mathfrak{p} \in \operatorname{Specm}_k B$ since B is good, therefore $A_{\mathfrak{q}} \simeq B_{\mathfrak{p}} \simeq (\Gamma_V)_{\mathfrak{p}^e}$.

(ii) if $\mathfrak{p} \notin \operatorname{Sing}(\operatorname{Spec} B)$ then $\mathfrak{q} \notin \operatorname{Sing}(\operatorname{Spec} A)$ since α^*: $\operatorname{Sing}(\operatorname{Spec} B) \to \operatorname{Sing}(\operatorname{Spec} A)$ is a homeomorphism. Hence $B_{\mathfrak{p}} \simeq (B - \mathfrak{p})^{-1}\bar{B} \simeq \bar{B}_{\mathfrak{p}^e}$ and $A_{\mathfrak{q}} \simeq (A - \mathfrak{q})^{-1}\bar{A} \simeq \bar{A}_{\mathfrak{q}^e}$. Our contention follows from the fact that $\bar{B}_{\mathfrak{p}^e}$ is flat over $\bar{A}_{\mathfrak{q}^e}$ and $\mathfrak{q}^e\bar{B} \simeq \mathfrak{p}^e$.

REMARKS 2.3. (a) The above result fails if A is not good. For instance let A be the affine representation of the real line given in example (1.1) and let $B = \mathbf{R}[X]$. It is clear that $A \prec B$ but B is not flat over A, which in fact is not good.

(b) The above lemma is also false if $\dim_k V > 1$, even if the involved completions are smooth. For example let $A = \mathbf{R}[X, Y]$ (affine representation of $\mathbf{A}^2_{\mathbf{R}}$) and let ϕ: $A \to A$ be the $\mathbf{R}$-homomorphism given by $\phi(X) = X$, $\phi(Y) = (X^2 + 1)Y$. Plainly ϕ extends to an isomorphism $\tilde{\phi}$: $\Gamma_{\mathbf{A}^2_{\mathbf{R}}} \to \Gamma_{\mathbf{A}^2_{\mathbf{R}}}$, but it is not flat.

LEMMA 2.4. *Let A and B be affine representations of a given Γ_V. Then*

(i) *if $A \prec B$ then $\Xi(A) \leq \Xi(B)$.*

Let in addition A and B be both good. Then

(ii) *if $A \prec B$ and $\Xi(A) = \Xi(B)$ then $A \simeq B$,*

(iii) *if $A \prec B$ and $B \prec A$ then $A \simeq B$.*

Proof. (i) Let ϕ: $A \to B$ be the inclusion given by $A \prec B$. Then ϕ extends to an automorphism $\tilde{\phi}$: $\Gamma_V \to \Gamma_V$ and therefore to an automorphism Φ: $K = \operatorname{Fract}(A) \to \operatorname{Fract}(B) = K$ such that $\Phi(A) \subset B$. Since $\Xi(A) = \Xi(\Phi(A))$ and $\{v \in \mathcal{V} \mid R_v \not\supset \Phi(A)\} \subset \{v \in \mathcal{V} \mid R_v \not\supset B\}$; we get the wanted inequality.

(ii) Let now ϕ^* be the induced map on the spectra. Clearly we need only to prove that ϕ^* is a homeomorphism. Now ϕ^* maps $\operatorname{Sing}(\operatorname{Spec} B) = \{\mathfrak{M}_1, \ldots, \mathfrak{M}_r\} \subset \operatorname{Specm}_k B$ bijectively into $\operatorname{Sing}(\operatorname{Spec} A) = \{\mathfrak{M}_1, \ldots, \mathfrak{M}_r\} \subset \operatorname{Specm}_k A$. Therefore, for every i, $i = 1, \ldots, r$, we have $A_{\mathfrak{M}_i} \simeq A_{\phi^*(\mathfrak{M}_j)} \simeq B_{\mathfrak{M}_j}$ for some j, $j = 1, \ldots, r$. On the other hand, for every $\mathfrak{M} \in \operatorname{Spec} A - \operatorname{Sing}(\operatorname{Spec} A)$, $A_{\mathfrak{M}} \simeq R_v$ for some $v \in \mathcal{V}$ and $R_v \simeq \Phi(R_v) = R_w$ for some $w \in \mathcal{V}$ (by (i), since $\Xi(A) = \Xi(B)$). Now $R_w \simeq B_{\mathfrak{N}}$ for some $\mathfrak{N} \in \operatorname{Specm} B - \operatorname{Sing}(\operatorname{Spec} B)$ and $\mathfrak{M} = \phi^*(\mathfrak{N})$. So ϕ^* turns out to be surjective; it is also an open immersion since it is flat (by 2.2.) and birational (by hypothesis), thus ϕ^* is a homeomorphism.

(iii) It follows immediately from (i) and (ii).

THEOREM 2.5. *The relation $\prec$ is a partial ordering on the set of all affine representations of a given Γ_V.*

Proof. We need only to check that $\prec$ is antisymmetric. Let $A \prec B$ via $\phi: A \to B$ and $B \prec A$ via $\psi: B \to A$. Consider the following commutative diagram

$$\begin{array}{ccccccccc} \Gamma_V & \simeq & N_A^{-1}A & \tilde{\to} & N_B^{-1}B & \tilde{\to} & N_A^{-1}A & \simeq & \Gamma_V \\ & & \cup\!\!\uparrow & & \cup\!\!\uparrow & & \cup\!\!\uparrow & & \\ & & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & A & & \end{array}$$

We want to show that $\eta = \psi \cdot \phi$ is an isomorphism. Call $\tilde{\eta}$ the isomorphism on the top row (which turns out to be induced by η). It is easy to check that its restriction $\eta' = \tilde{\eta}\,|_{A'}$ to the integral closure of A in Γ_V maps A into itself. Therefore by (2.4) η' is an isomorphism. The map η^*: $\operatorname{Spec} A \to \operatorname{Spec} A$ induced by η is a homeomorphism on an open set since it is birational, on the other hand η'^*: $\operatorname{Spec} A' \to \operatorname{Spec} A'$ is injective so η^* is also surjective. Our thesis follows then like in Lemma 2.4 (ii).

3. We show now that in order to find a sort of canonical completion the right context seems to be given by the set of good affine representations.

THEOREM 3.1. *The partial ordering $\prec$ has minimal elements on the set of good affine representations of a given Γ_V.*

Proof. If $A \prec B$ strictly, by (2.4) $\Xi(A) < \Xi(B)$. Therefore, because of the finiteness of Ξ, there exist minimal affine representations corresponding to the minimal values of Ξ.

We point out that $\prec$ is an ordering which is not inductive in the ascending way even if we consider only good affine representations:

EXAMPLE 3.2. Consider the $\mathbf{R}$-algebras $A_n = \mathbf{R}[X, \prod_{j=1}^{n} 1/(X^2 + j^2)]$ for $n > 0$. The A_n's give rise to an ascending chain of good affine representations of $\Gamma_{\mathbf{A}^1_{\mathbf{R}}}$ which is clearly non-stationary. We also remark that $\prec$ is not inductive in the descending way on the set of *all* affine representations:

EXAMPLE 3.3. We generalize example (1.1) by constructing a strictly descending chain $\cdots \prec A_n \prec A_{n-1} \prec \cdots \prec A_1 \prec A_0 = \mathbf{R}[t]$ of affine representations of $\mathbf{A}^1_{\mathbf{R}}$ where A_n is obtained inductively from A_{n-1} by glueing pairwise the complex points $(2n-1)i$ with $2ni$ and $-(2n-i)i$ with $-2ni$. We get this as an explicit application of the theory of glueings developed in [3], namely A_n is the pullback in the diagram:

$$\begin{array}{ccc} A_n & \to & A_{n-1} \\ \downarrow & & \downarrow\phi \\ \mathbf{C} & \overset{\Delta}{\to} & \mathbf{C}\times\mathbf{C} \end{array}$$

where Δ is the diagonal map and, for every $f\in A_{n-1}$, $\phi(f)=(f(2ni), f(2n-1)i)$. Straightforward but tedious computations give the following explicit form for the A_n's: let $h\in\{1,\dots,n\}$ and let

$$F_h(t)=\left(t^2+(2h-1)^2\right)\left(t^2+(2h)^2\right),$$

$$G_{h,n}(t)=(2h(2h-1)-n)t^3+h(2h-1)(4h(2h-1)+1+2n)t,$$

$$X_n(t)=\prod_{h=1}^{n}F_h(t),\qquad Y_n(t)=tX_n(t),$$

$$T_{h,n}(t)=\left(\prod_{j=1}^{h-1}F_j(t)^{n-h+1}\right)\cdot\left(\prod_{s=h+1}^{n}F_s(t)^{n-s+1}\right)\cdot\left(\prod_{r=h}^{n}G_{h,r}\right).$$

Then $A_n=\mathbf{R}[X_n, Y_n, T_{1,n},\dots,T_{n,n}]$. Thus the rings A_n are affine representations of the ring of regular functions on $\mathbf{A}^1_{\mathbf{R}}$ which have the following properties: (i) $A_n'=\mathbf{R}[t]$; (ii) the complexification of $\mathbf{A}^1_{\mathbf{R}}$ corresponding to A_n has exactly $2n$ singular non-real points; (iii) $A_{n+1}\prec A_n$ and $\cdots\prec A_n\prec\cdots\prec A_1\prec A_0$ ($A_0=\mathbf{R}[t]$) is a non-stationary descending chain.

Definition 3.4. An affine representation A of Γ_V is said to be canonical if it is good and $A\prec B$ for every good affine representation B.

Clearly the canonical representation is defined up to isomorphism. The corresponding completion will be called canonical completion; we shall see (§4) that it does not always exist. Here we give a sufficient criterion. First we remark that if Γ_V has canonical representation A, then $\Xi(A)$ is minimal in the set $\{\Xi(B)\mid B \text{ good affine representation}\}$. Furthermore, since $\Xi_1(B)=\Xi_1(V)$ is independent of the representation B, A is minimal with respect to $\prec$ if $\sum_{i=1}^{\infty}\Xi_i(A)$ is minimal.

Theorem 3.5. *Let V be a smooth curve such that $\Xi_1(V)\neq 0$. Then Γ_V admits a canonical representation.*

Proof. Let A be a good representation which is minimal with respect to $\prec$. We claim that $\Xi_i(A) = 0$ for all $i > 1$. Suppose not. Let $\mathcal{V}$ be the abstract Riemann surface of K over k. Fix an embedding $\sigma\colon A \to \Gamma_V \subset K$, then there exists a valuation $\bar{v} \in \mathcal{V}$ with $\deg \bar{v} = i > 1$ such that $R_{\bar{v}} \not\supset \sigma(A)$. By assumption there are some valuations v such that $\deg v = 1$ and $\sigma(A) \not\subset R_v$ (since $\Gamma_V \not\subset R_v$) and the divisor associated with $\{v \in \mathcal{V} \mid \deg v = 1, \sigma(A) \not\subset R_v\}$ is ample so that a suitable multiple of it gives an embedding of V in k^m. Let C be the affine representation determined by this embedding. It is easy to check that $C \simeq \bigcap R_v(R_v \supset C)$, but $C \subset R_{\bar{v}}$ so that $C \prec A$, contradiction with the minimality of A. This proves the claim.

Suppose now that B is another minimal representation of Γ_V; by the claim we have $\Xi_i(B) = 0$ for each $i > 1$. Fix $\tau\colon B \to \Gamma_V \subset K$ then $\{v \in \mathcal{V} \mid R_v \supset \tau(B)\} = \{v \in \mathcal{V} \mid R_v \supset \sigma(A)\}$, therefore

$$A \simeq \sigma(A) = \bigcap R_v(R_v \supset \tau(B)) = \tau(B) \simeq B.$$

COROLLARY 3.6. *Let V be as in Theorem 3.5 and let A be a good affine representation. Then A is the canonical representation if and only if $\Xi_i(A) = 0$ for all $i > 1$.*

4. We conclude this note applying the above results to the case $k = \mathbf{R}$.

If V is a smooth real curve, then $\mathbf{R}$ is the field of constants of $K = \operatorname{Fract} \Gamma_V$ over $\mathbf{R}$ itself and the valuations of K over $\mathbf{R}$ have only degrees 1 or 2. We call genus of V ($g(V)$) the genus of K. Since we always take for granted that V has $\mathbf{R}$-points then V has genus 0 iff it is rational (cf. [**1**] Ch. II§2).

REMARK 4.1. Let V be a smooth real curve. In the following cases V has canonical complexification.

(i) $\Xi_1(V) \neq 0$. In fact from Theorem 3.5 it turns out that the canonical complexification $\tilde{V}$ of V is $\operatorname{Spec}(P_V \otimes_{\mathbf{R}} \mathbf{C})$ where $P_V = \bigcap R_v$ (all $v \in \mathcal{V}$ but those of degree 1 such that $R_v \not\supset \Gamma_V$).

(ii) $\Xi_1(V) = 0$ and V is rational. In fact if A is a minimal good affine representation of Γ_V, then A is of the form $S^{-1}(\mathbf{R}[X, Y]/(Y^2 + C(X)))$ where S is a multiplicative set, $C(X)$ is a polynomial of degree 2 with two distinct factors in $\mathbf{C}[X]$ and $Y^2 + C(X)$ is irreducible (see e.g. [**2**] §3). By the minimality of A, we may assume $S = U(\mathbf{R}[X, Y]/(Y^2 + C(X)))$ (otherwise $\Xi_1(A) \neq 0$). Actually $C(X)$ factors in $\mathbf{R}[X]$ (or else $\operatorname{Specm}_{\mathbf{R}} A$ would be empty). Thus, after a change of coordinates, we can assume

$A = \mathbf{R}[X, Y]/(X^2 + Y^2 - 1)$. Therefore all good minimal affine representations of Γ_V are isomorphic.

The cases of Remark 4.1 are the only possible ones as follows from the following.

PROPOSITION 4.2. *Let V be a smooth real curve such that $\Xi_1(V) = 0$ and $g(V) \geq 1$. Then there exist infinitely many non-isomorphic complexifications corresponding to the minimal good affine representations of V.*

Proof. Consider first the case $g(V) = 1$. Let A be an affine representation of Γ_V such that $\Xi(A) = \Xi_2(A)$ and let $\bar{V}$ be a non-singular projective closure of the complexification of V associated to A. Let σ be the conjugation in $\bar{V}$ and let $\{P, \sigma P\}$ be the support of the divisor at infinity of Spec$(A \otimes_{\mathbf{R}} \mathbf{C})$ in $\bar{V}$ which consists of two complex conjugated points. Fixed a group law $+$ in $\bar{V}$, any automorphism of $\bar{V}$ is a translation followed by a group automorphism. Namely if $\phi \in \mathrm{Aut}(\bar{V})$, $\phi(x) = \varepsilon x + x'$ where ε is a (fourth or sixth) root of 1. Clearly there exist (infinitely many) points Q's such that $\sigma Q - Q \neq \varepsilon(\sigma P - P)$ for all ε, for which $\phi(\sigma P) \neq \sigma Q$ whenever $\phi(P) = Q$. Choose one of these Q's and let w be the valuation of degree two associated to $Q + \sigma Q$. Then the ring $B = \bigcap_{v \neq w} R_v$ is a minimal representation and $B \not\simeq A$ otherwise an isomorphism between A and B would extend to an automorphism of $\bar{V}$ sending $\{Q, \sigma Q\}$ into $\{P, \sigma P\}$.

If $g(V) > 1$, the group of automorphisms of any projective closure of a complexification of V has order at most $84(g-1)$. The same argument as above will prove our contention.

All together we have proved:

THEOREM 4.3. *A smooth affine real curve V has canonical complexification if and only if it is either rational or embeddable as a non-compact algebraic subspace of some $\mathbf{R}^n$ (in the usual topology).*

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