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INTEGRAL CLOSURE AND GENERALIZED TRANSFORMS IN GRADED DOMAINS

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In this article we consider the integral closure of integral domains by using the generalized transform and valuation rings. The first section establishes the basic theory in a general setting while the second deals with applications to graded rings, ending with a generalization of theorems due to Kuan and Seidenberg on integral closure in Z^+ graded rings. As in a number of recent articles, we investigate the idea that if a property holds in the graded case, and it holds for $R_s = \{a/b \mid a, b \in R, b$ a homogeneous non-zero divisor$\}$, then the property holds for the ring.

The notation will be fairly standard: all rings are commutative with identity; for an integral domain R, $\bar{R}$ is the integral closure of R; valuation rings will often be written (V, M) where M is the maximal ideal; $V(I)$ denotes the variety of I; and $V(\mathfrak{S})$ is $\bigcup_{I \in \mathfrak{S}} V(I)$.

1. Integral closure and the generalized transform. Let R be a commutative ring with identity and K the total quotient ring of R. In [**4**] Arnold and Brewer defined the *generalized transform* of a ring R at a multiplicatively closed set of ideals $\mathfrak{S}$ as $\{x \in K \mid xI \subseteq R$ for some $I \in \mathfrak{S}\}$ and used the notation $R_{\mathfrak{S}}$. $R_{\mathfrak{S}}$ is also called the $\mathfrak{S}$-transform of R.

DEFINITION 1.1. For an integral domain R, the *normal locus of* R is the set of all prime ideals $p \in \operatorname{Spec}(R)$ so that R_p is integrally closed. The *non-normal locus* of R is the set of prime ideals $q \in \operatorname{Spec} R$ so that R_q is not integrally closed.

We'll be using the following easy result.

PROPOSITION 1.2. *If $\mathfrak{S}$ contains the non-normal locus and $R_{\mathfrak{S}} = \bigcap_{p \notin V(\mathfrak{S})} R_p$ then $R_{\mathfrak{S}}$ is integrally closed.*

The next definition will be mainly used in graded domains where the relation "$\mathfrak{S}$-related" is an equivalence relation.

DEFINITION 1.3. Let $\mathfrak{S}$ be a multiplicatively closed set of ideals and $\mathfrak{P}$ the set of prime ideals in $V(\mathfrak{S})$. We say that for two valuation rings

(V_1, M_1) and (V_2, M_2) *V_1 and V_2 are $\mathfrak{S}$-related* (or *$\mathcal{P}$-related*) if there exists a valuation ring (V, M) so that $V_i \cap R_{\mathfrak{S}} \supseteq V \cap R_{\mathfrak{S}}$ and $M_i \cap R \supseteq M \cap R$ for $i = 1, 2$.

In general this will not be an equivalence relation. However, the valuation rings that are $\mathfrak{S}$-related are downwardly directed in that $V_1 > V_2$ if $V_1 \cap R_{\mathfrak{S}} \supset V_2 \cap R_{\mathfrak{S}}$.

THEOREM 1.4. *Let R be an integral domain, $\mathcal{P}$ the non-normal locus of R, $\mathfrak{S}$ the multiplicative set of ideals generated by products of primes in $\mathcal{P}$, and assume that $R_{\mathfrak{S}} = \bigcap_{p \notin V(\mathfrak{S})} R_p$. Then $\bar{R} = R_{\mathfrak{S}} \cap (\bigcap V_\alpha) = \bigcap (R_{\mathfrak{S}} \cap V_\alpha)$ where the V_α's can be chosen to be minimal elements in the $\mathfrak{S}$-related classes on valuation rings, if the minimal representatives exist.*

Proof. With the assumptions as stated in the Theorem, $R_{\mathfrak{S}} = \bigcap_{p \notin V(\mathfrak{S})} R_p$ is integrally closed by Proposition 1.2 and so $\bar{R} \subseteq R_{\mathfrak{S}}$. For $\{V_\beta\}$ the set of all valuation rings containing R, $\bar{R} = \bigcap V_\beta = R_{\mathfrak{S}} \cap (\bigcap V_\beta) \subseteq R_{\mathfrak{S}} \cap (\bigcap V_\alpha)$ where the V_α's are minimal representatives. To show equality, let $x \in R_{\mathfrak{S}} \cap (\bigcap V_\alpha)$ and let (V, M) be a valuation ring with $P = M \cap R$. If P is in the non-normal locus, there exists a valuation ring (V', M') minimal (we are assuming that minimal representatives exist) in the $\mathfrak{S}$-relation class containing (V, M) and $V \cap R_{\mathfrak{S}} \supseteq V' \cap R_{\mathfrak{S}}$. Hence $x \in V \cap R_{\mathfrak{S}}$. On the other hand, if (V, M) is from the normal locus then $x \in R_{\mathfrak{S}} \subseteq R_p \subseteq V$ since $p \in P$. In either case we have $x \in R_{\mathfrak{S}} \cap (\bigcap V_\alpha)$ implies $x \in \bar{\mathrm{R}}$. Thus $\bar{R} = R_{\mathfrak{S}} \cap (\bigcap V_\alpha)$.

2. Application to graded rings. In this section, $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ will be an integral domain graded by an arbitrary torsionless grading monoid Γ. By this we mean that R is an integral domain, Γ a commutative cancellative monoid, the quotient group $\langle \Gamma \rangle$ generated by Γ is a torsion free ordered abelian group, and if $r_\alpha \in R_\alpha$, $r_\beta \in R_\beta$, $r_\alpha \cdot r_\beta \in R_{\alpha+\beta}$. For such an R we let $R_S = \{a/b \mid a, b \in R\ b \neq 0 \text{ homogeneous}\}$ and call it the homogeneous quotient ring of R. We let $\mathfrak{S}$ be the set of all nonzero homogeneous or graded ideals (those generated by homogeneous elements).

PROPOSITION 2.1. $R_{\mathfrak{S}} = R_S$.

Proof. If $a/s \in R_S$ where $a \in R$ and $s \in S$, then $a/s \cdot (s) \subseteq R$. Since $(s) \in \mathfrak{S}$, $a/s \in R_{\mathfrak{S}}$. Conversely, if $x \in R_{\mathfrak{S}}$ then $xI \subseteq R$ for some $I \in \mathfrak{S}$. Let $i \in I \cap S$ then $xi \in R$ so $x = xi/i \in R_S$.

As in [**6**, **7**, **9**] one is able to define a graded valuation ring (or g-valuation ring) for Γ grading as well as Z or Z^+ grading. This is done by calling $R = \bigoplus_{\alpha\varepsilon\Gamma} R$ a Γ-graded valuation domain if for each homogeneous element $x \in R_S$, x or $1/x \in R$. Equivalently if for each pair of homogeneous ideal I and J we have $I \supseteq J$ or $J \supseteq I$ (the homogeneous ideals are totally ordered under inclusion). Note that for a grading monoid Γ to admit a graded valuation domain $g \in \langle \Gamma \rangle$ must imply that g or $-g \in \Gamma$. Thus, when we speak of a Γ-graded valuation ring (or domain) we are assuming that the grading is done by the group $\langle \Gamma \rangle$ or that Γ admits a Γ-graded valuation ring. We list three results that carry over from the Z or Z^+ grading to Γ grading. The proofs are identical to those given in [**7**, Lemma 1.6 through Proposition 1.9] with R_S substituted for $K[x, 1/x]$.

LEMMA 2.2. *Let $D = \bigoplus_{\alpha\in\Gamma} D_\alpha$ be a Γ graded integral domain with quotient field L and let G be an ordered abelian group. If f: $D \to G$ is defined so that the $f \mid D_\alpha = f_\alpha$ have the properties*:

(1) $f_\alpha(d_\alpha + g_\alpha) \geq \inf\{f_\alpha(d_\alpha), f_\alpha(g_\alpha)\}$ *for* $d_\alpha, g_\alpha \in D_\alpha$;

(2) $f_\alpha(d_\alpha d_\beta) = f_\alpha(d_\alpha) + f_\beta(d_\beta)$ *for* $d_\alpha \in D_\alpha$, $d_\beta \in D_\beta$; *and*

(3) *for* $r = \Sigma r_\alpha$, $r_\alpha \in D_\alpha$, $f(r) = \inf\{f_\alpha(r_\alpha)\}$, *then f can be extended to a valuation on* L_S.

THEOREM 2.3. *Let V^* be a Γ graded g-valuation ring with homogeneous quotient ring R_S. Then there exists a valuation ring V in the quotient field of V^* so that $V \cap R_S = V^*$.*

In a manner similar to that found in [**7**], we can define a *homogeneously defined* valuation as a valuation that satisfies $v(\Sigma r_\alpha) = \inf\{v(r_\alpha)\}$ for r_α homogeneous of degree α. The corresponding valuation ring V is called a *homogeneously defined valuation ring* [cf., **3**, inf valuation].

We also have:

PROPOSITION 2.4. *Let V_1 and V_2 be homogeneously defined valuation rings so that $V_1 \cap R_S = V_2 \cap R_S = V^*$. Then $V_1 = V_2$.*

Note that we are able to set up an equivalence relation on the valuation rings in the quotient field of R_S. We do this by first letting V be a valuation ring. $V \cap R_S$ is then a ring which contains a unique largest graded valuation ring V^* defined from the valuation v of V restricted to the homogeneous components as in Lemma 2.2. Thus there is a canonical homogeneously defined valuation ring which we denote by V'. The equivalence relation $\sim_{R_S}$ is defined by $V_1 \sim_{R_S} V_2$ means $V_1' = V_2'$. It is easy to check that this is an equivalence relation and that $V \cap R_S \supseteq V' \cap R_S$. Thus the homogeneously defined valuation ring will be a minimal representative of the equivalence class, minimal meaning minimal with respect to the intersection in R_S. We shall use these facts at a later time in this section.

DEFINITION 2.5. An ideal I in a Γ graded ring R is called *totally non-homogeneous* if I fails to contain a non-zero homogeneous element.

PROPOSITION 2.6. *Let I be a totally non-homogeneous ideal, then there exists a totally non-homogeneous prime ideal $J \supseteq I$.*

Proof. Since $I \cap S = \varnothing$ then I can be enlarged to an ideal J maximal with respect to $J \cap S = \varnothing$. Any such J is prime.

REMARKS. (1) If R is a Z or Z^+ graded domain, $S = \{$homogeneous non-zero elements in $R\}$, then the totally non-homogeneous primes of R are preserved in R_S. R_S is of the form $K[x, 1/x]$ for K a field and is hence of Krull dimension one. Thus if t is a non-zero non-homogeneous element of R, then t is contained in a height one totally non-homogeneous prime.

(2) If t is an element of an integral domain R and each prime which contains t is of height ≥ 2, then there fails to exist a non-trivial Z or Z^+ grading of R which makes t homogeneous. Equivalently, all Z and Z^+ gradings of R make t non-homogeneous.

The following material uses heavily the notation and ideas from [**5**, 4] and we refer the reader to that for the necessary background.

Let P be the set of totally non-homogeneous prime ideals, $\mathfrak{S}$ the set of non-zero homogeneous ideals in R, and $V(\mathfrak{S})$ the graded prime ideals and those primes which contain graded primes. Using the notation in [**5**], $G(P) = \{$ideals A in $R \mid A \not\subset Q\ \forall Q \in P\}$.

LEMMA 2.7. *With the notation as above, $G(P) = \{$ideals I of $R \mid I \supseteq$ graded ideal$\}$.*

Proof. It is clear that $G(P)$ contains all graded non-zero ideals since if A is a graded ideal then no totally non-homogeneous prime may contain it. So let I be an ideal which does not contain any graded elements. By Proposition 2.6, I is contained in a totally non-graded prime. Thus $I \in G(P)$ and we have equality.

LEMMA 2.8. $R_{G(P)} = R_{\mathfrak{S}}$.

Proof. From [**5**] we know that $G(P)$ is a multiplicatively closed set of ideals, and so we are comparing two generalized transforms. Let $x \in R_{G(P)}$, then $x \cdot I \subseteq R$ for some $I \in G(P)$. Let I^* be the ideal generated by the homogeneous elements in I. $I^* \subseteq I$ so $x \cdot I^* \subseteq R$. This means that $x \in R_{\mathfrak{S}}$ and we obtain $R_{G(P)} \subseteq R_{\mathfrak{S}}$. Since $G(P) \supseteq \mathfrak{S}$ we have $R_{G(P)} \supseteq R_{\mathfrak{S}}$. Thus $R_{\mathfrak{S}} = R_{G(P)}$.

PROPOSITION 2.9. *With* R, P *and* $\mathfrak{S}$ *as above*, $R_{\mathfrak{S}} = \bigcap_{p \in P} R_p$.

Proof. $R_{\mathfrak{S}} = R_{G(P)}$ by Lemma 2.8 and $R_{G(P)} = \bigcap \{R_q \mid q \in P\}$ by [**5**, Proposition 4.3].

We are now able to apply Theorem 1.4 to Γ-graded rings.

THEOREM 2.10. *If* R *is a* Γ *graded integral domain*, *then the integral closure of* R *is the intersection of all g-valuation rings containing* R.

Proof. Let $\mathfrak{S}$ be the set of non-zero homogeneous ideals and P the set of totally non-graded prime ideals, then $R_S = R_{\mathfrak{S}} = \bigcap_{p \in P} R_p$ by Propositions 2.1 and 2.9. R_S is integrally closed by [**1**, Propositions 2.1 and 3.2] and we apply Theorem 1.4 to obtain $\overline{R} = \bigcap (R_{\mathfrak{S}} \cap V_\alpha 0)$ where the V_α's are chosen to be minimal. The discussion following Proposition 2.4 shows that each V_α is a homogeneously defined valuation ring and so each $R_{\mathfrak{S}} \cap V_\alpha$ is a graded g-valuation ring.

We conclude with a theorem that generalizes Theorem 1 of [**10**] and Lemma 1 of [**11**]:

THEOREM 2.11. *If* R *is a* Γ *graded domain then for each totally non-graded prime* P, R_P *is integrally closed*.

Proof. Let P be a totally nonhomogeneous prime ideal. $P \cap S = \varnothing$ implies that $R_P = R_{SP_S}$, which is a localization of an integrally closed GCD domain and hence integrally closed.

REMARK. The referee noted that R_S is also completely integrally closed and that when P is height one, R_P will be a one dimensional GCD domain and hence completely integrally closed.

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