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AN INTERPOLATION THEOREM FOR ANALYTIC FAMILIES OF OPERATORS ACTING ON CERTAIN H^p SPACES

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The main objective of this paper is to obtain an interpolation theorem for families of operators acting on atomic H^p spaces, $0 < p \leq 1$. We prove that if $0 < p_0 < p_1 \leq 1$ and $\{T_z\}$, $z \in \bar{S} = \{z \in \mathbf{C}/0 \leq \text{Real } z \leq 1\}$, is an analytic and admissible family of linear transformations such that T_{j+iy} maps H^{p_j} into L^{p_j}, where $-\infty < y < \infty$, with norm not exceeding A_j, $j = 0, 1$, then for all θ, $0 \leq \theta \leq 1$, T_θ maps H^r into L^r with norm not exceeding $cA_0^{1-\theta}A_1^\theta$, where $1/r = (1-\theta)/p_0 + \theta/p_1$.

1. Introduction. For $x \in \mathbf{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index, with α_j a natural number, $j = 1, \ldots, n$, we write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $0 < p \leq 1$ and an integer $s \geq [n(1/p - 1)]$[1] we say that a function $a(x)$ is a *(p, s)-atom* if:

(i) the support of a is contained in a ball $B \subset \mathbf{R}^n$.

(ii) $|a(x)| \leq |B|^{-1/p}$ for all $x \in \mathbf{R}^n$ ($|B|$ denotes the measure of B).

(iii) $\int_{\mathbf{R}^n} a(x)x^\alpha\, dx = 0$ for all multi-indexes α with $|\alpha| \leq s$.

The space $H^{p,s}(\mathbf{R}^n)$ consists of those linear functionals f defined on an appropriate Lipschitz class of test functions (see [6] for details, where more general atoms are also considered) that can be expressed in the form $f = \sum_{i=1}^\infty \lambda_i a_i$, where the a_i are (p, s)-atoms and $\sum_{i=1}^\infty |\lambda_i|^p < \infty$. The *quasi-norm* of f is defined to be the infimum of the numbers $(\sum_{i=1}^\infty |\lambda_i|^p)^{1/p}$ taken over all the above representations of f; it is denoted by $\|f\|_{H^{p,s}}$. For fixed p, the spaces $H^{p,s}$ are independent of s and their quasi-norms are equivalent (for this and the relation of these spaces to the classical H^p spaces see [6]). For this reason we shall denote each of these equivalent spaces by H^p and $\|\ \|_{H^p}$ will denote the quasi-norm.

Another aspect of this theorem that we must make precise is the type of transformations we are dealing with.

A function Φ defined on $\bar{S}$, the closure of S, is said to be of *admissible growth* if there exist positive constants B and b, $b < \pi$, such that

$$\Phi(z) \leq Be^{b|\text{Im } z|}$$

for all $z \in \bar{S}$.

[1] $[t]$ denotes the integer part of the non-negative real number t.

A family of linear transformations $\{T_z\}$, $z \in \bar{S}$, each mapping simple functions to measurable functions on $\mathbf{R}^n$, is called *analytic* if the function

$$z \to \int_{\mathbf{R}^n} (T_z\varphi)\psi \, dx$$

is analytic for each pair of simple functions φ and ψ on $\mathbf{R}^n$.

This family is said to be *admissible* if the function $\log\|T_z\varphi\|_1$ is of admissible growth for each simple φ on $\mathbf{R}^n$.

We are now ready to state the main theorem.

THEOREM I. *Let $0 < p_0 < p_1 \leq 1$ and $\{T_z\}$, $z \in \bar{S}$, be an analytic and admissible family of linear transformations. We suppose*

$$\|T_{j+iy}f\|_{L^{p_j}} \leq A_j \|f\|_{H^{p_j}}, \qquad j = 0, 1, \tag{1.1}$$

for all $f \in H^{p_j} \cap L^{p_j}$, where $-\infty < y < \infty$ and A_j, $j = 0, 1$, are positive constants.

Then, for each θ, $0 \leq \theta \leq 1$, we have

$$\|T_\theta f\|_{L^r} \leq c A_0^{1-\theta} A_1^{\theta} \|f\|_{H^r} \tag{1.2}$$

for all $f \in H^r$, where $1/r = (1-\theta)/p_0 + \theta/p_1$ and c depends only on the dimension n and the quasi-norm used in H^r.

A similar theorem for transformations acting on H^p of the unit disk was proved by E. M. Stein and G. Weiss [5].[2]

2. Proof of Theorem I. Due to the equivalence of the spaces $H^{p,s}$ for p fixed, Theorem I follows from the theorem below.

THEOREM II. *Let $0 < p_0 < p_1 \leq 1$ and s an integer satisfying $s \geq [n(1/p_0 - 1)]$. Let $\{T_z\}$, $z \in \bar{S}$, be an analytic and admissible family of linear transformations such that*

$$\|T_{j+iy}f\|_{L^{p_j}} \leq A_j \|f\|_{H^{p_j,s}}, \qquad j = 0, 1, \tag{2.1}$$

for all $f \in H^{p_j,s} \cap L^{p_j}$, where $-\infty < y < \infty$ and A_j, $j = 0, 1$, are positive constants.

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[2] In [4] E. M. Stein proved a similar theorem for transformations acting on L^p spaces.

Then, for each θ, $0 \leq \theta \leq 1$, *we have*

$$\|T_\theta f\|_{L^r} \leq A_0^{1-\theta} A_1^\theta \|f\|_{H^{r,s}} \tag{2.2}$$

for all $f \in H^{r,s}$, *where* $1/r = (1-\theta)/p_0 + \theta/p_1$.

Three lemmas are used to prove this theorem.

LEMMA 1. *Let* $\Phi: S \to \mathbf{R}$ *be an upper semi-continuous function of admissible growth and subharmonic in* S. *Then for* $z_0 = x_0 + iy_0 \in S$ *we have*

$$\Phi(z_0) \leq \int_{-\infty}^{\infty} \Phi(i[y_0 + y])\omega(1 - x_0, y)\, dy + \int_{-\infty}^{\infty} \Phi(1 + i[y_0 + y])\omega(x_0, y)\, dy,$$

where

$$\omega(x, y) = \frac{1}{2} \frac{\sin \pi x}{\cos \pi x + \cosh \pi y}.$$

For a proof see **[5]**.

LEMMA 2. *If* $V(z)$ *is an analytic function mapping* S *into* L^1, *then, for any positive* $c \leq 1$, $\Phi(z) = \int_{\mathbf{R}^n} |V(z)|^c\, dx$ *is continuous and* $\log \Phi(z)$ *is subharmonic.*

The proof can be found in **[5]**.

LEMMA 3. *Let* $0 < p_0 < p_1 \leq 1$ *and* s *an integer satisfying* $s \geq [n(1/p_0 - 1)]$. *Let* a *be an* (r, s)*-atom, where* $1/r = (1-\theta)/p_0 + \theta/p_1$ *and* $0 \leq \theta \leq 1$.

Then there exists an analytic function $h(z)$ *mapping* S *into* L^1 *such that*

$$\|h(iy)\|_{H^{p_0,s}} \leq 1, \qquad \|h(1 + iy)\|_{H^{p_1,s}} \leq 1$$

for any real y *and* $h(\theta) = a$.

Proof. Let B be the support of a and define $h(z) = |B|^{\alpha(z)} a$ where $\alpha(z) = (1/r - 1/p_0)(1 - z) + (1/r - 1/p_1)z$. Observe that

$$\alpha(\theta) = 0. \tag{2.2}$$

$$\text{Real}\{\alpha(iy)\} = 1/r - 1/p_0 \quad \text{for all real } y. \tag{2.3}$$

$$\text{Real}\{\alpha(1 + iy)\} = 1/r - 1/p_1 \quad \text{for all real } y. \tag{2.4}$$

Using the fact (easily verified) that a belongs to $H^{p_0,s}$ with quasi-norm not exceeding $|B|^{1/p_0-1/r}$ and (2.3), we obtain

$$\|h(iy)\|_{H^{p_0,s}} = |B|^{1/r-1/p_0}\|a\|_{H^{p_0,s}} \leq 1.$$

Similarly, $\|h(1+iy)\|_{H^{p_1,s}} \leq 1$. Finally, (2.2) implies $h(\theta) = a$. □

Proof of Theorem II. Let a be an (r, s)-atom and denote by $h(z)$ the analytic function associated with a as in Lemma 3. Choose an integer k such that $kp_0 > r$ and g a positive simple function with $\|g\|_{k'} \leq 1$ where $1/k + 1/k' = 1$. Consider

$$\Phi(z) = \int_{\mathbf{R}^n} \left|[g(x)]^{\beta(z)}\right| \left|(T_z h(z))(x)\right|^{r/k} dx$$

where

$$\beta(z) = k' - \frac{k'r}{k}\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right).$$

Note that

$$\beta(\theta) = 1. \tag{2.5}$$

$$\text{Real}\,\beta(iy) = \frac{k'(kp_0 - r)}{kp_0} \quad \text{for all real } y. \tag{2.6}$$

$$\text{Real}\,\beta(1+iy) = \frac{k'(kp_1 - r)}{kp_1} \quad \text{for all real } y. \tag{2.7}$$

Using Hölder's inequality with index $kp_0/r > 1$, (2.6) and the fact that the $L^{k'}$ norm of g is bounded by 1, we obtain

$$|\Phi(iy)| \leq \left\{\int_{\mathbf{R}^n} \left|(T_{iy}h(iy))(x)\right|^{p_0} dx\right\}^{r/kp_0}.$$

Using this inequality, hypothesis (2.1) and Lemma 3, we obtain

$$|\Phi(iy)| \leq A_0^{r/k}\|h(iy)\|_{H^{p_0,s}} \leq A_0^{r/k}. \tag{2.8}$$

Similarly, using Hölder's inequality with index $kp_1/r > 1$, we obtain

$$|\Phi(1+iy)| \leq A_1^{r/k}. \tag{2.9}$$

Lemma 2 implies that $\log \Phi(z)$ is subharmonic. Consequently, Lemma 1 and inequalities (2.8) and (2.9) imply[3]

$$\log \Phi(\theta) \leq \log A_0^{r(1-\theta)/k} A_1^{r\theta/k}.$$

[3] Observe that $\int_{-\infty}^{\infty} \omega(1-\theta, y)\,dy = 1 - \theta$ and $\int_{-\infty}^{\infty} \omega(\theta, y)\,dy = \theta$.

Noticing that $\Phi(\theta) = \int_{\mathbf{R}^n} g(x)\,|(T_\theta a)(x)|^{r/k}\,dx$, the above inequality, together with

$$\left\{\int_{\mathbf{R}^n} |(T_\theta a)(x)|^r\,dx\right\}^{1/k}$$

$$= \sup\left\{\left|\int_{\mathbf{R}^n} |(T_\theta a)(x)|^{r/k} g(x)\,dx\right|,\ g \geq 0,\ g \text{ simple},\ \|g\|_{k'} \leq 1\right\},$$

implies

$$\left\{\int_{\mathbf{R}^n} |(T_\theta a)(x)|^r\,dx\right\}^{1/k} \leq A_0^{r(1-\theta)/k} A_1^{r\theta/k},$$

which proves Theorem II for atoms. From here the theorem follows from the definition of $H^{r,s}$ in terms of atoms. □

3. Remarks. We start with some historical remarks. H^p spaces can be defined on spaces of homogeneous type (see [**1**]). The main difference with the H^p spaces that we consider here is that (iii) in the definition of an atom is replaced by $\int a(x)\,dx = 0$. The reader will have no trouble verifying that Theorem I can be extended to include the homogeneous type case. In this context R. Macías proved in his dissertation (1973–1974) a particular case of Theorem I (see [**3**]). An interpolation theorem for linear operators acting on $H^p(\mathbf{R}^n)$ was published in 1974 by C. Fefferman, N. Rivière and Y. Sagher ([**2**]). Their proof uses the real method of interpolation and is quite different from Macías' proof. For this reason, it is unfortunate that Macías' dissertation was never published. Our work fills the gap left by the unpublished work of R. Macías.

It is worthwhile to notice that the fact proved by M. Taibleson and G. Weiss ([**6**]) concerning the equivalence of $H^{p,s}$ spaces for fixed p plays an important role in the proof of Theorem I.

Theorem I has several extensions. First, $\mathbf{R}^n$ can be replaced by any other space for which an atomic theory similar to the one described above can be defined. Second, condition (1.1) can be replaced by

$$\|T_{j+iy} f\|_{L^{q_j}} \leq A_j(y) \|f\|_{H^{p_j}}, \qquad j = 0, 1,$$

where $0 < q_j < \infty$, $\log A_j(y) \leq C_j e^{d_j|y|}$, $0 < d_j < \pi$ and $C_j > 0$, $j = 0, 1$. Then, it is easy to check that the conclusion of the theorem becomes

$$\|T_\theta f\|_{L^q} \leq C \|f\|_{H^r}$$

where C depends on θ, p_j, q_j, C_j, d_j ($j = 0, 1$) and the dimension n and $1/q = (1-\theta)/q_0 + \theta/q_1$.

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