

ON LOCAL CONVEXITY  
OF BOUNDED WEAK TOPOLOGIES  
ON BANACH SPACES

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**In this paper we prove that the bw topology on a Banach space  $E$ , i.e. the finest topology which agrees with the weak topology on bounded sets of  $E$ , is a locally convex topology if and only if the Banach space  $E$  is reflexive.**

**1. Introduction.** If  $E$  is a Banach space, the bounded weak (bw) topology is the finest topology which agrees with the weak topology on bounded sets. Wheeler in [7, p. 251] proves that the bw topology on  $c_0$  is not locally convex. This result gives a counterexample to a remark of Day [2, p. 42] which said that the bw topology is locally convex always. This fact suggests a question: Under what conditions on  $E$  is it true that bw is a locally convex topology?. The theorem of Banach and Dieudonné (2.2) shows that reflexivity is a sufficient condition. In this paper we obtain that reflexivity is also a necessary condition.

**2. Notations, definitions and preliminary results.** The notations for topological vector spaces are taken primarily from [6], but we employ the definition of polarity found in [4].

DEFINITION 2.1. If  $E$  is a locally convex space (lcs), the equicontinuous weak\* (ew\*) topology on  $E'$  is the finest topology on  $E'$  which coincides with the weak\* topology  $\sigma(E', E)$  on equicontinuous sets of  $E'$ .

The following result characterizes this topology when  $E$  is a metrizable lcs.

THEOREM 2.2. (*Banach-Dieudonné.*) *Let  $E$  be a metrizable locally convex space and  $E'$  its dual. The ew\* topology on  $E'$  is the topology of the uniform convergence on precompact subsets of  $E$ .*

For a demonstration of this theorem we refer the reader to [4] or [6].

As an immediate consequence of this theorem we have that if  $E$  is a metrizable lcs, the ew\* topology on  $E'$  is locally convex.

Other results about  $ew^*$  may be found in [1] and [2].

**DEFINITION 2.3.** If  $E$  is a locally convex space, the bounded weak (bw) topology on  $E$  is the finest topology on  $E$  which agrees with the weak topology  $\sigma(E, E')$  on bounded sets.

This definition is equivalent to Day's [2, p. 41]:

“The bw topology is the collection of all subsets  $U$  of  $E$  satisfying: for each bounded set  $B$  of  $E$ , there is a  $\sigma(E, E')$ -open  $V$  with  $U \cap B = V \cap B$ .”

Obviously the last definition is not changed if we choose the bounded sets on a fundamental family of bounded sets in  $E$ .

As follows from [1, p. 265], the bw topology is semi-linear, i.e. addition and scalar multiplication functions are separately continuous. Moreover, if  $E$  is a Banach space, it can be shown [3, p. 21] that bw is a vectorial topology if and only if it is a locally convex one.

A general result of Collins [1, p. 266], which can be extended to the complex case, makes the following definition valid.

**DEFINITION 2.4.** The convex bw (cbw) topology on a locally convex space  $E$  is the unique locally convex topology with a base of all convex neighborhoods of 0 in the bw topology.

It is easy to see that the cbw topology is the finest locally convex topology which agrees with the weak topology on bounded sets. In [7, p. 251] may be found the following result which characterizes the cbw topology:

**THEOREM 2.5.** *If  $E$  is a lcs, the cbw topology on  $E$  is that of uniform convergence on compact subsets of the completion of  $(E', \beta(E', E))$ .*

As consequence of this result and Theorem 2.2 we obtain:

**COROLLARY 2.6.** *If  $E$  is a Banach space, the cbw topology is the restriction to  $E$  of the  $ew^*$  topology on  $E''$ .*

In particular if  $E$  is reflexive we have:

**COROLLARY 2.7.** *If  $E$  is a reflexive Banach space, the bw topology on  $E$  is a locally convex topology.*

A different introduction and other results about this topology may be seen in [3].

If  $E$  is a Banach space, we denote by  $B, B'', S$  the closed unit ball of  $E$  and  $E''$  and the unit sphere of  $E$ , respectively, and we will write, for each  $n \in \mathbb{N}$ ,  $B_n = nB, B''_n = nB''$  and  $S_n = nS$ .

### 3. The bw topology and local convexity.

LEMMA 3.1. *Let  $E$  be a separable, non-reflexive Banach space. If  $E$  contains no subspace isomorphic to  $l^1$ , there exists a subset  $A$  of  $E$  which is bw-closed but is not closed in the restriction to  $E$  of the  $ew^*$  topology on  $E''$ .*

*Proof.* It is well known that for each  $n \in \mathbb{N}$  the sphere  $S_n$  is  $\sigma(E'', E')$ -dense in the closed ball  $B''_n$ . As  $E$  is a separable Banach space that contains no subspace isomorphic to  $l^1$  it follows from Rosenthal ([5], Theorem 3) that  $S_n$  is  $\sigma(E'', E')$ -sequentially dense in the ball  $B''_n$ , i.e. each  $z \in B''_n$  can be approximated in  $\sigma(E'', E')$  by a sequence contained in  $S_n$ . Hence if  $\phi \in E'' \setminus E$  and  $\|\phi\| = 1$ , there exists for each  $n \in \mathbb{N}$  a sequence  $(x_{k,n})_{k \in \mathbb{N}}$  contained in  $S_n$  and converging to  $n^{-1}\phi$  in  $\sigma(E'', E')$ .

We define  $A = \{x_{k,n}: k, n \in \mathbb{N}\}$ . For each  $m \in \mathbb{N}$  we have

$$\begin{aligned} A \cap B_m &= \{x_{k,n}: k \in \mathbb{N}, n \leq m\} \\ &= (\{x_{k,n}: k \in \mathbb{N}, n \leq m\} \cup \{n^{-1}\phi: n \leq m\}) \cap B_m, \end{aligned}$$

and since the set  $\{x_{k,n}: k \in \mathbb{N}, n \leq m\} \cup \{n^{-1}\phi: n \leq m\}$  is  $\sigma(E'', E')$ -compact, it is  $\sigma(E'', E')$ -closed; then the set  $A \cap B_m$  is closed in the restriction of  $\sigma(E'', E')$  to  $B_m$ , but this topology is the same as the restriction of  $\sigma(E, E')$  to  $B_m$ . This proves that  $A$  is bw-closed.

On the other hand, let  $U$  be a neighborhood of 0 in the  $ew^*$  topology; there exists  $W$ ,  $ew^*$ -neighborhood of 0 such that  $W + W \subset U$ , and as  $W$  is absorbent, there exists  $n_0 \in \mathbb{N}$  such that  $n_0^{-1}\phi \in W$ . By the definition of  $ew^*$  topology we know there exists a  $V$ ,  $\sigma(E'', E')$ -neighborhood of 0 satisfying

$$W \cap B''_{2n_0} = V \cap B''_{2n_0}.$$

As  $(x_{k,n_0})_{k \in \mathbb{N}}$  converges to  $n^{-1}\phi$  in the  $\sigma(E'', E')$ -topology there exists  $k_0 \in \mathbb{N}$  such that  $x_{k_0,n_0} - n_0^{-1}\phi \in V$  and then

$$x_{k_0,n_0} = x_{k_0,n_0} - n_0^{-1}\phi + n_0^{-1}\phi \in (V \cap B''_{2n_0}) + W \subset W + W \subset U.$$

This proves that 0 belongs to the closure of  $A$  in the  $ew^*$  topology, and since  $0 \in E$ , 0 is in the closure of  $A$  in the restriction of  $ew^*$  to  $E$  (we denote this topology  $rew^*$ ). Thus  $A$  is not closed in  $rew^*$ .  $\square$

**PROPOSITION 3.2.** *Let  $E$  be a separable Banach space that contains no subspace isomorphic to  $l^1$ . The bw topology on  $E$  is locally convex if and only if  $E$  is reflexive.*

*Proof.* If  $E$  is reflexive we saw in (2.7) that bw is a locally convex topology. Conversely, if  $E$  is not reflexive, (3.1) and (2.6) prove that the bw topology is not locally convex.  $\square$

**LEMMA 3.3.** *Let  $E$  be a Banach space and  $F$  a closed linear subspace of  $E$ . The bw topology of  $F$  is the restriction to  $F$  of the bw topology on  $E$ .*

*Proof.* We denote bw( $E$ ) and bw( $F$ ) the bw topology on  $E$  and  $F$  respectively. It is clear that the restriction of bw( $E$ ) to  $F$  is coarser than bw( $F$ ).

On the other hand, if  $U$  is bw( $F$ )-open, let  $V$  be the union of  $U$  and  $E \setminus F$ . It is sufficient to prove that  $V$  is bw( $E$ )-open. If  $B$  is a bounded subset of  $E$ , as  $\sigma(F, F')$  coincides with the restriction of  $\sigma(E, E')$  to  $F$ , there exists  $W$ ,  $\sigma(E, E')$ -open, such that

$$U \cap B = U \cap (B \cap F) = (W \cap F) \cap (B \cap F) = (W \cap F) \cap B,$$

and then

$$V \cap B = (W \cup (E \setminus F)) \cap B,$$

and since  $W \cup (E \setminus F)$  is  $\sigma(E, E')$ -open,  $V$  is bw( $E$ )-open.  $\square$

**PROPOSITION 3.4.** *Let  $E$  be a Banach space that contains no subspace isomorphic to  $l^1$ . The bw topology on  $E$  is locally convex if and only if  $E$  is reflexive.*

*Proof.* If  $E$  is not reflexive, there exists a separable nonreflexive subspace  $F$  of  $E$ . Obviously  $F$  contains no subspace isomorphic to  $l^1$ . From (3.2) it follows that the bw topology on  $F$  is not locally convex and then (3.3) shows that bw is not a locally convex topology on  $E$ . This fact and (2.7) prove the theorem.  $\square$

**LEMMA 3.5.** *There exists a subset  $A$  of  $l^1$  which is bw-closed but is not closed in the restriction to  $l^1$  of the ew\*-topology of  $(l^1)''$ .*

*Proof.* For each  $n \in \mathbf{N}$ , we denote by  $e_n$  the sequence of  $l^1$   $(0, 0, \dots, 1, 0, \dots)$  where the one is in the  $n$ th place.

Let  $A_0$  be the set  $A_0 = \{e_n; n \in \mathbf{N}\}$ . For each  $n \in \mathbf{N}$ ,  $e_n$  is a  $\sigma(l^1, l^\infty)$ -isolated point of  $A_0$ , and then  $A_0$  is a  $\sigma(l^1, l^\infty)$ -closed set which is not  $\sigma(l^1, l^\infty)$ -compact. Consequently  $A_0$  is not  $\sigma((l^1)'' , l^\infty)$ -closed. Thus there exists a linear form  $\phi$  that belongs to the  $\sigma((l^1)'' , l^\infty)$ -closure of  $A_0$  and  $\phi \notin A_0$ . Obviously  $\|\phi\| \leq 1$  and  $\phi \in (l^1)'' \setminus l^1$ .

Now, for each  $n \in \mathbf{N}$ , we define

$$A_n = \left\{ n(e_p - e_q) + e_k/n : p, q, k \in \mathbf{N}, p \neq q, p < k, q < k \right\}.$$

$A_n$  is contained in the sphere of radius  $2n + 1/n$  of  $l^1$  and if  $n \in \mathbf{N}$ ,  $n \geq 2$ , it is not hard to check that  $A_n$  is  $\sigma(l^1, l^\infty)$ -closed.

Let  $V$  be a balanced, convex  $\sigma((l^1)'', l^\infty)$ -neighborhood of 0. As  $\phi$  is an accumulation point of  $A_0$ ,  $\phi + V/3n$  contains an infinite number of points of  $A_0$ . If  $e_p, e_q, e_k \in (\phi + V/3n) \cap A_0$  with  $p < q < k$ , we have

$$\begin{aligned} \left[ n(e_p - e_q) + \frac{1}{n}e_k \right] - \frac{1}{n}\phi &= n(e_p - \phi) + n(\phi - e_q) + \frac{1}{n}(e_k - \phi) \\ &\in \frac{1}{3}V + \frac{1}{3}V + \frac{1}{3n^2}V \subset V. \end{aligned}$$

Thus  $(\phi/n + V) \cap A_n$  is a nonempty set. This proves that  $\phi/n$  belongs to the  $\sigma((l^1)'', l^\infty)$ -closure of  $A_n$ .

Now, we define  $A = \bigcup_{n=2}^\infty A_n$ . For each  $m \in \mathbf{N}$ , we have

$$A \cap B_m = \bigcup \{A_n : n \in \mathbf{N}, 5 \leq 2n^2 + 1 \leq mn\}.$$

This set is obviously  $\sigma(l^1, l^\infty)$ -closed, and thus  $A$  is a bw-closed set.

On the other hand, as for each  $n \in \mathbf{N}$ ,  $\phi/n$  belongs to the  $\sigma((l^1)'', l^\infty)$ -closure of  $A_n$ ; reasoning as in the last part of the proof of Lemma 3.1 proves that 0 belongs to the closure on  $\text{ew}^*$  of  $A$ , and as 0 does not belong to  $A$ , we see that  $A$  is not closed in the topology restriction to  $l^1$  of  $\text{ew}^*$  on  $(l^1)''$ .  $\square$

**PROPOSITION 3.6.** *Let  $E$  be a Banach space that contains a subspace isomorphic to  $l^1$ . Then bw is not a locally convex topology on  $E$ .*

*Proof.* From (3.5) we get that the bw topology on  $l^1$  is not locally convex. Hence if  $E$  contains a subspace isomorphic to  $l^1$ , Lemma 3.3 and the conservation of bw topologies under isomorphisms prove that bw is not a locally convex topology on  $E$ .  $\square$

**THEOREM 3.7.** *Let  $E$  be a Banach space. bw is a locally convex topology on  $E$  if and only if  $E$  is reflexive.*

*Proof.* If  $E$  is reflexive, (2.7) gives us the result. Conversely, if  $\text{bw}$  is a locally convex topology on  $E$ , from (3.6) it follows that  $E$  contains no subspace isomorphic to  $l^1$  and (3.4) shows that  $E$  must be reflexive.  $\square$

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