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INTERPOLATION OF BANACH SPACES AND NEGATIVELY CURVED VECTOR BUNDLES

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Let D be the open disk of the complex plane and T the unit circle. Let $\{B_{e^{i\theta}}\}$ be a family of Banach spaces parametrized by the points $e^{i\theta}$ of T. The fundamental construction in the theory of complex interpolation of Banach spaces produces from this data a family of Banach spaces $\{B_z\}$ which is parametrized by the points z of D and which has the given $\{B_{e^{i\theta}}\}$ as boundary values. Basic facts about this construction are summarized in §2. $B = \bigcup_{z\in D}\{B_z\}$ can be regarded as a complex vector bundle with base manifold D. In this paper we study the differential geometry of B and related vector bundles. We show relationships between interpolation theoretic inequalities for families of Banach spaces and the signs of certain curvatures of the associated vector bundles.

1. Introduction and Summary. One consequence of the construction of the $\{B_z\}$ is that the norm function of the vector space B_z regarded as a function of the fiber variable *and* the base variable is plurisubharmonic. The general relation between plurisubharmonicity and negative curvature suggests that B might have non-positive curvature in some appropriate sense. In fact, if all the $B_{e^{i\theta}}$ are Hilbert spaces (i.e. have norms given by inner products) then the bundle B is the unique Hermitian holomorphic vector bundle over D with curvature zero and with the given boundary values on T.

In general the bundles produced by the interpolation construction will be convex Finsler bundles rather than Hermitian bundles; that is, the norms in the fibers will be Banach space norms but need not be Hilbert space norms. In §3 we present two extensions of the notion of Ricci curvature to Finsler bundles and develop the elementary properties of these curvatures. Our main result is that both of the curvatures are non-positive for the bundle B.

The Ricci type curvatures in §3 describe behavior of the bundle which involves all fiber directions simultaneously. A more refined notion of curvature, closely modeled on the curvature of Hermitian holomorphic bundles, has been presented by Kobayashi [**8**]. In §4 we develop the relationship between that curvature and inequalities involving complex interpolation of Banach spaces. The main results are that the vector bundles produced by the interpolation construction are exactly those with zero curvature and the subinterpolation bundles are exactly those with non-positive curvature.

§5 contains real variable analogs of the results of §§3 and 4. The vector bundles considered are real vector bundles and the base space is the unit interval. The fibers are normed by a version of the K-functional of Peetre. (That functional is the starting point for the real variable interpolation theory for Banach spaces.) In that context the results concerning the signs of Ricci type curvatures are related to the classical geometric inequalities of Minkowski and of Brunn and Minkowski. The norms of affine section of such bundles satisfy a local maximum principle. That maximum principle can be formulated as a differential inequality analogous to the curvature results of §4.

I would like to express my thanks to Gary Jensen for his patience and his great help with my questions about geometry.

2. Complex interpolation spaces. We now present, without proof and a bit informally, some of the features of the theory of complex interpolation of finite-dimensional Banach spaces. More details, including proofs of the statements of this section and applications of this theory are in [**4**], [**5**], and [**6**]. A comprehensive view of interpolation theory can be found in [**1**] or [**3**].

Let R be an open subset of $\mathbf{C}$ and regard $\mathbf{C}^n \times R$ as a family of vector spaces parametrized by R. We suppose that for each z in R there is given a Banach space norm $\|\cdot\|_z$ defined on the vector space $\mathbf{C}^n \times \{z\} \approx \mathbf{C}^n$. This norm may vary (smoothly) from point to point and need not be given by an inner product. We denote the normed vector space $(\mathbf{C}^n, \|\cdot\|_z)$ by C_z. Thus $\{C_z\}_{z\in R}$ is a family of Banach spaces parametrized by points of R. We will say that such a family of Banach spaces is a *subinterpolation family* if given any holomorphic $\mathbf{C}^n$-valued function F defined on a subdomain of R, $\log\|F(z)\|_z$ is a subharmonic function of z. Given a Banach space C we denote the dual space by C^*. We establish the duality with respect to the bilinear pairing $(v_1,\ldots,v_n)\cdot(w_1,\ldots,w_n) = \sum v_i w_i$ and, if the norm of C is denoted by $\|\cdot\|$, we denote the norm on C^* by $\|\cdot\|^*$. In particular, $\{C_z^*\}_{z\in R}$ is the family of Banach spaces $\{\mathbf{C}^n \times \{z\}\}$ normed by $\|v\|_z^* = \sup\{|v\cdot w|;\ w \in C_z, \|w\|_z = 1\}$. We say $\{C_z\}$ is an *interpolation family* if both $\{C_z\}$ and $\{C_z^*\}$ are subinterpolation families.

The results of [**5**] insure the existence of interpolation families with given boundary values. Suppose Γ is a smooth simple closed curve in $\mathbf{C}$ which bounds a region R, that for each z in Γ a norm function $\|\cdot\|_z$ is specified on the vector space $\mathbf{C}^n$ and this norm varies smoothly with z. There is a unique interpolation family $\{C_z\}_{z\in R}$ which gives a continuous extension of the norm functions $\|\cdot\|_z$ from Γ to $\Gamma \cup R$. Furthermore, if $\{D_z\}_{z\in R}$ is any subinterpolation family which extends continuously to $\Gamma \cup R$ and is smaller than the given family on Γ, that is, for all z in Γ,

$$\|v\|_{D_z} \le \|v\|_z \quad \forall\, v \in \mathbf{C}^n, \tag{2.1}$$

then the inequality (2.1) also holds for all z in R. Finally, given any z_0 in R and any v in $\mathbf{C}^n$, there is an extremal function for v at z_0, a holomorphic $\mathbf{C}^n$-valued function F_{v,z_0} which satisfies $F_{v,z_0}(z_0) = v$ and $\|F_{v,z_0}(z)\|_z = \|v\|_{z_0}$ for all z in $R \cup \Gamma$. The fundamental step in the construction of the C_z is to define the norm on C_{z_0} for z_0 in R by setting, for v in $\mathbf{C}^n$,

$$\|v\|_{z_0} = \inf\Big\{ \sup_{z\in\Gamma} \|F(z)\|_z;\quad F \text{ a } \mathbf{C}^n\text{-valued holomorphic function on } R \cup \Gamma \text{ for which } F(z_0) = v\Big\}. \tag{2.2}$$

Here is a complete description for $n = 1$. A norm function $\|\cdot\|_z$ is completely specified if we know $w(z) = \|1\|_z$ and $w(z)$ can be any smooth positive function. The family of Banach spaces $(\mathbf{C}, \|\cdot\|_z)$ is a subinterpolation family exactly if $\log w(z)$ is subharmonic. $\|1\|_z^* = w(z)^{-1}$. Hence the family is an interpolation family exactly if $\log w(z)$ is harmonic. The existence theorem of the previous paragraph specializes to the existence of a function w which satisfies the equation $\Delta \log w = 0$ and has specified boundary values.

One class of interpolation families is especially simple to describe. Suppose $\|\cdot\|$ is a given fixed norm on $\mathbf{C}^n$ and T_z is a family of invertible linear maps of $\mathbf{C}^n$ to $\mathbf{C}^n$ which vary analytically with z for z in R. Define $\|\cdot\|_z$ by $\|\cdot\|_z = \|T_z v\|$. We will call interpolation families of this form *flat*. Every one-dimensional interpolation family is locally flat. That is because every solution of $\Delta \log w(z) = 0$ is locally of the form $w(z) = |f(z)|$ for some holomorphic function f. The same phenomenon persists in higher dimension if we restrict our attention to Hilbert spaces (i.e. spaces where the norm is given by an inner product). Let Γ be a smooth simple closed curve which bounds a region R. Suppose Hilbert space norms on $\mathbf{C}^n$ are specified for each point of Γ and let $\{C_z\}$ be the interpolation family which extends these norms to $\bar{R}$. Then all the C_z are Hilbert spaces and they form a flat family. (A differential equation interpretation of this fact is given at the end of §4.) (This is proved in [5]. The crucial step of the proof is that in this case the extremal functions F_{v,z_0}, which are obtained by solving the extremal problem implicit in (2.2), depend linearly on v.)

Not every interpolation family is flat. For p, $1 \leq p < \infty$, let l_n^p be $\mathbf{C}^n$-normed by $\|v\|_p = \|(v_1, \ldots, v_n)\|_p = (\Sigma |v_i|^p)^{1/p}$. Let l_n^∞ be $\mathbf{C}^n$-normed by $\|v\|_\infty = \max |v_i|$. Suppose R is a region in $\mathbf{C}$ and $p(z)$ is a function defined on R with $1 \leq p(z) \leq \infty$. The family $\{C_z\}_{z \in R}$, given by $C_z = l_n^{p(z)}$, can be shown to be an interpolation family exactly if $1/p(z)$ is harmonic. Since the spaces $l_n^{p_1}$ and $l_n^{p_2}$, $p_1 \neq p_z$, are not linearly isometric this cannot be a flat family.

3. Analogs of Ricci curvature. A family of Banach spaces parametrized by points of a domain in $\mathbf{C}$ can be regarded as a holomorphic vector bundle. Even when such bundles are topologically trivial, the presence of a norm function which may vary from point to point keeps the bundle from being geometrically trivial. In this section we introduce two measures of the departure of the bundle from being geometrically trivial. These measures are modeled on (and in the case of Hermitian bundles reduce to) the Ricci curvature for Hermitian holomorphic vector bundles.

The considerations in this section are all local. For convenience we suppose that we have a family of Banach spaces parametrized by the open disk. That is, we have Banach spaces C_z for z in D and we identify $\{C_z\}_{z \in D}$ with the complex manifold $\mathbf{C}^n \times D$.

For any Banach space B we denote by $(B)_1$ the closed unit ball of B and by $(B)\check{}_1$ the closed complex ellipsoid of maximum Euclidean volume contained in B_1. (By a complex ellipsoid we mean the image of the Euclidean unit ball of $\mathbf{C}^n$ under a complex linear map.) (We will omit the minor modifications needed to deal with the possible lack of uniqueness of $(B)\check{}_1$.) Let $v_*(B) =$ Euclidean volume of $(B)\check{}_1$. For the family of Banach spaces $\{C_z\}$ define

$$K_*(C_z) = \Delta \log v_*(C_z). \tag{3.1}$$

Note that although the definition of $v_*(C_z)$ involves a choice of volume form on C_z, all holomorphically varying choices will produce the same values of K_*.

Suppose T_z, z in D, is a family of invertible linear maps of $\mathbf{C}^n$ to $\mathbf{C}^n$. We define the family of Banach spaces $\{T_z C_z\}_{z \in D}$ by specifying the norm at each point z: for all v in $\mathbf{C}^n$, $\|v\|_{T_z C_z} = \|T_z^{-1} v\|_z$.

THEOREM 3.1. *For* $\{C_z\}$, $\{T_z\}$ *as just described*:
(a) *If all the* C_z *have the same norm then* $K_*(C_z) \equiv 0$.
(b) *If the* T_z *vary analytically with* z *then* $K_*(T_z C_z) \equiv K_*(C_z)$.
(c) *If the family* $\{C_z\}$ *is flat then* $K_*(C_z) \equiv 0$.

(d) *If the family* $\{C_z\}$ *consists of Hilbert spaces then* $K_*(C_z)$ *is twice the Ricci curvature of the Hermitian holomorphic vector bundle* $\{C_z\}$.

(e) *If* $\{C_z\}$ *is a subinterpolation family then* $K_*(C_z) \leq 0$ *for all* z *in* D.

Proof. (a) is immediate. To prove (b) note that T_z is a linear map of $(C_z)_1$ to $(T_z C_z)_1$. Hence T_z is a linear map of $(C_z)\check{}_1$ to $(T_z C_z)\check{}_1$. Thus $v_*(T_z C_z) = |\det(T_z)|^2 v_*(C_z)$. If T_z varies holomorphically in z then $\det(T_z)$ is a non-vanishing holomorphic function and thus $\log|\det(T_z)|^2$ is harmonic. Claim (b) follows. Part (c) is an immediate consequence of (a), (b), and the definition of flat. Suppose now that all of the C_z are Hilbert spaces and denote the inner product on C_z by $\langle\ ,\ \rangle_z$. For $i = 1,\ldots,n$ let $e_i = (0,\ldots,0,1,0,\ldots,0)$ be the ith standard basis vector. The e_i are a spanning set of holomorphic sections and we may use them in computing the Ricci curvature. Let $g_{i\bar{j}}(z) = \langle e_i, e_j\rangle_z$, $i, j = 1,\ldots,n$, and let $G(z) = \det(g_{i\bar{j}}(z))$. The base manifold of the vector bundle $\{C_z\}$ has one (complex) dimension. In this case the Ricci curvature is a scalar, namely $-\Delta \log G(z)$ (see, e.g., [7]). Now note that $(C_z)_1 = \{\sum \alpha_i e_i;\ \sum \alpha_i \bar{\alpha}_i g_{i\bar{j}}(z) \leq 1\}$. Thus the volume of $(C_z)_1$ is $(G(z))^{-2}C$, where C is the volume of the Euclidean unit ball of $\mathbf{C}^n$. Because all the C_z are Hilbert spaces, $(C_z)\check{}_1 = (C_z)_1$. Thus $v_*(C_z) = G(z)^{-2}C$ and $K_*(C_z) = -2\Delta \log G(z)$. This yields (d).

We now prove (e). Pick and fix z_0 in D and r with $0 < r < 1 - |z_0|$. Let $\gamma = \{z_0 + re^{i\theta};\ 0 \leq \theta < 2\pi\}$. For ζ in γ, let B_ζ be the Hilbert space with unit ball $(C_\zeta)\check{}_1$. For ζ interior to γ let $\{B_\zeta\}$ be the interpolation family defined inside γ which agrees with the B_ζ for ζ on γ. For ζ on γ, $(B_\zeta)_1 = (C_\zeta)\check{}_1 \subseteq (C_\zeta)_1$. Thus, for any ζ in γ, v in $\mathbf{C}^n$, $\|v\|_{B_\zeta} \geq \|v\|_\zeta$. By hypothesis the $\{C_z\}$ form a subinterpolation family and hence this inequality persists for ζ interior to γ. Thus for any ζ_0 interior to γ, $(B_{\zeta_0})_1 \subseteq (C_{\zeta_0})_1$. The B_ζ, ζ in γ, are all Hilbert spaces. Hence the family $\{B_\zeta\}$, for ζ interior to γ, is a flat family. In particular, B_{ζ_0} is a Hilbert space and thus $(B_{\zeta_0})_1$ is an ellipsoid. Thus $(B_{\zeta_0})_1 \subseteq (C_{\zeta_0})\check{}_1$. Thus $\log v_*(B_{\zeta_0}) \leq \log v_*(C_{\zeta_0})$. By part (c), $\log v_*(B_\zeta)$ is harmonic. By construction $\log v_*(B_\zeta) = \log v_*(C_\zeta)$ for ζ in γ. Thus we have shown $\log v_*(C_\zeta)$ has the super-mean value property and hence is superharmonic. This shows $K_*(C_z)$ is negative. Ths proof is complete.

We now describe the natural dual of K_*. For any Banach space B let $(B)\hat{}_1$ be the ellipsoid of minimal volume which contains $(B)_1$ and let $v^*(B)$ be the volume of $(B)\hat{}_1$. For any family of Banach spaces $\{C_z\}$ define

$$K^*(C_z) = -\Delta \log v^*(C_z). \tag{3.2}$$

The close relationship between K_* and K^* follows from the close relationship between v_* and v^*.

LEMMA 3.2. *There are constants α_n, β_n which depend only on n so that if B is any n-dimensional Banach space, then*

(a) $v_*(B)v^*(B) = \alpha_n$,

(b) $1 \leq v^*(B)/v_*(B) \leq \beta_n$.

Proof. Let $\|\cdot\|$ be the usual Euclidean norm on $\mathbf{C}^n$ and set $C = (\mathbf{C}^n, \|\cdot\|)$. Define α_n by $\alpha_n = v_*(C)v^*(C)$. If D is any n-dimensional Hilbert space then there is a linear isometry T of C to D. T will map $(C)\check{}_1 = (C)_1$ to $(D)\check{}_1 = (D)_1$ and thus $v_*(D) = |\det T|^2 v_*(C)$. Similarly

$$v^*(D^*) = |\det(T^{*-1})|^2 v^*(C^*) = |\det T|^{-2} v^*(C^*) = |\det T|^{-2} v^*(C).$$

Thus (a) holds for D. If B is any Banach space and A is the same vector space with a Hilbert space norm, then $(A)_1 \subseteq (B)_1$ if and only if $(A^*)_1 \supseteq (B^*)_1$. Also volume$(A)_1 \cdot$ volume$(A^*)_1 = \alpha_n$. Hence if we select the norm of A so that $(A)_1 = (B)\check{}_1$ then $(A^*)_1 = (B^*)\hat{}_1$ will follow. Thus (a) holds for general B.

Part (b) follows from a result of F. John (Ch. 9 of **[18]**) which insures that for any Banach space B, $(B)\check{}_1 \subseteq (B)_1 \subseteq \sqrt{n}(B)\check{}_1$ (where $\sqrt{n}(B)\check{}_1 = \{\sqrt{n}\,;\, v \in (B)\check{}_1\}$).

Combining part (a) of the lemma with the previous theorem gives

THEOREM 3.3. *Let $\{C_z\}$ be a family of Banach spaces. Then*

(a) $K_*(C_z) \equiv K^*(C_z^*)$.

(b) *If $\{C_z\}$ is an interpolation family then $K_*(C_z) \leq 0$ and $K^*(C_z) \leq 0$.*

(c) *If $\{C_z\}$ is a flat interpolation family (in particular an interpolation family of Hilbert spaces) then $K_*(C_z) \equiv K^*(C_z) \equiv 0$.*

If all of the $\{C_z\}$ are Hilbert spaces then, by (a) of the lemma, $K_*(C_z) = -K_*(C_z^*)$. In particular the dual of a family with negative K_* will have positive K_*. There seems to be no reason why this should be true for general families or even interpolation families. We pose this as a problem

Question. If $\{C_z\}$ is an interpolation family must $K_*(C_z) = K^*(C_z) = 0$?

Suppose now that $\{C_z\}$ is an interpolation family. By part (b) of the theorem $\log v^*(C_z)$ is subharmonic and $\log v_*(C_z)$ is superharmonic. Thus $L(z) = \log v^* - \log v_* = \log v^*/v_*$ is subharmonic. By part (b) of the lemma we have bounds on L, $0 \leq L \leq \log \beta_n$. Bounds for the subharmonic function L can be used in conjunction with Green's Theorem to obtain lower bounds on area integrals of $-\Delta L = K_* + K^*$. However the full relation between such estimates and the question just posed is not clear. In the extreme case when $\{C_z\}$ is an interpolation family on all of $\mathbf{C}$, L is bounded, subharmonic, and defined on all of $\mathbf{C}$. Such an L must be constant and an affirmative answer to the question follows quickly. (In fact, in that case one can show that the family $\{C_z\}$ is flat.)

Finally, a comment about the choices of v_* and v^*. Although these quantities are of some interest in general Banach space theory (e.g. Ch. 2 of **[10]**) a more tempting choice of a functional to extend the Ricci curvature would be $K(C_z) = \Delta \log(\text{Volume}(B_z)_1)$. However $K(C_z)$ has no simple relation to $K(C_z^*)$ and $K(C_z)$ need not be negative for interpolation families. Here is an example which shows that. Let $R = \{z \in \mathbf{C};\ 0 < \operatorname{Re} z < 1\}$. For z in R let $C_z = l_2^{1/x}$ (with $z = x + iy$). If $K(C_z)$ were negative then $\log(\text{volume}((l_2^{1/x})_1))$ would be a convex function of x for $0 < x < 1$ and hence also for $0 \leq x \leq 1$. However, explicit computation of $\text{volume}((l_2^{1/x})_1)$ for $x = 0, \frac{1}{2}\ 1$ shows the three quantities are in the ratio 6:3:1.

4. Curvature of Finsler bundles. A family of n-dimensional complex vector spaces $C = \{C_z\}_{z \in D}$ parametrized by points of the unit disk D can be regarded as a holomorphic vector bundle of rank (i.e. fiber dimension) n over the complex base manifold D. If all the spaces C_z are Hilbert spaces then C is a Hermitian holomorphic vector bundle and can be studied using the techniques of Hermitian differential geometry. One could, for example, ask if the interpolation families of Hilbert spaces are exactly the Hermitian holomorphic vector bundles with zero curvature. (The answer is yes; Theorem 4.4 below.) When the C_z are Banach spaces the C is still a holomorphic vector bundle with enough additional geometric structure to allow the introduction of a natural differential geometric notion of curvature. The appropriate notion of curvature is presented and developed by S. Kobayashi in **[8]**. In this section we present his definitions and some of his results and we will see a close relation between the geometric and interpolation-theoretic points of view. In particular, families of Banach spaces have negative curvature exactly if they are subinterpolation families and have zero curvature exactly if they are interpolation families.

We assume throughout this section that the norm functions considered are highly differentiable. That is, we assume that the function $\|(\zeta_1,\ldots,\zeta_n)\|_z$ is smooth on $C \backslash (\{(0,\ldots,0)\} \times D)$.

We now introduce the notation and terminology of [8] as it applies in this case. The holomorphic vector bundle $C = \mathbf{C}^n \times D = \{(\zeta, z);\ \zeta \in \mathbf{C}^n,\ z \in D\}$ is said to have a *complex Finsler structure* if there is a smooth real-valued function F defined on C which satisfies

$$(4.1) \qquad F(\zeta, z) \geq 0, \qquad F(\zeta, z) = 0 \text{ if and only if } \zeta = 0,$$

and

$$(4.2) \qquad F(\lambda\zeta, z) = |\lambda|^2 F(\zeta, z) \quad \text{for } \lambda \text{ in } \mathbf{C}.$$

The function of interest to us is $F(\zeta, z) = (\|\zeta\|_z)^2$. This function satisfies an additional condition which is a consequence of the triangle inequality for the norm $\|\cdot\|_z$, or, equivalently, the fact that for each z in D the set $(C_z)_1 = \{\zeta \in \mathbf{C}^n;\ \|\zeta\|_z \leq 1\}$ is a convex set. This condition implies

$$(4.3) \qquad \text{The matrix } \left(F_{i\bar{j}}\right) = \left(\partial^2 F/\partial\zeta_i\partial\bar{\zeta}_j\right) \text{ is positive definite.}$$

Such a Finsler structure is called *convex*. Thus our families of Banach spaces are holomorphic vector bundles with convex Finsler structure — for short, convex Finsler bundles.

Not all convex Finsler structures on C correspond to families of Banach spaces. Given a convex Finsler structure one can define $\|\cdot\|_z$ by

$$(4.4) \qquad F(\zeta, z) = (\|\zeta\|_z)^2.$$

By (4.1) and (4.2), $\|\cdot\|_z$ is positive semidefinite and has the correct homogeneity to be a norm function on $\mathbf{C}^n$. However (4.3) insures that for each z in D the set $\{\zeta \in \mathbf{C}^n;\ F(\zeta, z) \leq 1\}$ is pseudoconvex, but not necessarily convex. (The extension of the complex interpolation theory to such quasinormed spaces has not yet been developed.)

In [8] Kobayashi presents a theory of curvature for convex Finsler bundles. We will take the conclusions of his analysis as the basis of our definitions and must refer to [8] for the motivation and background of the definitions. We start with a family of Banach spaces $C = \{C_z\}_{z \in D}$ and by (4.4) we regard C as a convex Finsler bundle. Given a point (ζ_0, z_0) in C we introduce special coordinates for computation of curvature at (ζ_0, z_0). For $i = 1, \ldots, n$ let $w_i(z)$ be linear functionals on $\mathbf{C}^n$ which depend holomorphically on z. We suppose that $w_1(z_0), \ldots, w_n(z_0)$ are linearly independent. Thus for z at or near z_0 we can regard $w_1(z), \ldots, w_n(z)$ as coordinate functionals on C_z. We say that the w_i form a normal coordinate system at (ζ_0, z_0) if

$$(4.5) \qquad \left(\frac{\partial}{\partial w_i}\frac{\partial}{\partial \bar{w}_j}F\right)(\zeta_0, z_0) = \delta_{ij}, \qquad i, j = 1, \ldots, n,$$

and

$$\left(\frac{\partial}{\partial z}\frac{\partial}{\partial w_i}\frac{\partial}{\partial \overline{w}_j}F\right)(\zeta_0, z_0) = 0, \qquad i, j = 1,\ldots,n. \tag{4.6}$$

It is always possible to find such coordinates. In fact the w_i can be chosen as affine (in z) combinations of any given set of n linearly independent analytic (in z) linear (in ζ) functionals ((3.22) of [**8**]).

We say that C is negatively curved at (ζ_0, z_0) if given a normal coordinate system $w_1,\ldots,w_n$ at (ζ_0, z_0) the quantity

$$K(\zeta_0, z_0) = \frac{-1}{F(\zeta_0, z_0)}\left(\sum_{ij}\left(\frac{\partial^2}{\partial z\partial \bar{z}}\frac{\partial}{\partial w_i}\frac{\partial}{\partial \overline{w}_j}F\right)w_i\overline{w}_j\right)(\zeta_0, z_0) \tag{4.7}$$

is negative at (ζ_0, z_0) (positively curved if K is positive, etc.). We refer to [**8**], especially to Theorem 4.1, for proof that this definition doesn't depend on the choice of normal coordinates, and for the relation of this definition to other, more conceptual, definitions.

We now express K in terms of the norm function on the Banach spaces C_z. First note that the homogeneity of F forces the derivatives of F to satisfy certain identities. If (4.2) is differentiated with respect to λ and then $\bar{\lambda}$ and then evaluated at $\lambda = 1$ we obtain ((3.3) of [**8**])

$$\sum_{ij}\left(\frac{\partial}{\partial w_i}\frac{\partial}{\partial \overline{w}_j}F\right)w_i\overline{w}_j = F. \tag{4.8}$$

Let $\zeta_0(z)$ be the $\mathbf{C}^n$-valued function with $\zeta_0(z_0) = \zeta_0$ and which has constant $w_i(z)$ coordinates; $w_i(z)(\zeta_0(z)) = w_i(z_0)(\zeta_0)$. Substituting (4.8) into (4.7) gives

$$K(\zeta_0, z_0) = -\frac{1}{F(\zeta_0, z_0)}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}F(\zeta_0(z), z)\,\big|_{z=z_0}.$$

When this is combined with (4.4) we obtain

$$K(\zeta_0, z_0) = -2\big(\Delta \log\|\zeta_0(z)\|_z + 2\,|\nabla \log\|\zeta_0(z)\|_z|^2\big). \tag{4.9}$$

Suppose $h(z)$ is a $\mathbf{C}^n$-valued holomorphic function defined in a neighborhood N of z_0. Let $[h(z)]$ be the span of $h(z)$ regarded as a subspace of C_z. The family $H = \{[h(z)]\}_{z\in N}$ forms a line bundle over N and is an analytic subbundle of $\{C_z\}_{z\in N}$. The norm function on C_z induces a norm on the vector space $[h(z)]$. With this norm H is a convex Finsler bundle and we may consider its curvature. The curvature of H can be computed using the analog of (4.9) or by noting that a Finsler line bundle is also a Hermitian line bundle and the curvature defined by (4.9) agrees in this case with the curvature for Hermitian line bundles. Thus the curvature of H at z_0 is proportional to

$$K_H(h(z_0), z_0) = -\Delta \log\|h(z)\|_z. \tag{4.10}$$

A fundamental fact in the geometry of Hermitian holomorphic vector bundles is that curvature decreases on passage to subbundles. This feature is also present in the theory of [**8**]. By Theorem 6.1 of [**8**] we know that there is a universal positive constant C so that, if $\|h(z_0)\|_{z_0} = 1$,

$$K_H(h(z_0), z_0) \le CK(h(z_0), z_0). \tag{4.11}$$

In §2 we defined subinterpolation families. We now define the dual notion. We will call a family of Banach spaces $\{C_z\}$ a superinterpolation family if $\{C_z^*\}$ is a subinterpolation family. Thus $\{C_z\}$ is an interpolation family exactly if it is both a superinterpolation family and a subinterpolation family. The choice of terminology is supported by the following lemma.

LEMMA 4.1. *Let $\{C_z\}_{z\in D}$ be a family of Banach spaces. The following are equivalent*:

(a) *$\{C_z\}$ is a superinterpolation family.*

(b) *Given any smooth simple closed curve γ in D, let $\{D_z\}$ be the interpolation family defined on the interior of γ which agrees with C_z on γ. Then for all z interior to γ and all v in $\mathbf{C}^n$, $\|v\|_{C_z} \ge \|v\|_{D_z}$.*

(c) *For any z_0 in D, and ζ_0 in $\mathbf{C}^n$ there is an $\varepsilon = \varepsilon(\zeta_0, z_0) > 0$ so that for all r, $0 < r < \varepsilon$, if D_z is the interpolation family defined on $\{z;\ |z - z_0| < r\}$ which agrees with C_z on $\{z;\ |z - z_0| = r\}$, then $\|\zeta_0\|_{C_{z_0}} \ge \|\zeta_0\|_{D_{z_0}}$.*

Proof. These equivalences follow directly from the corresponding facts about subinterpolation families and the fact that the passage to dual spaces reverses inequalities (i.e. if the inclusion of A into B is norm decreasing then the same is true for the inclusion of the dual spaces B^* into A^*).

THEOREM 4.2. *Let $\{C_z\}_{z\in D}$ be a family of Banach spaces and C the corresponding convex Finsler bundle. Let $C^* = C\backslash(\{0\} \times D)$. Then*:

(a) *$K(\zeta, z) \le 0$ on C^* if and only if $\{C_z\}$ is a subinterpolation family.*

(b) *If $K(\zeta, z) \ge 0$ on C^* then $\{C_z\}$ is a superinterpolation family.*

(c) *$K(\zeta, z) = 0$ on all of C^* if and only if $\{C_z\}$ is an interpolation family.*

Proof. If $\{C_z\}$ is a subinterpolation family then, by definition, if f is any holomorphic $\mathbf{C}^n$-valued function then $\log\|f(z)\|_z$ is subharmonic. Thus, by (4.9), $K \leq 0$. If $K \leq 0$ then, by (4.11), the curvature of all one-dimensional holomorphic subbundles is negative. Hence, by (4.10) $\log\|f(z)\|_z$ is subharmonic for any holomorphic $\mathbf{C}^n$-valued function f. This proves (a).

To prove (b) we first suppose that the curvature is strictly positive, $K(\zeta, z) > 0$ on C^*. Given (ζ_0, z_0) in C^* we can, by (4.9), find a holomorphic $\mathbf{C}^n$-valued function f such that $f(z_0) = \zeta_0$ and $\Delta \log\|f(z)\|_z < 0$ at $z = z_0$. Pick r so small that $\Delta \log\|f(z)\|_z < 0$ interior to $\gamma = \{z; |z - z_0| = r\}$. For any r', $0 < r' < r$, let $\gamma' = \{z; |z - z_0| = r'\}$. Let $\{D_z\}$ be the interpolation family defined inside γ' which agrees with $\{C_z\}$ on γ'. Thus, for z on γ', $\log\|f(z)\|_z = \log\|f(z)\|_{D_z}$. For z interior to γ' the first term is superharmonic (because its Laplacian is negative) and the second is subharmonic (because $\{D_z\}$ is an interpolation family). Hence at z_0, $\|f(z_0)\|_{z_0} \geq \|f(z_0)\|_{D_{z_0}}$. We have verified condition (c) of Lemma 4.1 and hence $\{C_z\}$ is a superinterpolation family. We now pass to the general case by approximation. Let $h(z)$ be a fixed smooth positive superharmonic function and let $\{\hat{C}_z\}$ be the family of Banach spaces $\mathbf{C}^n \times \{z\}$, z in D, normed by $\|v\|_{\hat{C}_z} = h(z)\|v\|_z$. Suppose $K(\zeta, z) \geq 0$ on C^*. Hence by (4.9), given (ζ, z) in C^* there is a holomorphic $\mathbf{C}^n$-valued function f with $f(z) = \zeta$ and $\Delta \log\|f(z)\|_z \leq 0$. Thus

$$\Delta \log\|f(z)\|_{\hat{C}_z} = \Delta \log h(z) + \Delta \log\|f(z)\|_{C_z} < 0.$$

By (4.10) and (4.11), $-\Delta \log\|f(z)\|_{\hat{C}_z}$ gives a lower bound for $\hat{K}$, the curvature of $\{\hat{C}_z\}$. Thus $\hat{K}$ is strictly positive. Hence, by the first half of the argument, $\{\hat{C}_z\}$ is a superinterpolation family. $\{C_z\}$ is a limit of such families (as h is chosen near 1) and, hence, is a superinterpolation family.

Now suppose $K \equiv 0$ on C^*. By (a) and (b) $\{C_z\}$ is both a subinterpolation family and a superinterpolation family; hence it is an interpolation family. On the other hand, if $\{C_z\}$ is an interpolation family then, by (a), $K \leq 0$. Also, given (ζ_0, z_0), the extremal function $F = F_{\zeta_0, z_0}(z)$ is holomorphic, $F(z_0) = \zeta_0$ and $\|F(z)\|_z$ is constant. Thus the span of $F(z)$ forms a one-dimensional holomorphic subbundle with zero curvature. Thus, by (4.11), $K \geq 0$. This proves (c).

There is an obvious asymmetry in the theorem which we conjecture can be removed.

Conjecture. If $\{C_z\}$ is a superinterpolation family then $K \geq 0$ on C^*.

In light of the lemma and the other parts of the theorem, this is closely related to the question raised in [8] of whether the dual of a negatively curved convex Finsler bundle has positive curvature.

The following is a partial result which also suggests an interesting question in the interpolation theory.

THEOREM 4.3. *Suppose $\{C_z\}_{z\in D}$ is a superinterpolation family and C the corresponding convex Finsler bundle. Then for all z,* $\sup\{K(\zeta, z);\ \zeta \in C_z, \|\zeta\|_z = 1\} \geq 0$.

Proof. Pick and fix z_0 in D. Let γ_n be a small circle inside D with center z_0. Let $\{D_{n,z}\}$ be the interpolation family defined interior to γ_n which agrees with $\{C_z\}$ on γ_n. Pick ζ_0 in C_{z_0}, $\|\zeta_0\|_{z_0} = 1$. Let F_n be the corresponding extremal function; F_n is a $\mathbf{C}^n$-valued holomorphic function, $F_n(z_0) = \zeta_0$, and $\|F_n(z)\|_{D_{n,z}}$ is constant inside and on γ_n. $\log\|F_n(z)\|_z \geq \log\|F_n(z)\|_{D_n,z}$ interior to γ_n (because $\{C_z\}$ is a superinterpolation family) and equality holds on the boundary. Hence it cannot be true that $\log\|F_n(z)\|_z$ is strictly subharmonic interior to γ_n. Thus there is a z_n interior to γ_n such that $\Delta \log\|F_n(z)\|_z \leq 0$ at $z = z_n$. By (4.9) and (4.10) this gives $K(F_n(z_n), z_n) \geq 0$. As the γ_n shrink to z_0, z_n must converge to z_0. For large n,

$$\begin{aligned} \|F(z_n)\|_{z_0} \sim \|F(z_n)\|_{z_n} \sim \|F(z_n)\|_{D_n, z_n} &= \|F(z_0)\|_{D_n, z_0} \\ \sim \|F(z_0)\|_{z_0} = \|\zeta_0\|_{z_0} &= 1. \end{aligned}$$

Hence there is a subsequence of $\{(F_n(z_n), z_n)\}$ which converges to a point $(\hat{\zeta}_0, z_0)$ with $\|\hat{\zeta}_0\|_{z_0} = 1$. By continuity $K(\zeta_0, z_0) \geq 0$ and the proof is complete.

This proof fails to give the full conjecture because we are unable to show that $\hat{\zeta}_0 = \zeta_0$. What is missing is an appropriate uniform convergence theorem for the extremal functions. If such a result were available we would also be able to localize Theorems 4.2 and 4.3 in the fiber variable. That is, we would be able to show that $K(\zeta, z) \leq 0$ near (ζ_0, z_0) if and only if the $\{C_z\}$ form a local subinterpolation family at (ζ_0, z_0) (defined in the obvious way).

Finally, we note that if all the $\{C_z\}$ are Hilbert spaces then there is a complete pairing between the interpolation properties of $\{C_z\}$ and the curvature of C, now regarded as a Hermitian holomorphic vector bundle. (See [7] for appropriate definitions.)

THEOREM 4.4. *Let $\{C_z\}_{z\in D}$ be a family of Hilbert spaces and C the associated Hermitian holomorphic vector bundle.*

(a) *$\{C_z\}$ is a subinterpolation family if and only if C is negatively curved;*

(b) *$\{C_z\}$ is a superinterpolation family if and only if C is positively curved; and*

(c) *$\{C_z\}$ is an interpolation family if and only if C has zero curvature.*

Proof. We will only outline the proof. By direct calculation in local coordinates (or other ways) one checks that a Hermitian holomorphic vector bundle over the disk is negatively curved exactly if $\log\|f(z)\|_z$ is subharmonic for any holomorphic section f. This establishes (a). Part (b) follows from (a) by duality (that the dual of a negatively curved bundle is positively curved is Proposition 6.2 of **[9]**). (c) follows by combining (a) and (b).

The statement in part (c) of the theorem that the curvature vanishes can be reformulated as a statement that the norm function satisfies a partial differential equation. If the norm on C_z is described by the positive definite matrix $\Omega(z)$, $\|v\|_z^2 = \langle \Omega(z)v, v\rangle$, then the equation satisfied by $\Omega(z)$ for the bundle C to have vanishing curvature is

$$\bar\partial(\Omega^{-1}\partial\Omega) = 0. \tag{4.12}$$

This is the n-dimensional analog of the equation $\Delta \log w = 0$ discussed in §2. In this case the interpolation construction is equivalent to solving (4.12) for Ω with specified boundary data. Ω is obtained in the form

$$\Omega(z) = A^*(z)A(z)$$

for an appropriate holomorphic family of invertible matrices $A(z)$.

5. Real variable analogs. In the three subsections of this section we present real variable analogs of the results of the previous three sections. In §5.1 we define real interpolation and subinterpolation families. In §5.2 we study the volume of the unit balls of such families and their duals. We obtain a result analogous to, and in some ways sharper than, Theorem 3.3. In §5.3 we show that quantities analogous to K of §4 are negative for subinterpolation families.

5.1 Construction of spaces. Given a family of complex Banach spaces $\{C_{e^{i\theta}}\}$ defined for points $e^{i\theta}$ on the unit circle one can define the norms for the complex interpolation family having the given spaces as boundary values by setting, for z in D, v in $\mathbf{C}^n$.

$$(5.1)\quad \|v\|_z = \inf\Big\{ \sup_{e^{i\theta}\in\partial\bar{D}} \|F(e^{i\theta})\|_{e^{i\theta}};$$
$$F \text{ a holomorphic } \mathbf{C}^n\text{-valued function, } F(z) = v\Big\}.$$

It is one of the features of the complex interpolation theory that several seemingly different definitions produce the same norms. For example we could define the intermediate space C_z by specifying their unit balls by

$$(5.2)\qquad \bigcup_{z\in\bar{D}} (C_z)_1 = \text{holomorphically convex hull of } \bigcup_{z\in\partial\bar{D}} (C_z)_1$$

and this would produce the same norm. Alternatively, for fixed p, $1 \le p < \infty$, for z in D, $P_z(\theta)$ the Poisson kernel for z and v in $\mathbf{C}^n$ we could set

$$(5.3)\quad \|v\|_z = \inf\bigg\{ \Big(\int \|F(e^{i\theta})\|^p_{e^{i\theta}} P_z(\theta)\, d\theta \Big)^{1/p}$$
$$F \text{ a holomorphic } \mathbf{C}^n\text{-valued function, } F(z) = v\bigg\}$$

and obtain the same norm as that given by (5.1).

To form real variable analogs of these constructions we replace holomorphic functions by affine functions and considerations of holomorphic convexity with considerations of convexity. Rather than work with the unit ball of $\mathbf{R}$ we use the more convenient unit interval $I = (0, 1)$. For the rest of this section we will consider *real* Banach spaces.

We start with $\mathbf{R}^n \times \bar{I}$ and suppose that $A_i = \mathbf{R}^n \times \{i\}$ are given Banach spaces with norms $\|\cdot\|_i$, $i = 0, 1$. We wish to construct intermediate spaces A_t, $0 < t < 1$, by putting norms on $\mathbf{R}^n \times \{t\}$ in an appropriate way. One possibility is to mimic (5.1) and set, for v in $\mathbf{R}^n$, $0 < t < 1$.

$$(5.4)\quad \|v\|_t = \inf\Big\{ \sup_{s\in\partial\bar{D}} \|F(s)\|_s;\ F \text{ an affine}$$
$$\mathbf{R}^n\text{-valued function, } F(t) = v\Big\}.$$

This gives a norm on $\mathbf{R}^n$ which has many of the properties of the complex interpolation construction. The analog of (5.2) holds, that is, if we denote the unit ball of A_t by $(A_t)_1$ then

$$(5.5)\qquad \bigcup_{t\in\bar{I}} (A_t)_1 = \text{convex hull of } \bigcup_{t\in\partial\bar{I}} (A_t)_1.$$

When we need to exhibit the dependence of A_t on the boundary data we write $A_t = (A_0, A_1)_t$. It is a consequence of (5.5) that the construction of the A_t has a stability under repetition (a "reiteration property" in the language of interpolation theory). If $0 \le t_0 < s < t_1 \le 1$ and θ is defined by $s = t_0 + \theta(t_1 - t_0)$ then

$$(5.6) \qquad (A_0, A_1)_s = (A_{t_0}, A_{t_1})_\theta.$$

(Here $(A_{t_0}, A_{t_1})_\theta$ is short-hand for the norm on $\mathbf{R}^n \times \{s\}$ given by $(B_0, B_1)_\theta$ with B_0 the space $\mathbf{R}^n \times \{0\}$ normed by the norm of A_{t_0} carried to $\mathbf{R}^n \times \{0\}$ by the obvious isomorphism of $\mathbf{R}^n \times \{0\}$ with $\mathbf{R}^n \times \{t_0\}$, etc.) It follows from (5.5) and (5.6) that the norm of affine $\mathbf{R}^n$-valued functions will satisfy a maximum principle. That is, if F is an affine $\mathbf{R}^n$-valued function defined on $\mathbf{R}^n$ then

(5.7) $\quad \|F(t)\|_{A_t}$ will not have a interior strict local maximum.

We omit the direct verification of these properties.

Although (5.1) and (5.3) produce the same norms, that is not true for the real variable analogs. For p with $1 \le p < \infty$ we define $A_{t,p}$ to be $\mathbf{R}^n \times \{t\}$ normed by

$$(5.8) \qquad \|v\|_{t,p} = \inf\Big\{\big((1-t)\|F(0)\|_0^p + t\|F(1)\|_1^p\big)^{1/p}; \; F \text{ an affine } \mathbf{R}^n\text{-valued function, } F(t) = v\Big\}.$$

We set $A_{t,\infty} = A_t$.

These norms are not all the same. Another difference between these spaces and those of the complex variable theory is that in this case the dual spaces are of a different form. (In the complex theory the duals of interpolation spaces are the interpolation spaces of the duals.) For any norm $\|\cdot\|$ on $\mathbf{R}^n$ we denote the dual norm by $\|\cdot\|^*$. For p, $1 \le p \le \infty$, let q be the conjugate index defined by $1/p + 1/q = 1$ (with $1/\infty = 0$). For any linear functional L on $\mathbf{R}^n$ the norm of L as a functional on $A_{t,p}$ is

$$(5.9) \qquad \|L\|_{t,p}^* = \big((1-t)(\|L\|_0^*)^q + t(\|L\|_1^*)^q\big)^{1/q}.$$

In particular note that $\|L\|_{t,\infty}^* = \|L\|_t^*$ is an affine function of t.

We call a family of Banach spaces $\{A_t\}$ a *real interpolation family* if they are produced from some A_0 and A_1 by the construction (5.4). This is equivalent to requiring (5.6) to hold, that is, if $0 \le t_0 < t_1 \le 1$, $0 \le \theta \le 1$, then

$$(5.10) \qquad A_{t_0 + \theta(t_1 - t_0)} = (A_{t_0}, A_{t_1})_\theta.$$

Let $\{B_t\}_{t \in I}$ be a family of Banach spaces; that is, B_t is the vector space $\mathbf{R}^n \times \{t\}$ with norm $\|\cdot\|_{B_t}$. We say B_t is a *subinterpolation family* if for any subinterval of I, the B_t norm is dominated by the norm of the interpolation family which agrees with the family $\{B_t\}$ at the end points of the subinterval. This can be written as an inclusion of unit balls similar to (5.10). For $0 \leq t_0 < t_1 \leq 1$, $0 \leq \theta \leq 1$,

$$\left(B_{t_0+\theta(t_1-t_0)}\right)_1 \supseteq \left((B_{t_0}, B_{t_1})_\theta\right)_1. \tag{5.11}$$

There are a number of ways to reformulate this condition.

LEMMA 5.1. *Suppose $\{B_t\}_{t\in I}$ is a family of Banach spaces. The following are equivalent.*

(1) *$\{B_t\}_{t\in I}$ is a subinterpolation family.*

(2) *$(\bigcup_{t\in I}(B_t)_1)$ is a convex subset of $\mathbf{R}^{n+1}$.*

(3) *If F is an affine $\mathbf{R}^n$-valued function on I then $\|F(t)\|_{B_t}$ satisfies the local maximum principle; that is,* (5.7) *holds.*

(4) *For any linear functional L, $\|L\|^*_{B_t} = \|L\|_{B_t^*}$ is a concave function of t (i.e. $-\|L\|^*_{B_t}$ is convex).*

Proof. If (2) holds then, by (5.2), (5.10) holds, that is, (2) implies (1). If (1) holds, then (5.7) for interpolation families yields (5.7) for the $\{B_t\}$ so (1) implies (3). If (3) holds and $(v_0, t_0) \in (B_{t_0})_1$ and $(v_1, t_1) \in (B_{t_1})_1$, $t_0 < t_1$, then $F(t)$, the affine function which takes the values $F(t_i) = v_i$, $i = 0, 1$, must satisfy $\|F(t)\|_{B_t} \leq 1$ for t with $t_0 < t < t_1$. This, together with the observation that each set $(B_t)_1$ is convex, establishes (2). If (1) holds then a reverse inequality holds for the dual spaces. That is, if A_t is an interpolation family which agrees with B_t at t_0, t_1 with $t_0 < t_1$, then for any L, $\|L\|^*_{A_t} \leq \|L\|^*_{B_t}$ for t between t_0 and t_1. By (5.9) $\|L\|^*_{A_t}$ is affine. Thus $\|L\|^*_{B_t}$ sits above the affine function with the same values at t_0 and t_1. Thus $\|L\|^*_{B_t}$ is concave. Conversely, if $\|L\|^*_{B_t}$ is concave then $\|L\|^*_{A_t} \leq \|L\|^*_{B_t}$ and, hence, $\|v\|_{A_t} \geq \|v\|_{B_t}$.

As an example, note that for $1 \leq p < \infty$, the spaces $A_{t,p}$ form a subinterpolation family. The quickest way to verify this is to use condition (4) of the lemma and (5.9) and note that $d^2\|L\|^*_{t,p}/dt^2 < 0$.

We motivated the definitions of A_t and $A_{t,p}$ by the analogy with the holomorphic case. However these spaces are also closely related to real variable interpolation theory. A common starting point for real variable interpolation theory is the K-functional of Peetre. If A_0, A_1 are given n-dimensional normed spaces and v is in $\mathbf{R}^n$, $\lambda > 1$, then the K-functional $K_\infty(\lambda, v; A_0, A_1)$ is related to our $A_t = A_{t,\infty}$ by

$$K_\infty(\lambda, v; A_0, A_1) = (\lambda + 1)^{-1}\|v\|_{1/(\lambda+1)}.$$

(There are also functionals K_p related to our $A_{t,p}$. See page 75 of [**1**].) Thus the K-method of interpolation, which consists of constructing new norms by taking weighted averages of K-functionals, can be formulated in terms of the A_t spaces. For more on the real interpolation theory, see [**1**] or [**3**].

5.2 Volumes. The results in §3 involved subharmonicity of logarithms of volumes. The analogs for real spaces involve the convexity of nth roots of volumes. For any Banach space B, let $(B)_1$ be the unit ball and $v(B)$ the Euclidean volume of B. Suppose $\{B_t\}_{t\in I}$ is a family of n-dimensional Banach spaces parametrized by the points t of I. Analogously to (3.1) we define

$$K_{n,*}(B_t) = \frac{d^2}{dt^2} v(B_t)^{1/n} \tag{5.12}$$

and, in a similar spirit,

$$K_n^*(B_t) = -\frac{d^2}{dt^2} v(B_t^*)^{1/n}. \tag{5.13}$$

THEOREM 5.2. *Let* $\{B_t\}$ *be given.*

(a) *If* $\{B_t\}$ *is an interpolation family then*

$$K_{n,*}(B_t) \equiv 0, \qquad K_n^*(B_t) \le 0.$$

(b) *If* $\{B_t\}$ *is a subinterpolation family then, for all* t,

$$K_{n,*}(B_t) \le 0 \tag{5.14}$$

and

$$K_n^*(B_t) \le 0. \tag{5.15}$$

(c) *If* $\{B_t\}$ *is a subinterpolation family and equality holds in* (5.14) *for all* t *then* $\{B_t\}$ *is an interpolation family. If* $\{B_t\}$ *is a subinterpolation family and equality holds in* (5.15) *for all* t *then* $\{B_t\}$ *is a constant family* (*i.e.* $\|v\|_{B_t}$ *is independent of* t).

Proof. Suppose $\{B_t\}$ is an interpolation family. $\bigcup_{t\in\bar{I}}(B_t)_1$ is a subset of $\mathbf{R}^{n+1}$. The sets $S_i = \{(v, i);\ v \in (B_i)_1\}$, $i = 0, 1$, are convex sets in parallel affine hyperplanes. If the $\{B_t\}$ form an interpolation family then, by (5.5), $v(B_t)$ is the volume of the intersection of the hyperplane $\{(v, t);\ v \in \mathbf{R}^n\}$ with the convex hull of $S_0 \cup S_1$. The Brunn-Minkowski inequality [**2**] insures that the nth root of this volume is an affine function of t.

(This is the equality version of the inequality. It holds because the hyperplanes containing S_0 and S_1 are parallel.) That proves half of (a).

Let N be the norm function for the n-dimensional Banach space B. Let Σ be the Euclidean unit sphere in $\mathbf{R}^n$. Using polar coordinates we see that

$$v(B) = C\int_{\Sigma} N(v)^{-n}\, d\sigma(v), \tag{5.16}$$

where C is a normalization which depends only on the dimension and $d\sigma$ is surface measure on Σ. By (5.9) the norm function $N(t)$ on A_t^* satisfies

$$N(t) = (1-t)N(0) + tN(1). \tag{5.17}$$

Using (5.16), we see that the desired conclusion is that

$$r(t) = \left(C\int N(t)^{-n}\, d\sigma\right)^{1/n}$$

is a convex function of t. By (5.17) and the inequality between the harmonic and arithmetic means we have

$$\begin{aligned} r\left(\frac{1}{2}(t_1+t_2)\right) &= \left(C\int\left(\frac{1}{2}N(t_1)+\frac{1}{2}N(t_2)\right)^{-n} d\sigma\right)^{1/n} \\ &\le \left(C\int\left(\frac{1}{2N(t_1)}+\frac{1}{2N(t_2)}\right)^{n} d\sigma\right)^{1/n}. \end{aligned}$$

We now apply Minkowski's inequality, i.e. the triangle inequality for the space $L^n(\Sigma, d\sigma)$, and obtain

$$\begin{aligned} r\left(\frac{1}{2}(t_1+t_2)\right) &\le \left(C\int\left(\frac{1}{2N(t_1)}\right)^{n} d\sigma\right)^{1/n} + \left(C\int\left(\frac{1}{2N(t_2)}\right)^{n} d\sigma\right)^{1/n} \\ &= \frac{1}{2}r(t_1)+\frac{1}{2}r(t_2). \end{aligned}$$

Thus r is convex and (a) is proved. We also note that if $r(t)$ is affine then equality must hold between the harmonic and arithmetic means and thus $N(t_1)(v) = N(t_2)(v)$ for all t_1, t_2, v. That is, $\{B_t^*\}$ and hence also $\{B_t\}$, is constant.

Part (b) follows by using the results in part (a) to obtain support functions. Suppose $\{B_t\}$ is a given subinterpolation family, (t_0, t_1) a given subinterval of I and $\{A_t\}$ is the interpolation family which agrees with $\{B_t\}$ at $t = t_0$ and $t = t_1$. By the definition of subinterpolation family, (5.11), $(A_t)_1 \subseteq (B_t)_1$ for $t_0 \le t_1 \le t_1$. Hence $v(A_t)^{1/n} \le v(B_t)^{1/n}$ for $t_0 \le t \le t_1$ with equality at the end points. By (a), $v(A_t)^{1/n}$ is affine. Thus $v(B_t)^{1/n}$ sits above the affine function with which it agrees at the end points of the subinterval. This shows $K_{n,*}(B_t) \le 0$. Also note that if $K_{n,*}(B_t) \equiv 0$ then all of the inequalities must be equalities. This shows the first part of (c). By part (4) of Lemma 5.2, or by dualizing (5.11), $(B_t^*)_1 \subseteq (A_t^*)_1$. Hence $v((B_t^*)_1)^{1/n} \le v((A_t^*)_1)^{1/n}$. By part (a) the right-hand side is convex. Thus for any subinterval, $v((B_t^*)_1)^{1/n}$ is dominate by a convex function with the same values at the end points of the subinterval; hence $v((B_t^*)_1)^{1/n}$ is convex. This shows $K^*(B_t) \le 0$. Finally, if $K^*(B_t) \equiv 0$ then equality must hold in the previous inequalities and hence $B_t^* = A_t^*$ and $K^*(A_t) \equiv 0$. We noted earlier that implies $\{A_t\}$ is constant. The proof is complete.

5.3 Curvature. The maximum principle, (5.7), can be reformulated as a differential inequality. This gives estimates similar to some of those in §4. However, the fact that curvature decreases on passage to subbundles appears to be characteristic of the holomorphic theory. In the absence of such a principle we obtain only limited results in the real variable case.

If we combine (4.6) and (4.8) we find that the normal coordinates used in the definition of K in §4 satisfied (among other things)

$$\frac{\partial}{\partial z}\|w_i(z)\|_z = 0, \qquad i = 1, \ldots, n.$$

If we consider affine sections, $F(t)$, of the real vector bundle $\{B_t\}$, an analogous equation would be

$$\frac{d}{dt}\|F(t)\|_{B_t}\Big|_{t=t_0} = 0. \tag{5.18}$$

Analogously to (4.9) we could then define, for any $F(t)$ which satisfies (5.18),

$$K(F(t_0), t_0, F(t)) = -\frac{1}{\|F(t_0)\|_{t_0}^2}\frac{d^2}{dt^2}\|F(t)\|_{B_t}^2\Big|_{t=t_0}$$

or, equivalently, if $\|F(t_0)\|_{t_0} = 1$,

$$K(F(t_0), t_0, F(t)) = -2\frac{d^2}{dt^2}\|F(t)\|_{B_t}\Big|_{t=t_0}. \tag{5.19}$$

THEOREM 5.3. *If $\{B_t\}$ is a subinterpolation bundle then for any $F(t)$ which satisfies (5.18), $K(F(t_0), t_0, F(t)) \le 0$. Conversely if $K(F(t_0), t_0, F(t)) < 0$ for all such $F(t)$ then $\{B_t\}$ is a subinterpolation bundle.*

Proof. By Lemma 5.1, $\{B_t\}$ is a subinterpolation family if and only if $\|F(t)\|_{B_t}$ never has a strict interior local maximum for any affine function F. The theorem is a reformulation of that condition in terms of the second derivative test for a local maximum.

If all the $\{B_t\}$ are inner product spaces we can carry the analogy further. Suppose each $\{B_t\}$ is an inner product space with inner product $\langle v, w\rangle_t = v^t\Omega(t)w$ for column vectors v, w. Let t_0 be given. We will say that a collection of affine sections $v_1(t),\dots,v_n(t)$ give normal coordinates at t_0 if the analogs of (4.5) and (4.6) hold, namely

$$\langle v_i(t_0), v_j(t_0)\rangle_{t_0} = \delta_{ij}, \qquad i, j = 1,\dots,n, \tag{5.20}$$

$$\frac{d}{dt}\langle v_i(t), v_j(t)\rangle_t\big|_{t=t_0} = 0, \qquad k, j = 1,\dots,n. \tag{5.21}$$

We will be interested in the matrix K with components

$$K_{ij} = -\frac{d^2}{dt^2}\langle v_i(t), v_j(t)\rangle_t\big|_{t=t_0}, \qquad i, j = 1,\dots,n. \tag{5.22}$$

The verification that such sections can be chosen is direct. The curvature tensor of Hermitian holomorphic vector bundles can be computed by formulas completely analogous to the previous three (for that and the relation of the K to Finsler bundle curvature see **[8]** and **[9]**).

For t near t_0 write $t = t_0 + \varepsilon$. Let a dot denote t derivatives evaluated at t_0. Let $V(t)$ be the matrix composed of the column vectors $v_1(t),\dots,v_n(t)$. Let $V_0 = V(t_0)$, $V_1 = \dot V(t_0)$, $\Omega_0 = \Omega(t_0)$, $\Omega_1 = \dot\Omega(t_0)$ and $\Omega_2 = \ddot\Omega(t_0)$. Then $V(t) = V_0 + \varepsilon V_1$ (this is exact because $V(t)$ is affine) and the previous three equations become

$$(V_0 + \varepsilon V_1)^t(\Omega_0 + \varepsilon\Omega_1 + \tfrac{1}{2}\varepsilon^2\Omega_2)(V_0 + \varepsilon V_1) = I - \tfrac{1}{2}\varepsilon^2 K + O(\varepsilon^3). \tag{5.23}$$

Let C_0 and C_1 be two column vectors. Hence $F(t) = C_0 + \varepsilon C_1$ is the general affine section of $\{B_t\}$. Write

$$C_0 + \varepsilon C_1 = (V_0 + \varepsilon V_1)(V_0 + \varepsilon V_1)^{-1}(C_0 + \varepsilon C_1)$$

for ε near 0. Thus, using (5.23) near $\varepsilon = 0$,

$$\|C_0 + \varepsilon C_1\|^2_{B_{t_0+\varepsilon}} = (C_0 + \varepsilon C_1)^t(C_0 + \varepsilon C_1) - \tfrac{1}{2}\varepsilon^2 C_0^t(V_0^{-1})^t K V_0^{-1} C_0 + O(\varepsilon^3).$$

Thus, in order to have (5.18) for this section $F(t)$ we must have $C_0^t C_1 = 0$. If this is satisfied then the second derivative is

$$\frac{d^2}{dt^2}\|F(t)\|^2_{B_t}\big|_{t=t_0} = 2C_1^t C_1 - C_0^t(V_0^{-1})^t K V_0^{-1} C_0.$$

However $C_1^t C_1 \geq 0$ for any vector. Thus $K(F(t_0), t_0, F(t))$ will be negative for all $F(t)$ which satisfy (5.18) exactly if the matrix K, defined by (5.22) or, equivalently, (5.23), is negative definite. That is, we have the following analog of Theorem 4.4.

THEOREM 5.4. *If the family of real inner product spaces* $\{B_t\}$ *is a subinterpolation family then* K *is negative semidefinite. If* K *is negative definite then* $\{B_t\}$ *is a subinterpolation family.*

Finally, if Ω_0, Ω_1, and Ω_2 of (5.23) can be simultaneously diagonalized then we can obtain a simple formula for K. If we set $V_0 = \Omega_0^{-1/2}$ and $V_1 = -\frac{1}{2}\Omega_1\Omega_0^{-3/2}$ then the zeroth and first order terms in (5.23) match and we obtain (noting that everything commutes)

$$K = \Omega_2\Omega_0^{-1} + \tfrac{3}{2}\Omega_1^2\Omega_0^{-2}.$$

In the particular case that $\Omega(t) = W(t)^{-1}$ for some $W(t)$ we write $W_0 = W(t_0)$, $W_1 = \dot{W}(t_0)$, $W_2 = \ddot{W}(t_0)$. This becomes

$$K = W_0^{-1}W_2 - W_0^{-2}W_1^2. \tag{5.24}$$

Consider now the case of the family $A_{t,2}$ defined by (5.8). Suppose the norms on A_0 and A_1 are inner product spaces with norms specified by $\Omega(0)$ and $\Omega(1)$, and $\Omega(0)$ and $\Omega(1)$ commute. The spaces $A_{t,2}$ are then a subinterpolation family (by the comment after Lemma (5.1)) of inner product spaces (this by, for example, (5.9)). Also by (5.9) the matrix giving the norm of $A_{t,2}$ will be $\Omega(t) = ((1-t)\Omega(0)^{-1} + t\Omega(1)^{-1})^{-1}$. When we use (5.24) we have $W(t) = (1-t)\Omega(0)^{-1} + t\Omega(1)^{-1}$, $W_1 = \Omega(1)^{-1} - \Omega(0)^{-1}$, $W_2 = 0$. Thus, in this case,

$$K = -\left(\Omega(1)^{-1} - \Omega(0)^{-1}\right)^2\left((1-t_0)\Omega(0)^{-1} + t_0\Omega(1)^{-1}\right)^{-2}$$

and the negative definite nature of K is apparent.

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