

EQUIVALENT DEFINITIONS OF POSITIVE DEFINITENESS

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Equivalent conditions for linear functionals, on a commutative algebra \mathcal{A} with involution, to be positive are proved by elementary methods devoid of disintegration theorems. The theory is applied to present straightforward verifications of, heretofore subtle, equivalences of quadratic form and finite difference definitions of positive definiteness of functions on semigroups.

1. Introduction. Let \mathcal{A} be a commutative algebra with identity 1 and involution*. A subset τ of \mathcal{A} such that

- (i) $x^* = x$ for all $x \in \tau$,
- (ii) $1 - x$ is a positive linear combination of products of members of τ for each $x \in \tau$ ($1 - x \in \text{Alg span}^+ \tau$ for each $x \in \tau$), and
- (iii) each $x \in \mathcal{A}$ is a linear combination of products of members of τ ($\mathcal{A} = \text{Alg span } \tau$)

will be called *admissible*. A linear functional f on \mathcal{A} such that $f(x) \geq 0$ for all $x \in \text{Alg span}^+ \tau$ where τ is admissible will be called τ -*positive*. Following standard conventions, f is called *positive* if $f(xx^*) \geq 0$ for all $x \in \mathcal{A}$. If f is positive then we set $\|x\|_f^2 = \sup_y (f(xx^*yy^*)/f(yy^*))$ ($0/0 = 0$) and say f is *bounded* whenever $\|x\|_f < \infty$ for each $x \in \mathcal{A}$. It is shown in Maserick [8, Th. 1.1] that every τ -positive linear functional f admits a disintegration

$$(1.0.1) \quad f(x) = \int_{\Gamma} \rho(x) d\mu_f(\rho)$$

where μ_f is a non-negative regular Borel measure on a compact (weak*-topology) subset Γ of the τ -positive multiplicative linear functionals. From this it follows that every τ -positive linear functional is bounded and positive. The point of this paper is to give an elementary and almost completely algebraic proof of this latter fact, which does not appeal to the disintegration theorem cited above. By way of motivation, consider the following classical examples from analysis.

EXAMPLE 1.1. *The "little" moment problem.* Let \mathcal{A} be the real algebra of polynomials in one variable with $p^* = p$ for all $p \in \mathcal{A}$ and let τ consist of the two polynomials $p_1(t) = t$ and $p_2(t) = 1 - t$. The linear functionals

L on \mathcal{Q} can be identified with the sequences $a = \{a_n\}_n$ via the moment relation $L(\sum_n \alpha_n t^n) = \sum_n \alpha_n a_n$. Since $\text{Alg span}^+ \tau$ is the positive linear span of the polynomials of the form $t^m(1-t)^n$, a positive linear functional identifies with a sequence if and only if all finite differences of the sequences are non-negative, i.e. $\{a_n\}_n$ is a completely monotonic sequence cf. [12, p. 108]; the τ -positive multiplicative functionals being identified with the sequences $\{t^n\}_n$ for $0 \leq t \leq 1$ ($0^0 = 1$) cf. [8, p. 147]. Under this identification, (1.0.1) reduces to

$$(1.0.2) \quad a_n = \int_0^1 t^n d\mu_a(t),$$

the “little” moment problem of Hausdorff. On the other hand, a sequence becomes positive under this identification when and only when the moments of the squares of polynomials are non-negative, i.e. when all quadratic forms of the type $\sum_{i,j} c_i c_j a_{i+j}$ ($c_i \in \mathbf{R}$) are non-negative. It is well known and follows from (1.0.2) that a sequence $\{a_n\}_{n=0}$ is completely monotonic if and only if both $\{a_n\}_{n=0}$ and $\{a_n - a_{n+1}\}_{n=0}$ are positive definite. In [1, p. 325], Atzmon asks for a direct proof (i.e. a proof that does not use (1.0.2)) of this fact. The theory of reproducing kernels can be used to supply the direct proof that both positive definite conditions together imply complete monotonicity, while the main theorem of this paper yields the requested proof of the converse. This is done in Application (4.2). More generality is obtained by replacing \mathcal{Q} above by the algebra of shift operators on a commutative semigroup S . Atzmon’s question is then answered by specializing S to the additive semigroup of non-negative integers.

EXAMPLE 1.2. *The trigonometric moment problem of Herglotz.* Let \mathcal{Q} be the complex algebra of trigonometric polynomials $\sum_n \alpha_n e^{in\theta}$ and let

$$\tau = \left\{ \frac{1}{2} (1 + (\sigma/2)e^{i\theta} + (\bar{\sigma}/2)e^{-i\theta}) \mid \sigma^4 = 1 \right\}.$$

Similar to the previous example, the linear functionals L can be identified with functions f on the group I of integers via the moment relation $L(\sum_n \alpha_n e^{in\theta}) = \sum_n \alpha_n f(n)$; the positive multiplicative functionals being of the form $n \rightarrow e^{in\theta}$ ($n \in I$, $\theta \in [0, 2\pi)$). A characterization, in terms of finite differences, of those functions f on I which correspond to the τ -positive linear functionals on \mathcal{Q} is given in [7, p. 141] and the integral representation (1.0.1) reduces to the form of the trigonometric moment problem, namely

$$(1.0.3) \quad f(n) = \int_{[0, 2\pi)} e^{in\theta} d\mu_f(\theta).$$

But Hergoltz [5, p. 773] has characterized those functions f satisfying (1.0.3) as the positive definite functions on I . Thus a function f on I is positive definite if and only if it satisfies the appropriate finite difference condition. In Application (4.3) we give an alternate proof of this equivalence, in considerably more generality, which does not depend on (1.0.3).

EXAMPLE 1.3. *The non-negative, non-decreasing real functions on $(0, 1]$.* Let τ be the characteristic functions $\{1_{(a,b)} \mid 0 < a < b \leq 1\}$ and \mathcal{Q} the real algebra, with trivial involution, spanned by τ . The equation $L(1_{(0,a]}) = f(a)$ identifies the linear functionals L on \mathcal{Q} with the real functions f on $(0, 1]$; the positive linear functionals being identified with the non-negative, non-decreasing functions. The main theorem (Th. 3.1) then implies that a function f on $(0, 1]$ is non-negative and non-decreasing if and only if the matrix $(f(\min[x_i, x_j]))_{n \times n}$ is positive definite for every n .

2. τ -positiveness of bounded positive functionals. For each $x \in \mathcal{Q}$, define the shift operator E_x on the linear functionals f on \mathcal{Q} by $E_x f(y) = f(xy)$ ($y \in \mathcal{Q}$). Let f be a fixed bounded positive linear functional and set $H_f = \{E_x f \mid x \in \mathcal{Q}\}$. Parts (i) and (iii) of §1 imply $f(x^*) = \bar{f}(x)$. Thus H_f can be made into a pre-Hilbert space in a manner similar to the Gelfand-Segal-Naimark (GNS) construction [9, p. 215] by defining $\langle E_x f, E_y f \rangle = f(xy^*)$. The Cauchy-Schwarz inequality shows

$$|E_x f(y)|^2 = |\langle E_x f, E_{y^*} f \rangle|^2 \leq \langle E_x f, E_x f \rangle \cdot \langle E_{y^*} f, E_{y^*} f \rangle$$

or

$$(2.0.1) \quad |E_x f(y)| \leq \|E_{y^*} f\| \cdot \|E_x f\|.$$

Thus if $\|E_x f\| = 0$ then $E_x f = 0$, so $\langle \cdot, \cdot \rangle$ is a non-degenerate inner product. Substituting $(E_{x_n} - E_{x_m})$ for E_x in (2.0.1) shows that every Cauchy sequence in H_f is weak*-Cauchy and thus the completion \bar{H}_f is a subset of the linear functionals on \mathcal{Q} . For each $x \in \mathcal{Q}$, E_x is then a bounded linear operator on the pre-Hilbert space H_f (i.e. $E_x \in B(H_f)$) with operator norm

$$(2.0.2) \quad \|E_x\| = |x|_f.$$

Since Cauchy sequences in H_f are also weak*-Cauchy, it follows that the extension of E_x to the completion \bar{H}_f of H_f is again the translation operator E_x . Thus the map $x \rightarrow E_x$ of \mathcal{Q} onto $B(\bar{H}_f)$ is a (*-preserving) representation of \mathcal{Q} such that E_1 is the identity operator I . The following proposition summarizes the pertinent facts.

PROPOSITION 2.1. *Let f be a bounded positive linear functional on \mathcal{Q} . Then:*

(i) *The operator $E_x \in B(\overline{H}_f)$ is positive if and only if $E_x f$ is a positive linear functional on \mathcal{Q} .*

(ii) *If τ is admissible and $E_x f$ is positive for each $x \in \tau$ then f is τ -positive.*

(iii) *If α_x is any positive number such that $\alpha_x \geq |x|_f$ then*

$$\tau = \left\{ \frac{1}{2}(I + (\sigma/2\alpha_x)x + (\bar{\sigma}/2\alpha_x)x^*) \mid x \in \mathcal{Q}, \sigma^4 = 1 \right\}$$

is admissible and f is τ -positive.

Proof. Assertion (i) is obvious from the foregoing remarks. For (ii), let $x_1, \dots, x_k \in \tau$. If f satisfies the hypothesis of (ii) then E_{x_j} is a positive operator on \overline{H}_f for each $j = 1, \dots, k$ by (i). Since products of commuting bounded positive operators are again positive (cf. [6, p. 280] for a proof of this latter fact which does not depend on the spectral theorem), one has $f(\prod_j x_j) = \langle E_{\prod_j x_j} f, f \rangle \geq 0$ or f is τ -positive. Admissibility of τ in assertion (iii) is easily verified. But if A is any non-zero bounded operator on a Hilbert space and $\alpha \geq \|A\|$ the $I + (\sigma/2\alpha)A + (\bar{\sigma}/2\alpha)A^*$ is positive. Thus (2.0.2) implies the operator $I + (\sigma/2\alpha_x)E_x + (\bar{\sigma}/2\alpha_x)E_x^*$ is positive on \overline{H}_f so that (iii) follows from (ii). \square

3. Positiveness of bounded positive functionals. We now give the elementary proof promised in the introduction. Namely,

THEOREM 3.1. *If f is τ -positive for an admissible τ , then f is positive and bounded, and $E_x f$ is positive for each $x \in \tau$.*

Proof. (i) The central idea of the proof revolves around the theory of Bernstein polynomials. If $p(t_1, t_2)$ is any complex polynomial in two variables, $x_1, x_2 \in \mathcal{Q}$ and n is any positive integer, we define

$$(3.1.1) \quad B_n(p(x_1, x_2)) \\ = \sum_{i,j} p\left(\frac{i}{n}, \frac{j}{n}\right) \binom{n}{i} \binom{n}{j} x_1^i x_2^j (1-x_1)^{n-i} (1-x_2)^{n-j}.$$

Letting $p(x_1, x_2) = (x_1 - x_2)^2$, we get

$$B_n(x_1 - x_2)^2 = B_n(x_1^2 - 2x_1x_2 + x_2^2) \\ = (B_n(x_1^2))(B_n(1)) - 2(B_n(x_1))(B_n(x_2)) + (B_n(1))(B_n(x_2^2)) \\ = B_n(x_1^2) - 2x_1x_2 + B_n(x_2^2).$$

But, $B_n(x^2) = x/n + ((n - 1)/n)x^2$ (cf. Davis [4, p. 109]) so that

$$(3.1.2) \quad B_n(x_1 - x_2)^2 = \left[\frac{x_1}{n} + \left(\frac{n-1}{n} \right) x_1^2 \right] - [2x_1x_2] + \left[\frac{x_2}{n} + \left(\frac{n-1}{n} \right) x_2^2 \right].$$

If we impose any locally convex topology on \mathcal{Q} , such as the weakest in which all linear functionals are continuous, then we have

$$\begin{aligned} \lim_n E_x f(B_n(x_1 - x_2)^2) &= \lim_n f(xB_n(x_1 - x_2)^2) \\ &= f(x(x_1 - x_2)^2) = E_x f((x_1 - x_2)^2). \end{aligned}$$

But if $x, x_1, x_2 \in \text{Alg span}^+ \tau$ then [8, Prop. 2.1.1] implies both $1 - x_1/M, 1 - x_2/M \in \text{Alg span}^+ \tau$ for sufficiently large $M > 0$. Since the coefficients of $B_n(x_1/M - x_2/M)^2$ are non-negative, linearity of f implies $E_x f((x_1/M - x_2/M)^2) \geq 0$ so that

$$(3.1.3) \quad E_x f((x_1 - x_2)^2) \geq 0.$$

Let $y \in \mathcal{Q}$. By condition (iii) of the definition of τ , $y = (x_1 - x_2) + (x_3 - x_4)i$ where $x_j \in \text{Alg span}^+ \tau$ for each $j = 1, 2, 3$ or 4 . Thus $yy^* = (x_1 - x_2)^2 + (x_3 - x_4)^2$ so that (3.1.3) and linearity of $E_x f$ imply

$$(3.1.4) \quad E_x f(yy^*) \geq 0.$$

Hence $E_x f$ is positive for each $x \in \text{Alg span}^+ \tau$. But if $x \in \tau$ then $1 - x \in \text{Alg span}^+ \tau$, and since $E_{(1-x)} = I - E_x$ we have $f = E_x f + (I - E_x)f$, so that f is positive. This establishes the first and last assertions.

(ii) To see that f is bounded, let z be an arbitrary member of τ and replace x in (3.1.3) by $1 - z^2$. Clearly $1 - z^2 = (1 - z)(1 + z) \in \text{Alg span}^+ \tau$ so that (3.1.4) and Proposition 2.1 (i) imply $E_{1-z^2} = I - E_{z^2}$ is a positive operator on H_f . It follows that E_z is bounded in norm by 1 for all $z \in \tau$. But since τ generates \mathcal{Q} , E_x is bounded for all $x \in \mathcal{Q}$. Therefore f is bounded. □

4. Applications to semigroups. Let S be a commutative semigroup with identity e and involution*. To apply the theory presented in the foregoing sections, define the shift operator E_s on the space $\mathcal{F}(S)$ of complex-valued functions φ on S by $(E_s \varphi)(t) = \varphi(st)$ ($s, t \in S$) and set $\mathcal{Q} = \text{span}\{E_s | s \in S\}$. Since $E_{st} = E_s E_t$, \mathcal{Q} is an algebra with identity $I = E_e$ and involution $(\sum_i c_i E_{s_i})^* = \sum_i \bar{c}_i E_{s_i^*}$. The linear functionals f on \mathcal{Q}

can be biuniquely identified with the functions $\varphi \in \mathfrak{F}(S)$ by $f(\sum c_i E_{s_i}) = \sum c_i \varphi(s_i)$; the positive functionals being identified with those φ satisfying

$$(4.0.1) \quad \sum_{i,j} c_i \bar{c}_j \varphi(s_i s_j^*) \geq 0$$

for all finite subsets $\{c_i\}_i \subset \mathbf{C}$ and $\{s_i\}_i \subset S$. Consistent with standard terminology when S is a group, we call those functions which satisfy (4.0.1) for an arbitrary semigroup S , *positive definite*. In this generality, bounded positive definite functions were first introduced and studied in [5] under the name “*-definite”. The GNS-construction used in §2 carries over to the set $H_\varphi = \{T\varphi \mid T \in \mathcal{A}\}$ giving a pre-Hilbert space with $\langle T\varphi, R\varphi \rangle = TR^*\varphi(e)$. Moreover the operator E_T on H_φ corresponds to the operator T on H_φ defined by $T(T_1\varphi) = TT_1(\varphi)$. In particular E_s is an operator on H_φ which corresponds to the operator $E_{(E_s)}$ on H_f . This construction on \mathcal{A} reduces to the theory of reproducing kernels on $S \times S$ relative to the positive definite kernel $k(s, t) = \varphi(st^*)$. The following preliminary proposition is needed.

PROPOSITION 4.1. *If $\varphi \in \mathfrak{F}(S)$ is positive definite in the sense of (4.0.1) and bounded in the usual sense, then E_s , as an operator on H_φ , is bounded in norm by 1.*

Proof. (i) If ψ is any positive definite function on S then the Cauchy-Schwarz inequality yields

$$|\psi(st^*)|^2 = |\langle E_s \psi, E_t \psi \rangle|^2 \leq \|E_s \psi\|^2 \cdot \|E_t \psi\|^2 = \psi(ss^*)\psi(tt^*).$$

Thus

$$(4.1.1) \quad |\psi(st^*)|^2 \leq \psi(ss^*)\psi(tt^*)$$

so that setting $t = e$ gives

$$(4.1.2) \quad |\psi(s)|^2 \leq \psi(e)\psi(ss^*).$$

(ii) If ψ is bounded and positive definite then $|\psi(s)| \leq \psi(e)$ for all $s \in S$. Indeed, we may assume $\psi(e) = 1$ without loss of generality. For $s \in S$ and $a = ss^*$, we find by p successive applications of (4.1.2) that $|\psi(s)|^{2^p} \leq \psi(a^{2^{p-1}})$. But if ψ is bounded by K , then $|\psi(s)| \leq K^{2^{-p}} \rightarrow 1 = \psi(e)$ (cf. Szafraniec [10, 11] or Berg-Maserick [2]).

(iii) Let $T \in \mathcal{Q}$. Then $TT^*\varphi$ is positive definite since $E_{TT^*}f$ is a positive linear functional. Thus

$$\begin{aligned} TT^*\varphi(ss^*) &= \langle E_{ss^*}\varphi, TT^*\varphi \rangle \leq \|E_{ss^*}\varphi\| \cdot \|TT^*\varphi\| \\ &= \sqrt{\varphi((ss^*)^2)} \|TT^*\varphi\| \leq \sqrt{\varphi(e)} \|TT^*\varphi\|. \end{aligned}$$

Setting $\psi = TT^*\varphi$ in (4.1.2) shows $TT^*\varphi$ is bounded, from which (ii) implies $|TT^*\varphi(s)| \leq TT^*\varphi(e)$. Thus

$$|\langle E_s T\varphi, T\varphi \rangle| = |TT^*\varphi(s)| \leq TT^*\varphi(e) = \langle T\varphi, T\varphi \rangle$$

so $\|E_s\| \leq 1$. □

The applications depend on appropriate selections of τ which in turn may depend on the choice of S . In the interest of brevity we have selected two representative examples below for illustration. However, virtually all of the applications appearing in Berg-Maserick [2, §4] could be recast in this more elementary point of view, in lieu of the disintegration theorems used therein. Throughout G will denote a generator set for S in the sense that every $s \in S$ is a finite product of the form $\prod_i s_i$, where either $s_i \in G$ or $s_i^* \in G$ for each i .

(A) *Completely monotonic functions.* We assume $s = s^*$ for all $s \in S$ and recall that finite differences of a function $\varphi \in \mathcal{F}(S)$ are defined inductively by

$$\begin{aligned} (4.1.3) \quad \Delta_{k+1}\varphi(s_0; s_1, \dots, s_{k+1}) \\ = \Delta_k\varphi(s_0; s_1, \dots, s_k) - \Delta_k\varphi(s_0s_{k+1}; s_1, \dots, s_k) \quad (k = 1, 2, \dots), \end{aligned}$$

where $s_0, s_1, \dots, s_{k+1} \in S$. A non-negative $\varphi \in \mathcal{F}(S)$ is said to be *completely monotonic* if all of the differences (4.1.3) are non-negative. It is easy to see that these differences can be redefined in terms of the shift operator by

$$\Delta_k\varphi(s_0; s_1, \dots, s_k) = E_{s_0} \prod_i (I - E_{s_i})\varphi(e).$$

It follows that φ is completely monotonic if and only if the linear functional f corresponding to φ is τ -positive with respect to $\tau = \{E_s, I - E_s | s \in S\}$. However, we can reduce the size of τ to

$$(4.1.4) \quad \tau_G = \{E_s, I - E_s | s \in G\}$$

without adding more τ -positive functionals because it follows (algebraically, cf. Maserick [8, Prop. 2.1.1]) that $\{E_s, I - E_s\} \subset \text{Alg Span}^+ \tau_G$ for each $s \in S$. Hence $\text{Alg Span}^+ \tau_G = \text{Alg Span}^+ \tau$. Thus $\varphi \in \overline{\mathcal{F}}(S)$ is completely monotonic if and only if $\prod_{i=1}^k (I - E_{s_i})(s_0) \geq 0$ for all $s_0, s_1, \dots, s_k \in G$ where G is any generator set whatsoever.

The proof of the equivalence of (i) and (ii) below answers the question of Atzmon, mentioned in the introduction, by taking S to be the additive semigroup of non-negative integers.

Application 4.2. Let $\varphi \in \overline{\mathcal{F}}(S)$. The following are equivalent.

- (i) φ is completely monotonic,
- (ii) $E_s \varphi$ and $(I - E_s) \varphi$ are positive definite for each $s \in G$, and
- (iii) φ is bounded and $E_s \varphi$ is positive definite for each $s \in G \cup \{e\}$.

Proof. The implication (i) implies (ii) follows from the foregoing remarks and Theorem 3.1. Condition (ii) implies boundedness of φ on G . This, in turn, implies boundedness of φ on S since $I - E_s$ is in the positive linear span of $\{I - E_t \mid t \in G\}$ for all $s \in S$ as mentioned above. Thus (ii) implies (iii). Assuming (iii), Proposition (4.1) implies $\|E_s\| \leq 1$ for all $s \in G$ and since the operator E_s is positive by (i) of Proposition 2.1, it follows that $I - E_s$ is also positive. Commutativity implies all finite products from the set $\{E_s, I - E_s \mid s \in G\}$ are also positive operators so that

$$\prod_i (I - E_{s_i}) \varphi(s_0) = \left\langle E_{s_0} \prod_i (I - E_{s_i}) \varphi, \varphi \right\rangle \geq 0 \quad \text{for } s_0, s_1, \dots \in G$$

and (i) follows from (iii). □

REMARK. The proof of implications (i) \Rightarrow (ii) \Rightarrow (iii), without appealing to disintegration theory, is new to the best of our knowledge. We do not claim novelty here for the proof that (iii) implies (i); cf. [5, Cor. 4.4].

(B) *Trigonometric moment problem.* We now allow a nontrivial involution on S .

Application 4.3. Let φ be a bounded function on S , G a generator set of S and $\tau_G = \{\frac{1}{2}(I + (\sigma/2)E_s + (\bar{\sigma}/2)E_{s^*}) \mid \sigma^4 = 1, s \in G\}$. The following are equivalent:

- (i) φ is τ_G -positive.
- (ii) φ is positive definite.

Proof. Since τ_G is admissible, Theorem 3.1 shows (i) implies (ii). Assuming (ii), Proposition (4.1) implies $\|E_s\| \leq 1$ for all $s \in S$, since φ is bounded. Standard operator theory implies $(I + (\sigma/2)E_s + (\bar{\sigma}/2)E_{s^*})$ is positive for all 4th roots of unity σ and $s \in S$. Using commutativity as before, one finds

$$\prod_{j=1}^k T_j \varphi(e) = \left\langle \prod_{j=1}^k T_j \varphi, \varphi \right\rangle \geq 0$$

whenever $T_j \in \tau_G$ so that (i) follows from (ii).

REMARKS. (a) It now follows that the definition of τ_G -positivity is independent of the choice of G for bounded φ . This is not the case, however, if φ is unbounded [7, p. 142].

(b) If a positive definite function φ is bounded on $\{ss^* | s \in S\}$ then (4.1.2) shows φ to be bounded on S itself. If, for example, (ss^*) is idempotent for each $s \in S$, then a second application of (4.1.2) shows that every positive definite φ is bounded. Since without any boundedness restriction τ_G -positivity implies positive definiteness, boundedness can be waived for this rather general case. This idempotency occurs when s is a group with inversion as involution as in Example (1.2), or when the operation on S itself is idempotent such as Example (1.3) ($st = \min\{s, t\}$) and S has the trivial involution or more generally when S is an inverse semigroup with $s^* = s^{-1}$ (cf. Clifford and Preston [3]).

(c) If φ is τ_S -positive then φ is bounded on $\{ss^* | s \in S\}$ so that boundedness is not needed to prove that (i) implies (ii) but is needed to show the converse [7, p. 142].

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