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SUBSETS OF ${}^{\omega}\omega$ AND GENERALIZED METRIC SPACES

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In this note we use subsets of ${}^{\omega}\omega$ and certain set-theoretic statements about such sets to solve, up to consistency, several questions concerning generalized metric spaces. These results are improvements of some of Michael's work using the continuum hypothesis (hereafter CH) and some earlier work by the authors using CH and $P(c)$.

1. Introduction. As usual, ${}^{\omega}\omega$ denotes the set of all functions from the natural numbers (ω) into the natural numbers. An ordinal is equal to the set of all smaller ordinals, a cardinal is an initial ordinal, and $c = 2^{\omega}$.

DEFINITION 1.1. If $f, g \in {}^{\omega}\omega$, we write
$f <_* g$ iff there is $k \in \omega$ such that $f(n) < g(n)$ for all $n \geq k$, and
$f =_* g$ iff there is $k \in \omega$ such that $f(n) = g(n)$ for all $n \geq k$.

DEFINITION 1.2. Suppose κ is a cardinal. A set $S \subseteq {}^{\omega}\omega$ is called a *κ-scale* iff $S = \{g_\alpha\colon \alpha < \kappa\}$ where $g_\alpha <_* g_\beta$ whenever $\alpha < \beta < \kappa$ and if $f \in {}^{\omega}\omega$, then there is $\alpha < \kappa$ such that $f <_* g_\alpha$.

BF(κ) is the statement: "If $F \subseteq {}^{\omega}\omega$ and $|F| < \kappa$, then there is $g \in {}^{\omega}\omega$ such that $f <_* g$ for all $f \in F$."

We shall be mainly interested in using as additional axioms with ZFC the statements "There exists an ω_1-scale" and BF(c). We state without proof the following theorems regarding the consistency, independence, and strengths of these axioms.

THEOREM 1.3.
(a) **[Ha]** CH *implies that there exists an ω_1-scale.*
(b) BF(c) *implies there exists a c-scale.*
(c) **[He]** *It is consistent that there exists an ω_1-scale and* CH *is false* (*due to R. M. Solovay*).
(d) **[Bo]**, **[Ro$_2$]** CH $\Rightarrow$ MA $\Rightarrow P(c) \Rightarrow$ BF(c).
(e) **[So]** BF(c) *and not $P(c)$ is consistent.*
(f) **[He]** *There exists a κ-scale implies* BF(cfκ).

REMARK. In the notation used by Vaughan and van Douwen [Va], the statement BF(κ) is equivalent to "$b \geq \kappa$". In this notation b is the smallest cardinal κ such that there exists a family $F \subseteq {}^{\omega}\omega$ with $|F| = \kappa$ and there is no $g \in {}^{\omega}\omega$ with $f <_* g$ for all $f \in F$.

2. Scaling old problems. In this section we use the existence of ω_1-scales as an alternative to CH in the improvement of existing results on several old problems. The construction techniques which we shall be using are based upon the following lemmas. Analogous results using CH are Lemmas 3.1 and 3.5 of [$\mathbf{M_2}$]. It is pointed out in [$\mathbf{M_2}$] that the statements given there are actually equivalent to CH, and it is asked if CH is essential to the existence of examples of the type we shall describe (Question 7.1 of [$\mathbf{M_2}$]). Thus our results may be considered as an answer to this question.

LEMMA 2.1. (*Assume there exists an ω_1-scale.*) *Let $\{\tau_\alpha: \alpha < \omega_1\}$ be a family of first countable topologies on a set X and let D be a countable subset of X such that D is not the intersection of any countable subcollection from $\bigcup_{\alpha<\omega_1} \tau_\alpha$. There exists an uncountable set Z such that $D \subseteq Z \subseteq X$ and whenever $D \subseteq W \subseteq X$ and $W \in \bigcup_{\alpha<\omega_1} \tau_\alpha$, then $|Z \setminus W| \leq \omega$. (Hence Z is Lindelöf with respect to every τ_α.)*

Proof. Let $S = \{f_\alpha: \alpha < \omega_1\}$ denote an ω_1-scale. If $D = \{z_n: n \in \omega\}$ and $\beta < \omega_1$, let $\{U(z_n, k, \beta): k \in \omega\}$ be a decreasing open base at z_n with respect to τ_β. Now for any $\alpha < \omega_1$ and $h \in {}^{\omega}\omega$, let $W(h, \beta) = \bigcup_{n\in\omega} U(z_n, h(n), \beta)$ and $G_{\alpha\beta} = \bigcap\{W(h, \beta): h =_* f_\alpha\}$. Note that $D \subseteq G_{\alpha\beta}$, $G_{\alpha\beta}$ is a G_δ-set with respect to τ_β, and $G_{\alpha\beta} \supseteq G_{\gamma\beta}$ if $\alpha < \gamma < \omega_1$. For each γ, the set $\bigcap_{\beta\leq\gamma} G_{\gamma\beta}$ is uncountable; so it is possible to choose a set $\{x_\gamma: \gamma < \omega_1\}$ such that $x_\gamma \in \bigcap_{\beta\leq\gamma} G_{\gamma\beta} \setminus D$ and $x_\gamma \neq x_\theta$ if $\gamma \neq \theta$. Let $Z = \{x_\gamma: \gamma < \omega_1\} \cup D$. To see that Z satisfies the desired property, suppose there is $\theta < \omega_1$ and $W \in \tau_\theta$ with $D \subseteq W$. Choose $h \in {}^{\omega}\omega$ such that $W(h, \theta) \subseteq W$ and find $f_\alpha \in S$ such that $h <_* f_\alpha$ and $\theta \leq \alpha$. It is clear that for any $\gamma \geq \alpha$,

$$\bigcap_{\beta\leq\gamma} G_{\gamma\beta} \subseteq \bigcap_{\beta\leq\alpha} G_{\alpha\beta} \subseteq G_{\alpha\theta} \subseteq W(h, \theta) \subseteq W,$$

so that $\{x_\gamma: \alpha \leq \gamma < \omega_1\} \subseteq W$. Hence $|Z \setminus W| \leq \omega$.

REMARK. The hypothesis of the above lemma is somewhat stronger than necessary. Examination of the proof indicates that the topologies τ_β need not be first countable, but only that the points of D have countable character with respect to τ_β for each $\beta < \omega_1$.

For the uses made in the sequel, we find it convenient to have the following weaker form of Lemma 2.1 when there is only one topology given on the set X.

LEMMA 2.2. (*Assume there exists an ω_1-scale.*) *If X is a first countable T_1-space and D is a countable, non-G_δ-subset of X, then there exists an uncountable set Z such that $D \subseteq Z \subseteq X$ and whenever W is an open subset of X containing D, then $|Z \setminus W| \leq \omega$.*

EXAMPLE 2.3. (Assume there exists an ω_1-scale.) There is a regular Lindelöf space Y whose product with the irrationals is not normal.

Proof. Let Y be the set Z whose existence is given by 2.2 when $D = Q$ and $X = \mathbf{R}$, and give Y the subspace topology from the Michael line [**M**$_1$]. Michael constructed this space using CH (and the analogue of 2.2) in [**M**$_2$] and [**M**$_1$], and the verification of the properties claimed is given in [**M**$_2$].

To the authors' knowledge, it is unknown whether an example of this type can be constructed using no axioms beyond ZFC; see Problem 5.6 [**Pr**].

The remaining examples in this section are built upon H, Heath's "bow-tie space" [**Hh**]. The points of H are the points of the plane. For a point $x = (x_1, x_2)$, we let $B(x, n)$ be the set given by

$$B(x,n) = \left\{ y \in \mathbf{R} \times \mathbf{R} \colon \|x - y\| < \frac{1}{n} \text{ and } \left| \frac{y_2 - x_2}{y_1 - x_1} \right| < \tan \frac{1}{2n} \right\}.$$

Geometrically, $B(x, n)$ is a horizontal bow-tie centered at x. As n increases, both the radius and the vertex angle decrease.

It is known that the topology generated by using $\{B(x, n): n \in N, x \in H\}$ is a semimetrizable topology on H. We note that it is known that H is a Baire space, and that Lindelöf semimetric spaces are hereditarily Lindelöf. We define a "bow-tie" B to have orientation α if B can be obtained from $B(x, n)$, for some $x \in H$, $n \in N$, by a counter-clockwise rotation through the angle α. In this case, we write $B = B(x, n, \alpha)$. In particular, $B(x, n) = B(x, n, 0)$ for each $x \in H$, $n \in N$.

EXAMPLE 2.4. (Assume there exists an ω_1-scale.) There exists a Lindelöf semimetrizable space having no countable network.

Proof. Let τ_1 be the topology on H generated by $\{B(x, n, 0): x \in H, n \in N\}$ and τ_2 be the topology on H generated by $\{B(x, n, \pi/2): x \in H, n \in N\}$. Let Z be the set whose existence is given by 2.1 with $X = H$ and $D = Q \times Q$. Now let $Z_1 = (Z, \tau_1)$ and $Z_2 = (Z, \tau_2)$. Then both Z_1 and Z_2 are Lindelöf semimetric spaces. However $\{(x, x): x \in Z\}$ is a closed discrete subset of $Z_1 \times Z_2$. Hence $Z_1 \times Z_2$, and consequently at least one of Z_1 and Z_2, cannot have a countable network.

For our next example we would like to have a set $Z \subseteq H$ which is Lindelöf with respect to all possible orientations. Since 2.1 allows only ω_1 topologies, and we are admitting the possibility that $\omega_1 < c$, we use the following lemma.

LEMMA 2.5. *For $\alpha \in [0, \pi)$, let τ_α be the topology generated on H by $\{B(x, n, \alpha): x \in H, n \in N\}$. If $Z \subseteq H$ is Lindelöf with respect to τ_α for each rational α, then Z is Lindelöf with respect to τ_α for every α.*

Proof. Only the center point poses a problem, but if B is a bow-tie in Z with any orientation, then there is some rational α so that B is open with respect to τ_α. Now since Z is hereditarily Lindelöf with respect to τ_α for each rational α, the result follows.

EXAMPLE 2.6. (Assume there exists an ω_1-scale.) There is a nonmetrizable semimetrizable space which has a locally connected, perfectly normal compactification.

Proof. The desired example is given in Examples 3.2 and 3.3 of [$\mathbf{BD_1}$] using CH. The need for CH is to obtain the set which is called X in [$\mathbf{BD_1}$]. This set can be obtained by using an ω_1-scale from Lemma 2.1 by letting τ_α be the topology generated by $\{B(x, n, \alpha): x \in H, n \in N\}$ for each rational α, $X = H$ (in the notation of 2.1) and $D = Q \times Q$. Then the set Z (in the notation of 2.1) is the desired set. The remainder of the construction now follows from Lemma 2.5 and the work in [$\mathbf{BD_1}$] by first building a first countable compactification of H and then taking a quotient to get the compactification we are seeking.

It is not known if there exist ZFC examples which have the properties of those given above. However, it can be shown that the type of construction used here, as well as in [$\mathbf{M_2}$] and [**Ve**], cannot be done if $\mathrm{BF}(\omega_2)$ is assumed. Rothberger [$\mathbf{Ro_1}$] has shown that under $\mathrm{BF}(\omega_2)$ any subset A of the real numbers of cardinality ω_1 has the property that every countable subset of A is a relative G_δ. In fact, the setting of the real numbers is not needed for this theorem. The following result is well known (to at least van Douwen and some others).

THEOREM 2.7. (*Assume* BF(ω_2).) *If A is a subset of a first countable T_1 space and $|A| \leq \omega_1$, then every countable subset of A is a relative G_δ.*

In particular, one easily sees that the type of set given by 2.1, as is used in 2.3, 2.4 and 2.6, will not exist if BF(ω_2) is assumed.

3. Applications of BF(κ). It is often useful to specify a sequence converging to a given limit point of a set. We now turn our discussion to the problem of producing such sequences in spaces which may not be Fréchet, and show that in some cases one may actually obtain the Fréchet condition from sequential.

The following may be thought of as a topological characterization of BF(κ).

THEOREM 3.1. BF(κ) *if and only if whenever D is a conditionally compact subset of a regular space X, x_n is a limit of a sequence in D for each $n \in \omega$, and $x_n \to x$ where $\psi(x, X) < \kappa$, then x is a limit of a sequence in D.*

Proof. Assume BF(κ). Recall that $\psi(x, X) < \kappa$ implies there is a cardinal $\eta < \kappa$ and an open collection $\{U_\alpha: \alpha < \eta\}$ such that $\{x\} = \bigcap_{\alpha<\eta} U_\alpha$. Since X is regular, we may also assume $\{x\} = \bigcap_{\alpha<\eta} \overline{U}_\alpha$. For each $n \in \omega$, choose a sequence $\langle x_{n,k}: k \in \omega\rangle$ in D such that $x_{n,k} \to x_n$. For each U_α, choose a function $f_\alpha \in {}^{\omega}\omega$ such that whenever $x_n \in U_\alpha$ and $k \geq f_\alpha(n)$, then $x_{n,k} \in U_\alpha$. Apply BF(κ) to obtain $g \in {}^{\omega}\omega$ such that $f_\alpha <_* g$ for each $\alpha < \eta$. Now it is easy to see that the sequence $s = \langle x_{n,g(n)}: n \in \omega\rangle$ is eventually in every U_α. Hence every cluster point of s is in every $\overline{U}_\alpha$. Thus, since D is conditionally compact, x is the unique cluster point of every subsequence of s, and s must converge to x.

Conversely, suppose BF(κ) fails; then $b < \kappa$. Choose $\{f_\alpha: \alpha < b\} \subseteq {}^{\omega}\omega$ such that for each α, f_α is nondecreasing, $f_\alpha <_* f_\beta$ whenever $\alpha < \beta$, and there is no $g \in {}^{\omega}\omega$ such that $f_\alpha <_* g$ for every $\alpha < b$. Initially, we follow the construction of the space X given in Example 3 of **[BvD]**. Let $X = b \cup (\omega \times \omega)$, and topologize X as follows: Points of $\omega \times \omega$ are isolated. If $\alpha \in b\setminus\{0\}$, $\eta < \alpha$, $m \in \omega$, the set $U(\alpha, \eta, m) = (\eta, \alpha] \cup \{(k, n): k \geq m, f_\eta(k) < n \leq f_\alpha(k)\}$ is a basic neighborhood of α. If $\alpha = 0$, $m \in \omega$, the set

$$U(0, m) = \{0\} \cup \{(k, n): k \geq m, n \leq f_0(k)\}$$

is a basic neighborhood of 0. Now for each $n \in \omega$, let $A_n = \{n\} \times \omega$, and choose distinct points $x_n \notin X$. We let $Y = X \cup \{x_n: n \in \omega\}$ with X open in Y and basic neighborhoods of x_n will be cofinite subsets of A_n along with x_n. Note that Y is a locally compact T_2 space, and $\omega \times \omega$ is a dense conditionally compact subset of Y (see Lemma 4 of **[BvD]**). Let Z be the one-point compactification of Y, i.e., $Z = Y \cup \{p\}$. Now, since $|Z| = b$, $\psi(p, Z) < \kappa$. It is easy to see that $x_n \to p$, and, since $\omega \times \omega$ is conditionally compact in Y, no sequence in $\omega \times \omega$ can converge to p. This completes the proof.

A similar argument verifies the following result in which the conditionally compact assumption is removed by replacing the pseudocharacter restriction with a character restriction.

THEOREM 3.2. *Assume* BF(κ). *If X is sequential and $\chi(X) < \kappa$, then X is Fréchet.*

Proof. Suppose $A \subseteq X$. We will show that if, for each $n \in \omega$, $\langle x_{n,k}: k \in \omega \rangle$ is a sequence in A converging to a point x_n, and the sequence $\langle x_n: n \in \omega \rangle$ converges to a point x, then there is a sequence in A which converges to x. Since closure is obtained recursively from sequential closures, this suffices for the result.

Choose a local base $\{U_\alpha: \alpha < \theta\}$ at x where $\theta < \kappa$. For each U_α, we may choose a function $f_\alpha \in {}^\omega\omega$ such that, when $x_n \in U_\alpha$, $k \geq f_\alpha(n)$ implies $x_{n,k} \in U_\alpha$. Apply BF(κ) to obtain $g \in {}^\omega\omega$ such that $f_\alpha <_* g$ for every $\alpha < \theta$. Now the sequence $\langle x_{n,g(n)}: n \in \omega \rangle$ is eventually in every U_α and, hence, converges to x.

We now apply 3.1 and 3.2 to obtain improvements of certain results given in **[BD$_2$]**. For definitions of symmetrizable and weakly first countable spaces, see **[Ar$_1$]** or **[BD$_2$]**.

THEOREM 3.3 *Assume* BF(c). *Every regular symmetrizable space with a dense conditionally compact subset is first countable* (*hence, separable* **[St$_2$]**.)

Proof. Suppose X is regular and symmetrizable, and D is a dense conditionally compact subspace of X. By Lemma 3.4 of [**BD**$_2$] and the fact that, in a space with a dense conditionally compact subset, points which are G_δ's have countable character, it suffices to show that any point which is the limit of a sequence of sequential limit points of D must itself be a sequential limit point of D. This follows from 3.1.

THEOREM 3.4. *Assume* BF(c). *If X is an uncountable T_2 compact weakly first countable space, and $|X| \leq c$, then $|X| = c$.*

Proof. Suppose $\omega < |X| < c$. By 3.1, X must be Fréchet, and it is well known that a T_2 Fréchet weakly first countable space is first countable. Now from an old result in [**AU**], every uncountable, first countable, compact T_2 space has cardinality at least c, we have a contradiction.

Malýhin has announced in [**Ma**] that using forcing there are T_2 compact weakly first countable spaces with cardinality strictly between ω and c.

It is an open question whether every regular feebly compact symmetrizable space must be separable. Stephenson [**St**$_2$] has shown that this question has an affirmative answer for spaces which have a dense set of isolated points. We use BF(c) to give an affirmative answer for spaces with a dense set of points of countable character. The crux of the argument is contained in the following lemma.

LEMMA 3.5. *Assume* BF(c). *If X is regular, feebly compact, sequential, $\psi(x, X) < c$ for each $x \in X$, and there is a subset $M \subseteq X$ with each point of M having a countable local base, then every limit point of M is a sequential limit point of M.*

Proof. We show that any point which is a limit of a sequence of sequential limit points of M must be a sequential limit point of M. Since X is sequential, this gives the result. Let $x \in X$ and $\langle x_n : n \in \omega \rangle$ be a sequence in X such that $x_n \to x$ and for each $n \in \omega$ there is a sequence $\langle x_{n,k} : k \in \omega \rangle$ in M with $x_{n,k} \to x_n$. Choose a collection $\{U_\alpha : \alpha < \beta\}$ of open subsets of X such that $\beta < c$ and $\{x\} = \bigcap_{\alpha<\beta} U_\alpha = \bigcap_{\alpha<\beta} \overline{U}_\alpha$. For each $z \in M$, let $\{V(z, n) : n \in \omega\}$ be a decreasing local base at z.

For each $\alpha < \beta$, choose $f_\alpha \in {}^{\omega}\omega$ such that, when $x_n \in U_\alpha$, $k \geq f_\alpha(n)$ implies $x_{n,k} \in U_\alpha$. Apply BF(c) to $\{f_\alpha : \alpha < \beta\}$ to obtain $f \in {}^{\omega}\omega$ with $f_\alpha <_* f$ for each α. For each $\alpha < \beta$, choose $g_\alpha \in {}^{\omega}\omega$ such that, when

$x_{n,f(n)} \in U_\alpha$, $k \geq g_\alpha(n)$ implies $V(x_{n,f(n)}, k) \subseteq U_\alpha$. Apply BF($c$) to $\{g_\alpha: \alpha < \beta\}$ to obtain $g \in {}^\omega\omega$ such that $g_\alpha <_* g$ for each α. Consider the sequence $\langle V(x_{n,f(n)}, g(n)): n \in \omega \rangle$ of open sets. By the feeble compactness, the tails of this sequence must form a countable open filterbase with unique cluster point x. Hence the filterbase converges to x, and so must the sequence $\langle x_{n,f(n)}: n \in \omega \rangle$.

THEOREM 3.6. *Assume* BF(c). *Every regular feebly compact symmetrizable space with a dense set of points of countable character is first countable (and hence separable* [$\mathbf{St}_2$]).

Proof. Suppose X is regular and symmetrizable, and M is a dense subset of X with each point of M having countable character. By Lemma 3.4 of [$\mathbf{BD}_2$] and the fact that, in a feebly compact space, points which are G_δ's have countable character, it suffices to show that every limit point of M is a sequential limit of M. This follows from 3.5.

REFERENCES

[AU] P. S. Aleksandrov and P. S. Urysohn, *Sur les espaces topologiques compacts*, Bull. Intern. Acad. Pol. Sci. Ser. A, (1923), 5–8.

[Ar_1] A. Arhangel'skiĭ, *Mappings and spaces*, Russian Math. Surveys, **21** (1966), 115–162.

[Ar_2] ——, *On the cardinality of bicompacta satisfying the first axiom of countability*, Soviet Math. Dokl., **10** (1969), 951–955.

[Bo] D. Booth, *Countably indexed ultrafilters*, Ph. D. Thesis, University of Wisconsin, Madison, Wisconsin, 1969.

[BD_1] D. Burke and S. Davis, *Compactifications of symmetrizable spaces*, Proc. Amer. Math. Soc., **81** (1981), 647–651.

[BD_2] ——, *Separability in nearly compact symmetrizable spaces*, Topology Proc., **5** (1980), 59–69.

[BvD] D. Burke and E. van Douwen, *On Countably Compact Extensions of Normal Locally Compact M-Spaces*, in: Set-Theoretic Topology, edited by G. M. Reed, Academic Press, New York, (1977), 81–89.

[Ha] F. Hausdorff, *Untersuchen über ordnungstypen*, Ber. Sächs. Acad. Wiss., **59** (1907), 84–159.

[Hh] R. Heath, *Arc-wise connectedness in semi-metric spaces*, Pacific J. Math., **12** (1962), 1301–1319.

[He] S. Hechler, *On the existence of certain cofinal subsets of* ${}^\omega\omega$, Axiomatic Set Theory, Proc. Symp. Pure Math. (vol. 13, part 2), Amer. Math. Soc., (1974), 155–173.

[Ma] V. Malyhin, *On the power of weakly first countable bicompacta*, Abstracts Amer. Math. Soc., **1** (1980), 247.

[M_1] E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc., **69** (1963), 375–376.

[M_2] ——, *Paracompactness and the Lindelöf property in finite and countable Cartesian products*, Comp. Math., **23** (1971), 199–214.

[Pr] T. Przymusiński, *Normality and paracompactness in finite and countable Cartesian products*, Fund. Math., **105** (1980), 87–104.

[Ro_1] F. Rothberger, *Sur un ensemble toujours de première catégorie qui est dépourvu de la propriété* λ, Fund. Math., **32** (1939), 294–300.

[Ro_2] _____, *On some problems of Hausdorff and Sierpinski*, Fund. Math., **35** (1948), 29–46.

[So] R. C. Solomon, *Families of sets and functions*, Czech J. Math., **27** (1977), 556–559.

[St_1] R. M. Stephenson, *Symmetrizable, $\mathcal{F}$-, and weakly first countable spaces*, Canadian J. Math., **29** (1977), 480–488.

[St_2] _____, *Symmetrizable-closed spaces*, Pacific J. Math., **68** (1977), 507–514.

[Va] J. Vaughan, *Some cardinals related to c and topology*, 1979 Guilford College Topology Conference, pp. 39–49.

[Ve] N. Veličko, *Symmetrizable spaces*, Mat. Zametki, **12** (1972), 577–582 = Math. Notes, **12** (1972).

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