BIJECTIVELY RELATED SPACES I: MANIFOLDS

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The following equivalence relation is introduced: Two (Hausdorff) spaces X and Y are *bijectively related* if there exist continuous bijections $f: X \to Y$ and $g: X \to Y$. This first paper considers the case in which X and Y are connected manifolds. If either f or g is not a homeomorphism, then each space is necessarily non-reversible and hence this study produces more knowledge of such spaces. The chief results here are the existence theorem (Theorem 2) and, perhaps, Corollary 12, which states that a simply-connected manifold having only compact boundary components is reversible.

This is a continuation of a study of continuous bijections following the work of Rajagopalan and Wilansky [5], Petty [4], and Doyle and Hocking [2, 3]. We introduce here the following equivalence relation among topological spaces:

DEFINITION. Two spaces X and Y are *bijectively related* if there exist continuous bijections $f: X \to Y$ and $g: Y \to X$. Each space is then a *bijective relative* of the other, the maps f and g are *relating bijections* and, to be brief, we say that "$[X, Y, f, g]$ holds". We denote by $B(X)$ the equivalence class of all spaces (in the category under study) which are bijectively related to X.

1. Preliminaries. Throughout this study spaces will be assumed to be Hausdorff (at least). With this assumption we surely have $B(X) = \{X\}$ if X is compact. To provide a more general result in this direction, recall that the space X is said to be *reversible* [5] if the only continuous self-bijections $f: X \to X$ are the homeomorphisms. If X is reversible and if $[X, Y, f, g]$ holds, then $g \circ f: X \to X$ must be a homeomorphism. Then $f^{-1} = (g \circ f)^{-1} \circ g$ is continuous, so f is a homeomorphism. Thus $B(X) = \{X\}$ whenever X is reversible. However the condition $B(X) = \{X\}$ does not characterize reversible spaces, as we see next.

THEOREM 1. *Among metric spaces the rationals Q constitute a non-reversible space for which $B(Q) = \{Q\}$.*

Proof. We use as a lemma the known fact that every countable perfect metric space is homeomorphic to Q. Write

$$Q = [Q \cap (-\infty, \pi)] \cup [Q \cap (\pi, +\infty)]$$
$$= [Q \cap (-1, 1)] \cup [Q \cap (1, +\infty)].$$

Then there exist homeomorphisms

$$h_1 \colon Q \cap (-\infty, \pi) \to Q \cap (-\infty, 1)$$

and

$$h_2 \colon Q \cap (\pi, +\infty) \to Q \cap [1, +\infty).$$

when we put these together we get a continuous bijection $f \colon Q \to Q$ that is not a homeomorphsim. Thus Q is non-reversible.

If X is a continuous bijective image of Q, then X is countable and perfect. Thus we have $B(Q) = \{Q\}$ as claimed.   □

It is perhaps more surprising to find that there are non-reversible connected manifolds M for which $B(M) = \{M\}$ (see Example 2 below). First, however, we provide an existence theorem and a first example of bijectively related manifolds.

THEOREM 2. *For each $n \geq 2$ there exist non-homeomorphic connected bijectively related n-manifolds.*

Proof. In $\mathbf{R}^2$ consider the following submanifolds (see Figure 1):

$$M = \{(x, y) \colon -1 < y < 0\}$$
$$\cup \bigcup_{n \in \mathbf{Z}} \{(x, y) \colon 3n - 1 < x < 3n, 0 \leq y < 4\}$$
$$\cup \bigcup_{n \in \mathbf{Z}} \{(x, y) \colon 3n \leq x \leq 3n + 1, 1 < y < 2 \text{ or } 3 < y < 4\}$$
$$\cup \bigcup_{n \in \mathbf{Z}_+} \{(x, y) \colon -3n + 1 < x \leq -3n + 2, 1 < y < 2 \text{ or } 3 < y < 4\},$$

and

$$N = M \cup \{(x, y) \colon 1 < x \leq 2, 3 < y < 4\}.$$

The interior of N contains a simple closed curve J which separates the boundary of N. No such exists in M so the two are not homeomorphic. It is obvious from inspection that M and N are bijectively related, and the rest of the theorem follows from consideration of the manifolds $M \times S^k$ and $N \times S^k$ for $k = 1, 2, 3, \ldots$.   □
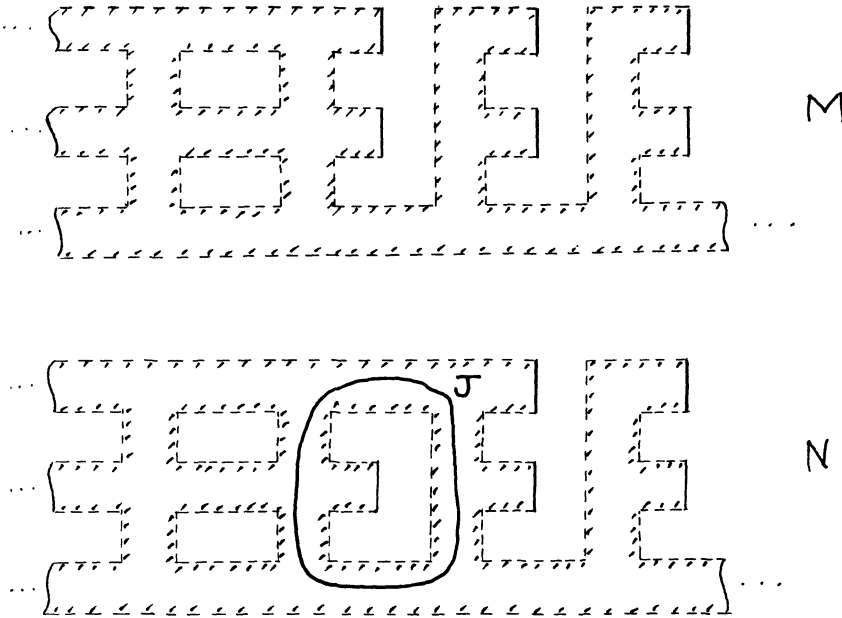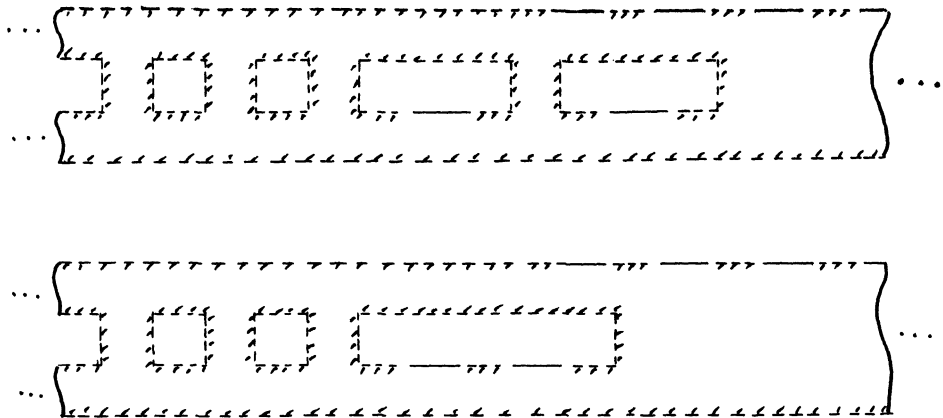
FIGURE 1

FIGURE 2

In Figure 2 we picture two more planar manifolds bijectively related to those in Figure 1. These clearly indicate that the class $B(M)$ is infinite for the manifold M of Theorem 2. This gives rise to a problem which seems to be difficult: Let M be a connected manifold for which $B(M) \neq \{M\}$. Is $B(M)$ necessarily infinite?

The non-reversible manifolds in Figures 1 and 2 might mislead the unwary into making false conjectures about the number of boundary components and ends which such manifolds must possess. The following easily proved result is instructive in this regard.

THEOREM 3. *If M is any connected non-reversible manifold, then M × [0, 1) is a non-reversible manifold having connected boundary and precisely one end.*

It is also interesting to note that for the manifolds M and N of Figure 1, $M \times [0, 1)$ and $N \times [0, 1)$ are homeomorphic, but $M \times [0, 1]$ and $N \times [0, 1]$ are not homeomorphic. (There is a copy of the simple closed curve J in the boundary of $N \times [0, 1]$ that fails to separate this boundary. No such nonseparating simple closed curve exists in the boundary of $M \times [0, 1]$.) These observations yield several more unsolved problems of the following nature: If $[M, N, f, g]$ holds and P is any other manifold, surely $M \times P$ and $N \times P$ are bijectively related. If we assume M and N are not homeomorphic, does there exist a manifold P such that $M \times P$ and $N \times P$ are homeomorphic? Can such a manifold P be compact?

For general information as well as subsequent use, we list the next five theorems. The proofs either are simple exercises or are already known.

THEOREM 4. *If $[M, N, f, g]$ holds, then each manifold embeds in the interior of the other.*

THEOREM 5. *If $[M, N, f, g]$ holds and if one manifold is orientable, then so is the other.*

THEOREM 6. *If $[M, N, f, g]$ holds and if ∂M has only compact components, then ∂N has only compact components.*

(We use ∂M and Int M to denote the boundary of M and the interior of M, respectively.)

THEOREM 7. *If $[M, N, f, g]$ holds and if $f(\partial M) = \partial N$, then f is a homeomorphism (Theorem 3.4 of [4].)*

THEOREM 8. *If $[M, N, f, g]$ holds, if every component of ∂M is compact and if f is not a homeomorphism, then there is at least one component C of ∂M such that $f(C) \subset$ Int N (Theorem 3 of [2]).*

THEOREM 9. *If the connected 2-manifold M has infinitely many handles, infinitely many compact boundary components and infinitely many annular ends, then M is non-reversible, and if M has only compact boundary components, then the converse also holds (see Figure 3).*

Proof. To prove the second statement, let $f: M \to M$ be a continuous bijection which is not a homeomorphism. Theorem 8 says that f "swallows" some component C of ∂M. Then $f^{-1}(C), f^{-1}(f^{-1}(C)), \ldots$ provides us with infinitely many compact boundary components. If U is a sufficiently small neighborhood of $f(C)$, then $f^{-1}(U)$ has one component V which contains an annular end of M and $f^{-1}(V), f^{-1}(f^{-1}(V)), \ldots$ gives us a sequence of such ends. There is a simple closed curve J in Int M that meets $f(C)$ transversely at a single point and is such that $f^{-1}(J)$ is connected. Then $f(J), f(f(J)), \ldots$ identify the required handles.

To prove the first statement we provide a continuous bijection f from M to a manifold N and then show that $N = M$. An annular end $S^1 \times [-1, 0)$ and a collar $S^1 \times [0, 1]$ on a boundary component $S^1 \times \{0\}$ are carried by local homeomorphisms to handle $S^1 \times [-1, 1]$ to form N. Thus f^{-1} is discontinuous along $S^1 \times \{0\}$. The details of this construction can be left to the reader.

Next we select a sequence of disjoint handles $H_1, H_2, \ldots$ which "converge" to an end ε of M. Then we choose a topological line l in Int M having both ends at ε and separating M into components U and V. Select l so that U contains the handles H_i, no other handles, no boundary components and no ends of M. This line l also lies in N, of course, and has the same properties there. Now run an arc from a point of l to a point in some simple closed curve in N cutting off the new handle H_0. Swell up this arc and add the disk containing H_0 to obtain an open set X in N bounded by a topological line l' separating N into components X and Y. We have constructed l' so that X and U are homeomorphic, and, in fact, there is a homeomorphism of X onto U which carries l' to l leaving $l' \cap l$ fixed. Analogously we may select topological lines in Int N, then alter them in Int M, to cut off sequences of annular ends and boundary components. This provides both four homeomorphic pieces of M and N and the means of fitting them together. $\square$

EXAMPLE 2. The 2-manifold M pictured in Figure 3 is an infinite tube with countably many handles to the right and countably many compact boundary components $\{C_{-n}\}$ and annular ends (at the tops of the chimneys) to the left. Theorem 9 tells us that M is non-reversible and we now
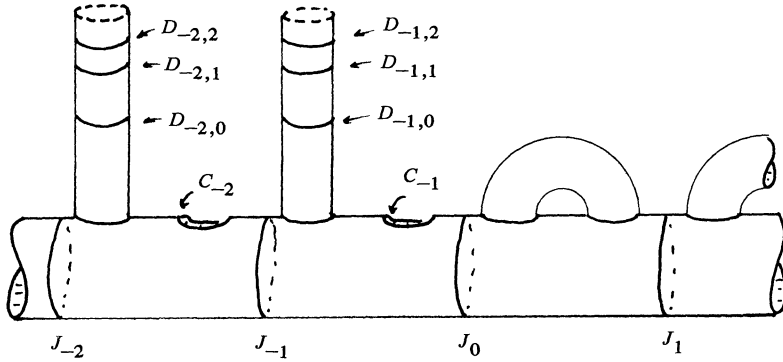
FIGURE 3

claim that $B(M) = \{M\}$. To prove this, suppose $[M, N, f, g]$ holds for some manifold N. Express M as a monotone increasing union $M = \bigcup K_n$ of compact submanifolds where $K_0 = J_0$ and

$$\partial K_1 = J_{-1} \cup D_{-1,0} \cup C_{-1} \cup J_1,$$
$$\partial K_2 = J_{-2} \cup D_{-2,0} \cup C_{-2} \cup D_{-1,1} \cup C_{-1} \cup J_2,$$
$$\vdots$$

Clearly, no component of $M - K_n$ has compact closure and all but two of such components are open annuli. Let U_n and V_n be the non-annular components of $M - K_n$ to the right of J_n and to the left of J_{-n}, respectively. Then $\{U_n\}$ is a sequence of domains defining the "end to the right" and $\{V_n\}$ similarly for the "end to the left".

We first claim $f| \overline{U}_n$ is a homeomorphism for all n. This is certainly true if $f(\overline{U}_n) = f(\overline{U}_n)$. But if there were a point $p \in f(\overline{U}_n) - f(\overline{U}_n)$ (at which f^{-1} would not be continuous, of course), then p would have to lie on the image $f(C)$ of some component C of ∂M. But $f(C) \subset \text{Int } N$, hence some neighborhood of this compact set would contain points from infinitely many handles, and this is impossible.

We treat the end to the left differently. First we note that if $f(\partial M) \cap \text{Int } N$ has finitely many components, then M and N are homeomorphic. To see this, suppose f "swallows" components $C_{i_1}, C_{i_2}, \ldots, C_{i_k}$ of ∂M by sewing them to annular ends $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$. Select n sufficiently large so that $M - J_{-n} \cup J_0$ consists of the three components V_n, P and U_0, where P contains all of the components $C_{i_1}, \ldots, C_{i_k}$ and all of the ends $\varepsilon_1, \ldots, \varepsilon_k$. Applied to P, f simply forms k handles and we may rearrange these, via a homeomorphism leaving $f(J_{-n} \cup J_0)$ fixed, so that $f(P)$ is homeomorphic to the bounded component of $M - J_{-n+k} \cup J_k$. The homeomorphism from M to N is now obvious.

If $f(\partial M) \cap$ Int N had infinitely many components, then surely $g \circ f$: $M \to M$ would "swallow" infinitely many components by creating infinitely many new handles. Because $g \circ f$ preserves the end to the right (homeomorphically) and the end to the left as well, this is not possible. $\square$

One question suggested by Example 2 seems to be difficult: Let M be a non-reversible manifold and $f: M \to N$ a continuous bijection of M to a manifold N. Is every (isolated) wild end of M duplicated in N?

2. Some structure theorems. The results in this section serve to elucidate the notion of bijectively related manifolds. There are, inevitably, some recent additions to the knowledge of non-reversible manifolds.

Any non-reversible 2-manifold M has non-trivial first homology and hence $\pi_1(M) \neq 1$. However there do exist simply connected non-reversible manifolds of higher dimensions. E. H. Kronheimer provided us with the following example: Let M consist of the lower open half-space $z < 0$ together with countably many open annular boundary patches on the plane $z = 0$. Using a well-known bijection due to K. Whyburn [6], it is easy to construct a self-bijection $f: M \to M$ that is not a homeomorphism. In Corollary 12 below, then, we seem to have the strongest result possible of its kind.

THEOREM 10. *Let J be a simple closed curve in a normal space M. Suppose some point $p \in J$ has an open neighborhood U with the following properties*: (1) $U \cap J$ *is an open arc A in J with endpoints p_1 and p_2, and* (2) $\overline{U} - U$ *is the union of separated sets C_1 and C_2 with $C_i \cap J = p_i$, $i = 1, 2$. Then J is essential in M.*

Proof. Define a retraction r_1: $\overline{U} \to \overline{A}$ such that $r_1(C_i) = p_i$, $i = 1, 2$. Setting $B = J - A$, repeat the construction of r_1 to obtain a retraction r_2: $M - U \to B$ with $r_2(C_i) = p_i$, $i = 1, 2$. Thus there is a retraction r: $M \to J$. $\square$

We shall say that the continuous bijection f: $M \to N$ "respects boundary components" if the following conditions hold:

(1) each component of ∂M is carried by f to a closed set in N;

(2) $f^{-1}(\text{Int } N) \cap \partial M$ is a union of components of ∂M; and

(3) if C is a component of ∂M with $f(C) \subset$ Int N, then $f(C)$ is bicollared in Int N with a bicollar that fails to meet all other components of $f(\partial M)$.

THEOREM 11. *Let f: $M \to N$ be a continuous bijection which respects boundary components. If $\pi_1(N) = 1$, f must be a homeomorphism.*

Proof. If f were not a homeomorphism, then by property (2) above and Theorem 3.4 of [4] there is a component C of ∂M with $f(C) \subset \text{Int } N$. We let U be an open bicollar on $f(C)$ and assume $\overline{U}$ is homeomorphic to $C \times [-1, 1]$. Surely U does not separate N. Thus by joining with an arc in Int $N - U$ the endpoints of a fiber in U, we construct a simple closed curve J which has the properties set out in Theorem 10. This tells us $\pi_1(N) \neq 1$. ☐

COROLLARY 12. *If the manifold M has only compact boundary components and if $\pi_1(M) = 1$, then M is reversible.*

Proof. Each component of ∂M is collared and these collars can be chosen to be pairwise disjoint. If f: $M \to M$ were not a homeomorphism and the boundary component C had $f(C) \subset \text{Int } M$, we use the f-image of the collar on C and the core of this collar just as the bicollar on $f(C)$ was used in Theorem 11. ☐

COROLLARY 13. *If M is a non-reversible manifold having only compact boundary components, then $\pi_1(M)$ is infinitely generated.*

Proof. Let f: $M \to M$ be a continuous bijection swallowing a boundary component C. Construct the simple closed curve J piercing $f(C)$ as in the proof of Theorem 11. Then consider $f(J), f(f(J))$, etc. ☐

COROLLARY 14. *If ∂M has only simply connected compact components, then the universal covering manifold $\tilde{M}$ is reversible. For a 2-manifold, $\tilde{M}$ is always reversible.*

As Theorem 3 indicates, the number of ends plays little role in the reversibility property of a manifold. The *nature* of the ends, however, is very important in this regard.

THEOREM 15. *If ∂M has only compact components and if M has only euclidean ends, then M is reversible.*

Proof. By assumption each end embeds in $\mathbf{R}^n$, where $n = \dim M$. If a boundary component C could be sewed to an end by some bijection f: $M \to M$, then C would also embed in $\mathbf{R}^n$.

Because $f(C)$ cannot separate M, f has constructed a "handle" H on M such that H does not embed in $\mathbf{R}^n$. By iterating f we see that M contains an infinite sequence of such handles and hence that M has a non-euclidean end.   □

To facilitate the next discussion let us briefly describe the ends of a non-compact connected manifold M. Represent M as a monotone increasing union $M = \cup_{n=1}^{\infty} C_n$ of compact submanifolds C_n of M. We may assume $C_n \subset \operatorname{Int}_M C_{n+1}$ for each n. Each end ε of M may now be represented by a monotone decreasing sequence $\{U_n\}$, where U_n is a component of $M - C_n$ for each n and $\overline{U}_n$ is non-compact. (In fact, any components of $M - C_n$ which have compact closure may be added to C_n without effect on the ends.)

DEFINITION. If, in addition to the above, the compact submanifolds C_n can be so chosen that $\operatorname{Fr} U_n \subset \operatorname{Int} M$ for each n, we shall say that ε is an *interior end* of M.

THEOREM 16. *Let M be a non-compact manifold. Every end of M is interior iff every component of ∂M is compact.*

Proof. Suppose B is a non-compact component of ∂M. Then B has at least one end η. If M is expressed as a monotone increasing union of compact submanifolds $M = \cup C_n$, surely $B = \cup (B \cap C_n)$. Hence there is a sequence $\{V_n\}$, each V_n being a component of $B - B \cap C_n$, which represents η. This identifies a sequence $\{U_n\}$ of components U_n of $M - C_n$, where $V_n \subset U_n$. Surely $\{U_n\}$ represents an end ε of M. We claim that the submanifolds C_n cannot be selected so that $\operatorname{Fr} U_n \cap B = \varnothing$. This is true because $B \cap C_n$ is a submanifold of B and therefore must contain points of $\operatorname{Fr} V_n \subset \operatorname{Fr} U_n$. It follows that ε is not an interior end.

On the other hand, suppose each component B_i of ∂M is compact. Let $p_i \colon B_i \times [0, 1)$ be an open collar C_i on B_i, $i = 1, 2, \ldots$. For each i and j, let $C_{ij} = p_i(B_i \times [0, 1/j))$ so $C_{i'j+1} \subset C_{ij}$ and $\cap_j C_{ij} = B_i$. Given any sequence of compact submanifolds M with $M = \cup M_n$, we can obviously adjust M_n so that

$$\overline{C}_1 \cup \overline{C}_2 \cup \cdots \cup \overline{C}_n \subset M_n$$

while

$$\bigcup_{j>n} C_{jj} \subset M - M_n.$$

Then if U is any component of $M - M_n$, Fr $U \subset$ Int M, and thus every end of M is interior. $\qquad\square$

THEOREM 17. *Suppose the manifold M has exactly one end and that end is interior. If $f: M \to N$ is any continuous bijection (to a manifold N) which is not a homeomorphism, then N is compact.*

Proof. The map f carries some component B of ∂M into Int N. It follows that M has a B-like end (see [3]). Let V be a connected neighborhood of $f(B)$ that is separated by $f(B)$ such that $\overline{V}$ is compact. Let U be the component of $f^{-1}(V)$ not containing B. Then $\overline{U}$ is not compact and is in some sequence representing the end. It follows that $f(M - U) \cup f(\overline{U} \cup B)$ presents N as a union of two compact sets. $\qquad\square$

THEOREM 18. *If $[M, N, f, g]$ holds, if every component of ∂M is compact and if M is euclidean (i.e. if M embeds in some euclidean space as a codimension zero submanifold), then M and N are homeomorphic.*

Proof. Each component of ∂M separates the euclidean space $\mathbf{R}^n$ in which M embeds. Since N embeds in Int M, N also embeds in $\mathbf{R}^n$, whence f cannot carry a component of ∂M into Int N. $\qquad\square$

THEOREM 19. *Suppose $[M, N, f, g]$ holds and ∂M has only compact components. Suppose further there is some set C, closed in Int M and having codimension ≥ 2, such that (Int M) $- C$ embeds in $\mathbf{R}^n$, where $n = \dim M$. Then M and N are homeomorphic.*

Proof. First notice that the set $C' = g^{-1}(C \cap g(\text{Int } N))$ is a set in Int N enjoying exactly the properties of C in Int M. If there were a component B of ∂M such that $f(B) \subset$ Int N, surely Int $N - f(B)$ is connected, whereas (Int $N - C') - f(B)$ is not connected. Thus the euclidean domain Int $N - B$ is separated by a set of codimension ≥ 2, which is impossible. $\qquad\square$

COROLLARY 20. *Suppose $[M, N, f, g]$ holds and ∂M has only compact components. If Int N has a residual set R (see [1]) of codimension ≥ 2, then M and N are homeomorphic.*

REFERENCES

[1]   P. H. Doyle and J. G. Hocking, *A decomposition theorem for n-dimensional manifolds*, Proc. Amer. Math. Soc., **13**, 469–471.

[2]   _____, *Continuous bijections on manifolds*, J. Austral. Math. Soc., **22** (1976), 257–263.

[3]   _____, *Strongly reversible manifolds*, J. Austral. Math. Soc. (Series A), **34** (1983), 172–176.

[4]   D. H. Pettey, *One-to-one mappings into the plane*, Fund. Math., **67** (1970), 209–218.

[5]   M. Rajagopalan and A. Wilansky, *Reversible topological spaces*, J. Austral. Math. Soc., **61** (1966), 129–138.

[6]   K. Whyburn, *A non-topological* 1-1 *mapping onto* E^3. Bull. Amer. Math. Soc., **71** (1965), 533–537.

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