

INEQUALITIES FOR EIGENVALUES OF THE BIHARMONIC OPERATOR

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Let D be a bounded domain in R^m with smooth boundary. The first $n + 1$ eigenvalues for the problem

$$\Delta^2 u - \mu u = 0 \quad \text{in } D, \quad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D$$

satisfy the inequality

$$\sum_{i=1}^n \frac{\sqrt{\mu_i}}{\mu_{n+1} - \mu_i} \geq \frac{m^2 n^{3/2}}{8(m+2)} \left(\sum_{i=1}^n \mu_i \right)^{-1/2}$$

For the first two eigenvalues we have the stronger bound

$$\begin{aligned} \mu_2 &\leq 7.103 \mu_1 \quad (\text{in } R^2), \\ \mu_2 &\leq 4.792 \mu_1 \quad (\text{in } R^3). \end{aligned}$$

The first two eigenvalues for the problem

$$\Delta^2 u + \nu \Delta u = 0 \quad \text{in } D, \quad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D$$

satisfy the inequality

$$\begin{aligned} \nu_2 &\leq 2.5 \nu_1 \quad (\text{in } R^2), \\ \nu_2 &\leq 2.12 \nu_1 \quad (\text{in } R^3). \end{aligned}$$

Introduction. Let D be a bounded domain in R^m , $m \geq 2$, with smooth boundary ∂D . For the case $m = 2$, Payne, Polya and Weinberger [6] obtained upper estimates, independent of the domain, for eigenvalues of the three well-known eigenvalue problems:

- (1) $\Delta u + \lambda u = 0 \quad \text{in } D, \quad u = 0 \quad \text{on } \partial D,$
- (2) $\Delta^2 u - \mu u = 0 \quad \text{in } D, \quad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D,$
- (3) $\Delta^2 u + \nu \Delta u = 0 \quad \text{in } D, \quad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D.$

Let

$$\begin{aligned} 0 &< \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \\ 0 &< \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots, \\ 0 &< \nu_1 \leq \nu_2 \leq \nu_3 \leq \cdots \end{aligned}$$

denote the successive eigenvalues for (1), (2) and (3), respectively. Payne, Polya and Weinberger showed that for domains in the plane,

$$(4) \quad \lambda_{n+1} \leq \lambda_n + \frac{2}{n} \sum_{i=1}^n \lambda_i \leq 3\lambda_n, \quad n = 1, 2, \dots,$$

$$(5) \quad \mu_{n+1} \leq \mu_n + \frac{8}{n} \sum_{i=1}^n \mu_i \leq 9\mu_n, \quad n = 1, 2, \dots,$$

$$(6) \quad \nu_2 \leq 3\nu_1.$$

Inequality (4) was improved and extended to $m \geq 2$ by Protter and Hile [4]. They showed that the eigenvalues for (1) satisfy the inequality

$$(7) \quad \sum_{i=1}^n \frac{\lambda_i}{\lambda_{n+1} - \lambda_i} \geq \frac{mn}{4},$$

which amounts to an implicit bound for λ_{n+1} in terms of the preceding eigenvalues. One can derive (4) from (7) by replacing each λ_i in the denominators of the left-hand side of (7) by the larger quantity λ_n and then solving the resulting inequality for λ_{n+1} .

In this paper we derive an analogue of (7) for the biharmonic eigenvalue problem (2). We obtain the implicit bound

$$(8) \quad \sum_{i=1}^n \frac{\sqrt{\mu_i}}{\mu_{n+1} - \mu_i} \geq \frac{m^2 n^{3/2}}{8(m+2)} \left(\sum_{i=1}^n \mu_i \right)^{-1/2}.$$

From this inequality one may derive the weaker, but explicit, bound

$$(9) \quad \mu_{n+1} \leq \mu_n + \frac{8(m+2)}{m^2 n^{3/2}} \left(\sum_{i=1}^n \mu_i \right)^{1/2} \left(\sum_{i=1}^n \sqrt{\mu_i} \right),$$

which in turn may be used to derive the still weaker, but perhaps aesthetically more appealing, bound

$$(10) \quad \mu_{n+1} \leq \mu_n + \frac{8(m+2)}{m^2 n} \sum_{i=1}^n \mu_i.$$

The last inequality (10) is the natural extension to dimensions $m \geq 2$ of the bound (5) for domains in the plane.

We also obtain improved estimates for some of the lower eigenvalues of (2). We show that for any $\sigma > 0$,

$$(11) \quad \mu_{n+1} \leq (1 + \sigma)\mu_n + q(\sigma) \frac{M(m)}{n} \sum_{i=1}^n \mu_i,$$

where

$$q(\sigma) = \left[\frac{(1 + \sigma)^3}{\sigma} \right]^{1/2}, \quad M(m) = \frac{32}{3} \sqrt{\frac{2}{3}} m^{-1} (m + 2)^{-1/2}.$$

For given values of m and n one is obliged to select an optimal value of σ so that the right side of (11) is minimized. For example, for $m = 2, n = 1$, one chooses $\sigma = .4$ to obtain the bound

$$\mu_2 \leq 7.103 \mu_1 \quad (\text{in } R^2).$$

For $m = 3, n = 1$, the choice $\sigma = .36$ gives

$$\mu_2 \leq 4.792 \mu_1 \quad (\text{in } R^3),$$

and for $m = 2, n = 2$, we may choose $\sigma = .34$ to obtain

$$\mu_3 \leq 2.897 \mu_1 + 4.237 \mu_2 \quad (\text{in } R^2).$$

The corresponding inequalities obtained from (10) are

$$\mu_2 \leq 9\mu_1 \quad (\text{in } R^2), \quad \mu_2 \leq 5.4\bar{4} \mu_1 \quad (\text{in } R^3)$$

$$\mu_3 \leq 4\mu_1 + 5\mu_2 \quad (\text{in } R^2).$$

Thus (11) is an improvement over (10) in these cases. We will show further that (11) is an improvement over (10) for $n = 1$ and *any* value of m , and that for $m = 2, 3, 4$, (11) is stronger than (10) for $n \leq 8, 3, 2$, respectively.

In the last section of the paper we give an improvement and an extension to higher dimensions of inequality (6) for the eigenvalue problem (3). We show that

$$(12) \quad \nu_2 \leq \frac{m^2 + 8m + 20}{(m + 2)^2} \nu_1.$$

Thus for $m = 2, 3$, for example, we have

$$\nu_2 \leq 2.5 \nu_1 \quad (\text{in } R^2),$$

$$\nu_2 \leq 2.12 \nu_1 \quad (\text{in } R^3).$$

We remark that there has been quite a bit of interest in obtaining the best estimate for the ratio λ_2/λ_1 for domains in the plane. Payne, Polya and Weinberger showed that

$$\lambda_2 \leq 3\lambda_1.$$

This bound has undergone a succession of improvements by Brands [1], DeVries [3], and Chiti [2], with the latest bound by Chiti being

$$\lambda_2 \leq 2.586 \lambda_1.$$

For the circular disk we have $\lambda_2/\lambda_1 = 2.539 \dots$. It has been conjectured [6], but not yet proved, that among all domains in the plane the circular disk maximizes the ratio λ_2/λ_1 .

There has also been some interest in bounds for the quantities λ_3/λ_1 , $(\lambda_2 + \lambda_3)/\lambda_1$. See for example [1], [4], [5], [6]. The best bounds to date appear to be those of Marcellini [5], who showed that for domains in the plane,

$$\lambda_2 + \lambda_3 \leq 5.596 \lambda_1, \quad \lambda_3 \leq 3.917 \lambda_1.$$

1. Inequalities for μ_n . Let D be a bounded domain in R^m , $m \geq 2$, with boundary ∂D . Let the eigenvalues of

$$\begin{aligned} \Delta^2 u &= \mu u \quad \text{in } D, \\ u &= \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial D \end{aligned}$$

be designated by

$$0 < \mu_1 \leq \mu_2 \leq \dots \leq \mu_n \leq \dots,$$

with corresponding eigenfunctions $u_1, u_2, \dots, u_n, \dots$, normalized so that

$$\int_D u_i u_j = \delta_{ij}, \quad i, j = 1, 2, \dots$$

The following theorem is an extension to higher dimensions of the inequalities (5). Later we will refine some of the techniques in the proof in order to obtain stronger inequalities.

THEOREM 1. For $m \geq 2$ and $n \geq 1$ we have

$$(1.1) \quad \mu_{n+1} \leq \mu_n + \frac{8(m+2)}{m^2 n} \sum_{i=1}^n \mu_i \leq \left(\frac{m+4}{m} \right)^2 \mu_n.$$

Proof. Following Payne, Polya and Weinberger [6] we consider the n trial functions

$$\varphi_i = x_1 u_i - \sum_{j=1}^n a_{ij} u_j, \quad i = 1, 2, \dots, n,$$

where $x = (x_1, \dots, x_m) \in R^m$, and the constants a_{ij} are defined by

$$a_{ij} = \int_D x_1 u_i u_j, \quad i, j = 1, \dots, n.$$

Then each φ_i is orthogonal to u_1, \dots, u_n . Moreover, since $\varphi_i = \partial\varphi_i/\partial n = 0$ on ∂D , we have the well-known inequality

$$(1.2) \quad \mu_{n+1} \leq \frac{\int \varphi_i \Delta^2 \varphi_i}{\int \varphi_i^2}, \quad i = 1, 2, \dots, n.$$

Now,

$$(1.3) \quad \begin{aligned} \int \varphi_i \Delta^2 \varphi_i &= \int \varphi_i \left[\Delta^2(x_1 u_i) - \sum_{j=1}^n a_{ij} \mu_j u_j \right] \\ &= \int \varphi_i [x_1 \Delta^2 u_i + 4 \Delta u_{ix_1}] = \mu_i \int \varphi_i^2 + 4 \int \varphi_i \Delta u_{ix_1}. \end{aligned}$$

After substituting into (1.2) and summing over i we have

$$(1.4) \quad \mu_{n+1} \sum_{i=1}^n \int \varphi_i^2 \leq \sum_{i=1}^n \mu_i \int \varphi_i^2 + 4 \sum_{i=1}^n \int \varphi_i \Delta u_{ix_1}.$$

Now we make the assumption that

$$(1.5) \quad \sum_{i=1}^n \int u_{ix_k}^2 = \frac{1}{m} \sum_{i=1}^n \int |\nabla u_i|^2, \quad k = 1, \dots, m.$$

This equality can be made to hold by rotating the coordinate system in R^m . Suppose, for example, that (1.5) does not hold, and that x_p and x_q denote two coordinate directions such that

$$\sum_{i=1}^n \int u_{ix_p}^2 < \frac{1}{m} \sum_{i=1}^n \int |\nabla u_i|^2 < \sum_{i=1}^n \int u_{ix_q}^2.$$

Then we may make a rotation of the x_p - x_q plane until (1.5) holds for, say, $k = p$, with the left-hand side of (1.5) remaining unaffected for values of k different from p and q . This operation can be repeated until (1.5) holds for all values of k .

We pause to make a few technical calculations. Let us define

$$J_1 = \sum_{i=1}^n \int \varphi_i \Delta u_{ix_1}, \quad J = \frac{m+2}{2m} \sum_{i=1}^n \int |\nabla u_i|^2.$$

LEMMA 1. *The quantities J_1 and J satisfy:*

- (i) $J_1 = J$,
- (ii) $J^2 \leq n((m+2)/2m)^2 \sum_{i=1}^n \mu_i$,
- (iii) $n^2(m+2)/8 \leq J \sum_{i=1}^n \int \varphi_i^2$.

Proof of Lemma 1. (i) We have

$$J_1 = \sum_{i=1}^n \int x_1 u_i \Delta u_{ix_1} - \sum_{i,j=1}^n a_{ij} \int u_j \Delta u_{ix_1}.$$

The last term above vanishes, since $a_{ij} = a_{ji}$, and, by integration by parts,

$$\int u_j \Delta u_{ix_1} = - \int u_i \Delta u_{jx_1}.$$

As for the first term, we show also by integration by parts that

$$\begin{aligned} \int x_1 u_i \Delta u_{ix_1} &= \int \Delta(x_1 u_i) u_{ix_1} = \int (x_1 \Delta u_i + 2u_{ix_1}) u_{ix_1} \\ &= - \int (x_1 \Delta u_i)_{x_1} u_i + 2 \int u_{ix_1}^2 \\ &= - \int x_1 u_i \Delta u_{ix_1} + \int |\nabla u_i|^2 + 2 \int u_{ix_1}^2. \end{aligned}$$

Transposing, summing over i , and applying (1.5) we have

$$J_1 = \sum_{i=1}^n \left[\frac{1}{2} \int |\nabla u_i|^2 + \int u_{ix_1}^2 \right] = J.$$

(ii) By the Schwarz inequality we have

$$\begin{aligned} J^2 &= \left(\frac{m+2}{2m} \right)^2 \left(\sum_{i=1}^n \int -u_i \Delta u_i \right)^2 \\ &\leq \left(\frac{m+2}{2m} \right)^2 \left(\sum_{i=1}^n \int u_i^2 \right) \left(\sum_{i=1}^n \int (\Delta u_i)^2 \right). \end{aligned}$$

Moreover,

$$\int u_i^2 = 1, \quad \int (\Delta u_i)^2 = \int u_i \Delta^2 u_i = \mu_i \int u_i^2 = \mu_i.$$

(iii) Let us compute

$$(1.6) \quad \sum_{i=1}^n \int \varphi_i u_{ix_1} = \sum_{i=1}^n \int x_1 u_i u_{ix_1} - \sum_{i,j=1}^n a_{ij} \int u_j u_{ix_1}.$$

We have, by integration by parts

$$\int x_1 u_i u_{ix_1} = -\frac{1}{2}, \quad \int u_j u_{ix_1} = -\int u_i u_{jx_1}.$$

Since $a_{ij} = a_{ji}$, the last term of (1.6) vanishes, and we obtain

$$(1.7) \quad \sum_{i=1}^n \int \varphi_i u_{ix_1} = -\frac{n}{2}.$$

Hence, upon squaring and applying (1.5) we have

$$\frac{n^2}{4} \leq \left(\sum_{i=1}^n \int \varphi_i^2 \right) \left(\sum_{i=1}^n \int u_{ix_1}^2 \right) = \left(\sum_{i=1}^n \int \varphi_i^2 \right) \left(\frac{2}{m+2} J \right),$$

which completes the proof of Lemma 1.

Returning now to (1.4), if we replace μ_i by μ_n , then in view of Lemma 1 we have

$$(1.8) \quad (\mu_{n+1} - \mu_n) \sum_{i=1}^n \int \varphi_i^2 \leq 4J.$$

Moreover, combination of (ii) and (iii) of Lemma 1 yields

$$(1.9) \quad J \leq \frac{2(m+2)}{nm^2} \left(\sum_{i=1}^n \mu_i \right) \left(\sum_{i=1}^n \int \varphi_i^2 \right).$$

Substitution of (1.9) into (1.8) yields

$$\mu_{n+1} \leq \mu_n + \frac{8(m+2)}{m^2 n} \sum_{i=1}^n \mu_i \leq \left(\frac{m+4}{m} \right)^2 \mu_n,$$

the last inequality being obtained by replacing μ_i by μ_n and simplifying. The proof of Theorem 1 is now complete.

Next we obtain a stronger result than Theorem 1 by a somewhat more lengthy argument.

THEOREM 2. *For $m \geq 2$ and $n \geq 1$ we have the implicit bound*

$$(1.10) \quad \sum_{i=1}^n \frac{\sqrt{\mu_i}}{\mu_{n+1} - \mu_i} \geq \frac{m^2 n^{3/2}}{8(m+2)} \left(\sum_{i=1}^n \mu_i \right)^{-1/2}$$

and the explicit bound

$$(1.11) \quad \mu_{n+1} \leq \mu_n + \frac{8(m+2)}{m^2 n^{3/2}} \left(\sum_{i=1}^n \mu_i \right)^{1/2} \left(\sum_{i=1}^n \sqrt{\mu_i} \right).$$

Inequality (1.10) is stronger than (1.11), and both are stronger than (1.1).

Proof. We return again to (1.4) but instead of replacing each μ_i by μ_n we introduce a new parameter α , $\alpha > \mu_n$, and write

$$(1.12) \quad \mu_{n+1} \sum_{i=1}^n \int \varphi_i^2 \leq \alpha \sum_{i=1}^n \int \varphi_i^2 + \sum_{i=1}^n (\mu_i - \alpha) \int \varphi_i^2 + 4J.$$

We also apply the Cauchy inequality to (1.7), obtaining for any $\delta > 0$,

$$(1.13) \quad \frac{n}{2} \leq \frac{\delta}{2} \sum_{i=1}^n (\alpha - \mu_i) \int \varphi_i^2 + \frac{1}{2\delta} \sum_{i=1}^n (\alpha - \mu_i)^{-1} \int u_{ix_1}^2.$$

Now we could use trial functions φ_{ik} , based on x_k instead of x_1 , for $k = 1, 2, \dots, m$, and obtain inequalities analogous to (1.12) and (1.13) of the type

$$(1.14) \quad (\mu_{n+1} - \alpha) \sum_{i=1}^n \int \varphi_{ik}^2 \\ \leq \sum_{i=1}^n (\mu_i - \alpha) \int \varphi_{ik}^2 + 4J, \quad k = 1, \dots, m,$$

$$(1.15) \quad n \leq \delta \sum_{i=1}^n (\alpha - \mu_i) \int \varphi_{ik}^2 \\ + \delta^{-1} \sum_{i=1}^n (\alpha - \mu_i)^{-1} \int u_{ix_k}^2, \quad k = 1, \dots, m.$$

(Because of (1.5) and its consequence (i) of Lemma 1, the quantity “ J ” is the same for each value of k .) We sum each of (1.14) and (1.15) over k , denoting

$$S = \sum_{k=1}^m \sum_{i=1}^n \int \varphi_{ik}^2, \quad T = \sum_{k=1}^m \sum_{i=1}^n (\alpha - \mu_i) \int \varphi_{ik}^2,$$

to obtain the inequalities

$$(1.16) \quad (\mu_{n+1} - \alpha)S + T \leq 4mJ,$$

$$(1.17) \quad mn \leq \delta T + \delta^{-1} \sum_{i=1}^n (\alpha - \mu_i)^{-1} \int |\nabla u_i|^2.$$

We also have the estimate

$$\int |\nabla u_i|^2 = \int -u_i(\Delta u_i) \leq \left(\int u_i^2 \right)^{1/2} \left(\int (\Delta u_i)^2 \right)^{1/2} = \sqrt{\mu_i},$$

which when applied to (1.17) yields

$$(1.18) \quad mn \leq \delta T + \delta^{-1} \sum_{i=1}^n \frac{\sqrt{\mu_i}}{\alpha - \mu_i}.$$

The right side of this inequality is minimized by choosing

$$\delta = T^{-1/2} \left(\sum_{i=1}^n \frac{\sqrt{\mu_i}}{\alpha - \mu_i} \right)^{1/2}.$$

Substitution of this value of δ into (1.18) and solving for T yields

$$(1.19) \quad T \geq \frac{m^2 n^2}{4} \left(\sum_{i=1}^n \frac{\sqrt{\mu_i}}{\alpha - \mu_i} \right)^{-1}.$$

From (ii) of Lemma 1 we also have

$$(1.20) \quad J \leq \sqrt{n} \frac{m + 2}{2m} \left(\sum_{i=1}^n \mu_i \right)^{1/2}.$$

Thus upon substituting (1.19) and (1.20) into (1.16) we arrive at the inequality

$$(1.21) \quad (\mu_{n+1} - \alpha)S \leq 2\sqrt{n}(m + 2) \left(\sum_{i=1}^n \mu_i \right)^{1/2} - \frac{m^2 n^2}{4} \left(\sum_{i=1}^n \frac{\sqrt{\mu_i}}{\alpha - \mu_i} \right)^{-1}.$$

Recall that α is restricted so that $\alpha > \mu_n$. We choose α so that the right side of (1.21) is zero. Thus

$$(1.22) \quad \sum_{i=1}^n \frac{\sqrt{\mu_i}}{\alpha - \mu_i} = \frac{m^2 n^{3/2}}{8(m + 2)} \left(\sum_{i=1}^n \mu_i \right)^{-1/2}.$$

Since the left side of (1.22) is a monotone decreasing function of α on (μ_n, ∞) , decreasing from $+\infty$ to 0^+ , such a choice of α exists and in fact is unique. With this choice of α we obtain from (1.21) that $\mu_{n+1} \leq \alpha$. Thus it is clear that the replacement of α in (1.22) by μ_{n+1} increases the left-hand side. Hence we obtain (1.10).

Inequality (1.11) is derived from (1.10) by replacing each μ_i with μ_n in the denominators of the left-hand side, and then solving for μ_{n+1} . Inequality (1.1) is in turn obtained from (1.11) by noting that

$$\sum_{i=1}^n \sqrt{\mu_i} \leq \sqrt{n} \left(\sum_{i=1}^n \mu_i \right)^{1/2}.$$

Thus the proof of Theorem 2 is complete.

2. Stronger inequalities for lower eigenvalues. For the case $n = 1$ the inequalities (1.1), (1.10), and (1.11) of §1 all reduce to the same estimate for μ_2 , namely

$$(2.1) \quad \mu_2 \leq \left(\frac{m+4}{m} \right)^2 \mu_1.$$

Thus in R^2 and R^3 , for example, we have the estimates

$$\mu_2 \leq 9\mu_1 \quad (\text{in } R^2), \quad \mu_2 \leq \left(\frac{7}{3} \right)^2 \mu_1 \doteq 5.445 \quad (\text{in } R^3).$$

We will now obtain improvements of (2.1) for all values of m . In particular, we will show that

$$\mu_2 \leq 7.103 \mu_1 \quad (\text{in } R^2), \quad \mu_2 \leq 4.792 \mu_1 \quad (\text{in } R^3).$$

We will also obtain improved estimates for certain other lower eigenvalues in dimensions $m = 2, 3, 4$. The method involves still further refinements of the arguments of §1. We retain the notation and terminology of the previous section.

The next theorem contains a partial improvement over Theorems 1 and 2 which will be stated explicitly in subsequent corollaries and theorems.

THEOREM 3. For $m \geq 2$, $n \geq 1$, and any constant $\sigma > 0$ we have

$$(2.2) \quad \begin{aligned} \mu_{n+1} &\leq (1 + \sigma)\mu_n + q(\sigma) \frac{M(m)}{n} \sum_{i=1}^n \mu_i \\ &\leq [1 + \sigma + q(\sigma)M(m)]\mu_n, \end{aligned}$$

where

$$q(\sigma) = \left[\frac{(1 + \sigma)^3}{\sigma} \right]^{1/2}, \quad M(m) = \frac{32}{3} \sqrt{\frac{2}{3}} m^{-1} (m + 2)^{-1/2}.$$

Proof. We begin again with inequality (1.4), apply (i) of Lemma 1, and introduce a real parameter β to obtain

$$(2.3) \quad \mu_{n+1} \sum_{i=1}^n \int \varphi_i^2 \leq \sum_{i=1}^n \mu_i \int \varphi_i^2 + 4(1 + \beta)J_1 - 4\beta J.$$

We also introduce parameters $\sigma > 0$, $\tau_i > 0$, $i = 1, \dots, n$, and apply Cauchy's inequality to J_1 to obtain

$$\begin{aligned}
 (2.4) \quad 4(1 + \beta)J_1 &= 4(1 + \beta) \sum_{i=1}^n \int -\nabla\varphi_i \cdot \nabla u_{ix_1} \\
 &\leq \sum_{i=1}^n 2\tau_i \int |\nabla\varphi_i|^2 + 2(1 + \beta)^2 \sum_{i=1}^n \tau_i^{-1} \int |\nabla u_{ix_1}|^2 \\
 &= \sum_{i=1}^n 2\tau_i \int -\varphi_i \Delta\varphi_i + 2(1 + \beta)^2 \sum_{i=1}^n \tau_i^{-1} \int u_{ix_1x_1} \Delta u_i.
 \end{aligned}$$

But by Cauchy's inequality and (1.3) we have

$$\begin{aligned}
 \sum_{i=1}^n 2\tau_i \int -\varphi_i \Delta\varphi_i &\leq \sum_{i=1}^n \sigma^{-1}\tau_i^2 \int \varphi_i^2 + \sum_{i=1}^n \sigma \int (\Delta\varphi_i)^2 \\
 &= \sum_{i=1}^n (\sigma\mu_i + \sigma^{-1}\tau_i^2) \int \varphi_i^2 + 4\sigma J.
 \end{aligned}$$

We substitute this expression into (2.4), and the resulting inequality into (2.3), and have as a result:

$$\begin{aligned}
 (2.5) \quad \mu_{n+1} \sum_{i=1}^n \int \varphi_i^2 &\leq \sum_{i=1}^n (\mu_i + \sigma\mu_i + \sigma^{-1}\tau_i^2) \int \varphi_i^2 \\
 &\quad + 2(1 + \beta)^2 \sum_{i=1}^n \tau_i^{-1} \int u_{ix_1x_1} \Delta u_i + 4(\sigma - \beta)J.
 \end{aligned}$$

In order to simplify (2.5) we choose each τ_i so that

$$(1 + \sigma)\mu_i + \sigma^{-1}\tau_i^2 = \tau, \quad i = 1, \dots, n,$$

where τ is a new parameter. The conditions $\tau_i, \sigma > 0$ require that $\tau > \mu_n$. In fact, we have

$$\begin{aligned}
 \tau &> (1 + \sigma)\mu_n > \mu_n, \\
 \tau_i &= [\tau - (1 + \sigma)\mu_i]^{1/2} \sigma^{1/2}, \quad i = 1, \dots, n, \\
 \tau_1 &\geq \tau_2 \geq \dots \geq \tau_n > 0.
 \end{aligned}$$

If we use trial functions φ_{ik} based on x_k instead of x_1 , inequality (2.5) has the counterpart

$$\begin{aligned}
 (2.6) \quad \mu_{n+1} \sum_{i=1}^n \int \varphi_{ik}^2 &\leq \tau \sum_{i=1}^n \int \varphi_{ik}^2 + 2(1 + \beta)^2 \sum_{i=1}^n \tau_i^{-1} \int u_{ix_kx_k} \Delta u_i \\
 &\quad + 4(\sigma - \beta)J, \quad k = 1, \dots, m.
 \end{aligned}$$

We sum these inequalities over k , using

$$\sum_{k=1}^m \int u_{ix_k x_k} \Delta u_i = \int (\Delta u_i)^2 = \mu_i,$$

and obtain

$$(2.7) \quad (\mu_{n+1} - \tau)S \leq 2(1 + \beta)^2 \sum_{i=1}^n \frac{\mu_i}{\tau_i} + 4m(\sigma - \beta)J.$$

The counterpart of (iii) of Lemma 1 for x_k is

$$\frac{n^2(m+2)}{8} \leq J \sum_{i=1}^n \varphi_{ik}^2, \quad k = 1, \dots, m.$$

Summing over k leads to

$$(2.8) \quad \frac{n^2 m(m+2)}{8} \leq JS.$$

By restricting $\sigma - \beta < 0$ we can use (2.8) to eliminate J in (2.7), and then multiply by S to obtain

$$(\mu_{n+1} - \tau)S^2 - 2(1 + \beta)^2 \sum_{i=1}^n \frac{\mu_i}{\tau_i} S + \frac{m^2}{2}(m+2)n^2(\beta - \sigma) \leq 0.$$

Therefore, we have a quadratic inequality in S of the form

$$aS^2 - 2bS + c \leq 0.$$

We can assert that

$$a \leq b^2 c^{-1},$$

for unless $a \leq 0$, in which case we are done, the quadratic equation $ax^2 - 2bx + c = 0$ must have a real root. Thus

$$\mu_{n+1} - \tau \leq \frac{2(1 + \beta)^4}{m^2(m+2)n^2(\beta - \sigma)} \left(\sum_{i=1}^n \frac{\mu_i}{\tau_i} \right)^2.$$

Since $f(\beta) = (1 + \beta)^4/(\beta - \sigma)$ is minimized by taking $\beta = (4\sigma + 1)/3$ we substitute this value in the preceding inequality to obtain

$$(2.9) \quad \mu_{n+1} \leq \tau + \frac{2(4^4)(1 + \sigma)^3}{3^3 m^2(m+2)n^2} \left(\sum_{i=1}^n \frac{\mu_i}{\tau_i} \right)^2.$$

Since $\tau_1 \geq \tau_2 \geq \dots \geq \tau_n$, replacing each τ_i by τ_n in (2.9) and further eliminating τ_n in favor of τ gives

$$(2.10) \quad \mu_{n+1} \leq \tau + \frac{2(4^4)(1 + \sigma)^3}{3^3 m^2(m+2)n^2 \sigma} \left(\sum_{i=1}^n \mu_i \right)^2 [\tau - (1 + \sigma)\mu_n]^{-1}.$$

The right side of inequality (2.10) has the form

$$\tau + A(\tau - B)^{-1},$$

which is minimized by letting $\tau = \sqrt{A} + B > (1 + \sigma)\mu_n$. Substitution of this value of τ into (2.10) gives the desired result (2.2). The weakened version of (2.2) is obtained by replacing each μ_i with μ_n , thereby completing the proof of Theorem 3.

If in inequality (2.2) of Theorem 3 we take $\sigma = .4$ for $m = 2, n = 1$, and $\sigma = .34$ for $m = 2, n = 2$, we obtain the following corollary.

COROLLARY 1. *For domains in R^2 we have*

$$\begin{aligned} \mu_2 &\leq 7.103 \mu_1, \\ \mu_3 &\leq 2.897 \mu_1 + 4.237 \mu_2. \end{aligned}$$

The corresponding inequalities obtained from (1.1) of Theorem 1 are

$$\mu_2 \leq 9 \mu_1, \quad \mu_3 \leq 4 \mu_1 + 5 \mu_2.$$

From (1.11) of Theorem 2 we also obtain

$$\mu_3 \leq \mu_2 + 2\sqrt{2}(\mu_1 + \mu_2)^{1/2}(\sqrt{\mu_1} + \sqrt{\mu_2}).$$

Inequality (1.10) of Theorem 2 also yields a different bound for μ_3 which is quite a bit more complicated.

General comparisons between Theorem 3 and Theorems 1 and 2 are difficult to make for general m and n . We will compare only Theorems 3 and 1, and only for a few simple cases. We first show that for μ_2 the best bound is given by Theorem 3 in all dimensions. We compare the two inequalities

$$(2.11) \quad \mu_{n+1} \leq \left(\frac{m+4}{m}\right)^2 \mu_n \quad (\text{Theorem 1}),$$

$$(2.12) \quad \mu_{n+1} \leq [1 + \sigma + q(\sigma)M(m)]\mu_n \quad (\text{Theorem 3}).$$

Inequality (2.12) holds for all $\sigma > 0$. Thus the best bound is obtained by choosing σ so that the right-hand side is minimized. In general a closed form expression for the optimal σ is difficult to attain, since one is required to solve a cubic equation. We will show nevertheless that this optimal σ always yields a better bound in (2.12) than the bound (2.11).

THEOREM 4. *For all $m \geq 2$ inequality (2.12) is strictly stronger than (2.11) provided that the optimal value of σ is chosen.*

Proof. Let us denote

$$H_m(\sigma) = 1 + \sigma + q(\sigma)M(m),$$

where $q(\sigma)$ and $M(m)$ are as described in Theorem 3. It must be shown that for some $\sigma > 0$ we have

$$H_m(\sigma) < \left(\frac{m+4}{m}\right)^2,$$

or, equivalently, for some $\sigma > 0$,

$$B_m(\sigma) > 1,$$

where we define

$$B_m(\sigma) = \sigma^{-1}[8(m+2)m^{-2} - M(m)q(\sigma)].$$

One can show after some lengthy computations that the maximum of $B_m(\sigma)$ occurs at

$$\sigma = \bar{\sigma}_m := \left[\frac{3}{8}(m+2)^3 m^{-2} - 1\right]^{-1},$$

where obviously $\bar{\sigma}_m > 0$, and the corresponding maximum of $B_m(\sigma)$ is

$$(2.13) \quad B_m(\bar{\sigma}_m) = m^{-4}[m^4 + 8m^2 + 32m + 16] > 1.$$

We denote by σ_m the optimal value of σ which minimizes $H_m(\sigma)$, and hence the right side of inequality (2.12). By programming a calculator we have estimated a few values of $H_m(\sigma_m)$, which we denote by H_m . We find

$$\begin{aligned} H_2 &\doteq 7.103, & H_3 &\doteq 4.792, & H_4 &\doteq 3.704, \\ H_5 &\doteq 3.081, & H_6 &\doteq 2.684, & H_7 &\doteq 2.409, \\ H_8 &\doteq 2.210, & H_9 &\doteq 2.058, & H_{10} &\doteq 1.939, \\ H_{100} &\doteq 1.081\ 578, & H_{1000} &\doteq 1.008\ 015\ 963. \end{aligned}$$

These numbers H_m give upper bounds for the ratios μ_{n+1}/μ_n for domains in R^m .

Next we compare in some special cases the following two inequalities, also appearing respectively in Theorems 1 and 3:

$$(2.14) \quad \mu_{n+1} \leq \mu_n + \frac{8(m+2)}{m^2 n} \sum_{i=1}^n \mu_i \quad (\text{Theorem 1})$$

$$(2.15) \quad \mu_{n+1} \leq (1 + \sigma)\mu_n + q(\sigma) \frac{M(m)}{n} \sum_{i=1}^n \mu_i \quad (\text{Theorem 3}).$$

THEOREM 5. *In the cases $m = 2, 3, 4$ and $n \leq 8, 3, 2$, respectively, inequality (2.15) is strictly stronger than (2.14) provided that the optimal value of σ is chosen.*

Proof. The right-hand sides of (2.14) and (2.15) may be rewritten as

$$P_{m,n} = \frac{8(m+2)}{m^2n} \sum_{i=1}^{n-1} \mu_i + \left[\frac{8(m+2)}{m^2n} \right] \mu_n,$$

$$H_{m,n}(\sigma) = \frac{1}{n} q(\sigma) M(m) \sum_{i=1}^{n-1} \mu_i + \left[1 + \sigma + \frac{1}{n} q(\sigma) M(m) \right] \mu_n.$$

Clearly, for any $\sigma > 0$ such that

$$(2.16) \quad \sigma + \frac{1}{n} q(\sigma) M(m) \leq \frac{8(m+2)}{m^2n},$$

we will have

$$H_{m,n}(\sigma) < P_{m,n}.$$

But (2.16) is equivalent to

$$(2.17) \quad B_m(\sigma) = \sigma^{-1} [8(m+2)m^{-2} - M(m)q(\sigma)] \geq n.$$

Now, from (2.13) we have

$$B_2(\bar{\sigma}_2) = 8, \quad B_3(\bar{\sigma}_3) \doteq 3.3, \quad B_4(\bar{\sigma}_4) \doteq 2.1.$$

Therefore, for example, for $m = 4$, if we restrict $n \leq 2$, then (2.71) is valid for $\sigma = \bar{\sigma}_4$. Similar statements hold for $m = 3, n \leq 3$, and for $m = 2, n \leq 8$.

3. Inequality for ν_2 . In a similar setting as before, let ν_1, ν_2 , with $\nu_1 \leq \nu_2$, be the first two eigenvalues of

$$\begin{aligned} \Delta^2 u + \nu \Delta u &= 0 \quad \text{in } D, \\ u = \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial D, \end{aligned}$$

where D is a bounded domain in R^m with sufficiently smooth boundary. The following theorem is a technical improvement and an extension to higher dimensions of a result in [6].

THEOREM 6. *For $m \geq 2$ we have*

$$\nu_2 \leq \frac{m^2 + 8m + 20}{(m+2)^2} \nu_1.$$

In particular, we have

$$\nu_2 \leq 2.5 \nu_1 \quad \text{if } m = 2, \quad \nu_2 \leq 2.12 \nu_1 \quad \text{if } m = 3.$$

Proof. Let $u = u_1$ be the eigenfunction corresponding to $\nu = \nu_1$, normalized so that

$$(3.1) \quad \int_D |\nabla u|^2 = 1.$$

As in §1, we may rotate the coordinate system so that

$$(3.2) \quad \int u_{x_1}^2 = \int u_{x_2}^2 = \cdots = \int u_{x_m}^2 = \frac{1}{m} \int |\nabla u|^2.$$

We may further perform a translation in order that

$$(3.3) \quad \int x_k |\nabla u|^2 = 0, \quad k = 1, \dots, m.$$

We start with the well-known inequality

$$(3.4) \quad \nu_2 \leq \frac{\int \varphi \Delta^2 \varphi}{\int |\nabla \varphi|^2},$$

which is satisfied by any sufficiently smooth function φ such that

$$\varphi = \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial D,$$

$$\int_D \nabla \varphi \cdot \nabla u = 0.$$

Following the method of Payne, Polya and Weinberger [6], we choose as our trial function

$$\varphi = x_1 u,$$

which clearly satisfies the above boundary condition, and also the orthogonality condition, since

$$\int \nabla \varphi \cdot \nabla u = \int x_1 |\nabla u|^2 + \int u u_{x_1} = 0 + 0,$$

in view of (3.3) and integration by parts. Hence $\varphi = x_1 u$ satisfies (3.4).

We next calculate the denominator of (3.4):

$$(3.5) \quad \int |\nabla \varphi|^2 = \int x_1^2 |\nabla u|^2 + 2 \int x_1 u u_{x_1} + \int u^2 = \int x_1^2 |\nabla u|^2.$$

As for the numerator of (3.4), we have

$$\int \varphi \Delta^2 \varphi = \int \varphi (x_1 \Delta^2 u + 4 \Delta u_{x_1}) = -\nu \int \varphi x_1 (\Delta u) + 4 \int \varphi \Delta u_{x_1}.$$

Also, by integration by parts and (3.5),

$$\begin{aligned} \int \varphi x_1 (\Delta u) &= - \int \nabla (x_1 \varphi) \cdot \nabla u \\ &= - \int x_1^2 |\nabla u|^2 - 2 \int x_1 u u_{x_1} = - \int |\nabla \varphi|^2 + \int u^2. \end{aligned}$$

Thus,

$$(3.6) \quad \int \varphi \Delta^2 \varphi = \nu \int |\nabla \varphi|^2 - \nu \int u^2 + 4 \int \varphi \Delta u_{x_1}.$$

We pause to make some technical calculations. We define

$$I = \int \varphi \Delta u_{x_1}.$$

LEMMA 2. *We have:*

- (i) $I = (m + 2)/2m$,
- (ii) $I^2 \leq \int |\nabla \varphi|^2 \int u_{x_1 x_1} (\Delta u)$,
- (iii) $\int u^2 \geq \nu^{-1}$.

Proof of Lemma 2. To prove (ii), we write

$$(3.7) \quad I = - \int \nabla \varphi \cdot \nabla u_{x_1}.$$

By the Schwarz inequality and integration by parts,

$$I^2 \leq \int |\nabla \varphi|^2 \int |\nabla u_{x_1}|^2 = \int |\nabla \varphi|^2 \int u_{x_1 x_1} (\Delta u).$$

To prove (i), we proceed from (3.7), using integration by parts and (3.2), (3.1), to obtain

$$\begin{aligned} I &= - \int (x_1 \nabla u \cdot \nabla u_{x_1} + u u_{x_1 x_1}) \\ &= \frac{1}{2} \int |\nabla u|^2 + \int u_{x_1}^2 = \frac{m+2}{2m} \int |\nabla u|^2 = \frac{m+2}{2m}. \end{aligned}$$

For (iii), working on (3.1) we have

$$\begin{aligned} 1 &= \int -u\Delta u = \left(\int -u\Delta u \right)^2 \leq \left(\int u^2 \right) \int (\Delta u)^2 \\ &= \int u^2 \left(\int u\Delta^2 u \right) = \left(\int u^2 \right) \nu \int |\nabla u|^2 = \nu \int u^2, \end{aligned}$$

which completes the proof of Lemma 2.

Returning to (3.4) and (3.6) we have, in view of Lemma 2,

$$\begin{aligned} \nu_2 \int |\nabla \varphi|^2 &\leq \int \varphi \Delta^2 \varphi \leq \nu \int |\nabla \varphi|^2 - \nu(\nu^{-1}) + 4I \\ &= \nu \int |\nabla \varphi|^2 + \frac{m+4}{m}. \end{aligned}$$

Now from Lemma 2 we also obtain

$$1 = \frac{2m}{m+2} I = \left(\frac{2m}{m+2} \right)^2 I^2 \leq \frac{4m^2}{(m+2)^2} \int |\nabla \varphi|^2 \int u_{x_1 x_1} \Delta u.$$

Thus division of (3.8) by $\int |\nabla \varphi|^2$ leads to

$$\nu_2 \leq \nu + \frac{4m(m+4)}{(m+2)^2} \int u_{x_1 x_1} \Delta u.$$

Had we used trial functions $\varphi_k = x_k u$, we would have obtained likewise

$$\nu_2 \leq \nu + \frac{4m(m+4)}{(m+2)^2} \int u_{x_k x_k} \Delta u, \quad k = 1, \dots, m.$$

Summing these inequalities over k we obtain

$$\nu_2 \leq \nu + \frac{4(m+4)}{(m+2)^2} \int (\Delta u)^2.$$

But, as we have seen,

$$\int (\Delta u)^2 = \nu.$$

Hence,

$$\nu_2 \leq \left[1 + \frac{4(m+4)}{(m+2)^2} \right] \nu = \frac{m^2 + 8m + 20}{(m+2)^2} \nu_1.$$

This completes the proof of Theorem 4.

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