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COMPACT QUOTIENTS BY C*-ACTIONS

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Let X be a compact normal complex space on which $\mathbf{C}^*$ acts 'in a nice manner'. We describe all invariant open subsets U of X such that the holomorphic map $U \to U/\mathbf{C}^*$ of U onto the categorical quotient for the category of compact complex spaces, $U/\mathbf{C}^*$, is locally Stein. The description depends on a partial ordering of the fixed point components which arises from the Bialynicki-Birula decompositions of X.

Introduction. Let ρ: $T \times X \to X$ be a meromorphic action, (cf. §1), of $T = \mathbf{C}^*$ on an irreducible compact normal complex analytic space X. Such an action is said to be locally linearizable if and only if given any $x \in X$ there is a T-invariant neighborhood V of x and a proper T-equivariant holomorphic embedding of V into $\mathbf{C}^n$ with T acting linearly on $\mathbf{C}^n$.

In this paper we solve the following problem:

Describe all T-invariant Zariski open subsets U of X, such that U/T is a compact complex analytic space and $U \to U/T$ is a semi-geometric quotient (i.e. a categorical quotient which is locally Stein cf. (1.8)).

This problem has been solved by A. Bialynicki-Birula and A. Sommese, [**B** − **B** + **S**], under the above setting when U contains no fixed points and by A. Bialynicki-Birula and J. Swiecieka, [**B** − **B** + **Sw**], when the action is algebraic and X is a compact algebraic variety.

As in [**B** − **B** + **S**], our description of semi-geometric quotients $U \to U/T$ is intimately linked to a certain partial ordering of the fixed point components $F_1,\ldots,F_r$. So that we can state our results precisely we shall introduce the following notation. We assume that all analytic spaces are Hausdorff, reduced and have countable topology.

Let $\{F_1,\ldots,F_r\}$ be the connected components of the fixed point set of T, X^T. Define ϕ^+, ϕ^-: $X \to X^T$ by $\phi^+(x) = \lim_{t\to 0} tx$ and $\phi^-(x) = \lim_{t\to\infty} tx$, respectively.

Let $X_i^+ = \{x \in X \mid \phi^+(x) \in F_i\}$, $i = 1,\ldots,r$, and $X_i^- = \{x \in X \mid \phi^-(x) \in F_i\}$, $i = 1,\ldots,r$.

An index i is said to be *directly less than* an index j if $C_{ij} = (X_i^+ - F_i) \cap (X_j^- - F_j) \neq \varnothing$. We say that i is *less than* j, denoted $i < j$, if there exists a sequence $i = i_0,\ldots,i_k = j$ such that i_l is directly less than i_{l+1} for $l = 0,\ldots,k-1$. This relation forms an ordering of the indices $\{1,\ldots,r\}$.

A *cross section* of $\{1,\ldots,r\}$ is a division of $\{1,\ldots,r\}$ into two non-empty disjoint subsets A^- and A^+ satisfying the condition that $i \in A^-$ and $j < i$ implies that $j \in A^-$.

A *semi-cross section* of $\{1,\ldots,r\}$ is a division of $\{1,\ldots,r\}$ into three disjoint subsets, A^-, A^0, A^+, at least two of which are nonempty, which satisfy the following two conditions:

(a) if $i < j$ and $j \in A^0$ then $i \notin A^0$

(b) if $A^+ \neq \varnothing$ then $(A^- \cup A^0, A^+)$ is a cross section and if $A^- \neq \varnothing$ then $(A^-, A^0 \cup A^+)$ is a cross section.

A subset B of X is a *semi-sectional set* if $B = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-$ for some semi-cross section (A^-, A^0, A^+).

MAIN THEOREM. *Let ρ: $T \times X \to X$ be as above and let U be a T-invariant Zariski open subset of X. Then U/T is a compact complex analytic space and $U \to U/T$ is a semi-geometric quotient if and only if U is a semi-sectional set with respect to some semi-cross section (A^-, A^0, A^+).* □

Our proof uses the techniques of [**B** − **B** + **S**].

We conclude this paper with a simple illustration of the Theorem for the case of a diagonal action of $\mathbf{C}^*$ on $\mathbf{P}^1 \times \mathbf{P}^1$.

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1. Notation and background material. In this section we establish the pertinent notation and background material needed for the proof of the Theorem. The principal reference for this material is [**B** − **B** + **S**].

Let T denote $\mathbf{C}^*$, the multiplicative group of non-zero complex numbers. A holomorphic action ρ: $T \times X \to X$ of T on a normal compact analytic space X is said to be a *meromorphic action* if ρ extends to a meromorphic map $\tilde{\rho}$: $\mathbf{P}^1 \times X \to X$, where $\mathbf{P}^1$ in one-dimensional complex projective space. This condition is satisfied if X is a Kaehler manifold and X^T has non-empty intersection with every connected component of X, [**So**].

The maps ϕ^+, ϕ^-: $X \to X^T$ as defined in the introduction always exist for meromorphic actions, [**Kor$_1$**]. The collections of subsets $\{X_i^+ \mid i = 1,\ldots,r\}$ and $\{X_i^- \mid i = 1,\ldots,r\}$ form two decompositions of the space X, called respectively the plus and the minus Bialynicki-Birula decompositions. They satisfy the following properties:

(1.1) (a) $X = \bigcup X_i^+ = \bigcup X_i^-$ is a disjoint union of T-invariant sets.

(b) There are two special components of X^T, F_1 called the *source* and F_r called the *sink* (renumbering if necessary), such that X_1^+ and X_r^- are Zariski open in X.

(c) Each X_i^+ and X_j^- is a constructible set, i.e., the finite union of locally closed sets.

These properties were proven in the algebraic category by Bialynicki-Birula, [**B** − **B**], and in the Kaehler category by Carrell and Sommese, [**C** + **S**], and Fujiki [**Fu**$_2$].

We will now state a result found in [**B** − **B** + **S**] which is modeled on a result of Fujiki [**Fu**$_2$]. It provides the basis for the proof of the Main Theorem.

THEOREM (1.2). *Let ρ: $T \times X \to X$ be a meromorphic action of T on an irreducible compact complex analytic space X. There is a diagram*:

$$\begin{array}{ccc} Z & \xrightarrow{\mu} & X \\ f\downarrow & & \\ Q & & \end{array}$$

with the following properties:

(a) *f is a flat morphism of irreducible compact complex spaces Z and Q.*

(b) *μ is a bimeromorphic holomorphic map of Z onto X such that the restriction of μ to each fiber $Z_q = f^{-1}(q)$ is an embedding.*

(c) *There is a natural holomorphic action of T on Z making f and μ T-equivariant with respect to the trivial action on Q and ρ on X respectively.*

(d) *There is a dense Zariski open subset $\mathcal{U}$ of Q such that for every $q \in \mathcal{U}$, Z_q is reduced and $\mu(Z_q)$ is the closure of a T-orbit from $X_1^+ \cap X_r^-$.*

(e) *Every fiber Z_q of f is one-dimensional and for fibers Z_q, $Z_{q'}$ that are reduced, $\mu(Z_q) = \mu(Z_{q'})$ if and only if $q = q'$.*

(f) *$\mu(Z_q)$ is connected and meets F_1, the source, and F_r, the sink, for all $q \in Q$.*

(g) *For all $q \in Q$, $Z_q \cap Z^T$ is finite.*

(h) *Any continuous map τ: $A \to Y$ of an open subset A of Q to a complex analytic space Y which is holomorphic on a Zariski open dense subset of A is holomorphic on all of A.* □

Let K be a compact complex space and let Comp(K) be the set of all compact subsets of K. The *Hausdorff metric* on Comp(K) is defined by:

$$\underline{\operatorname{dist}}(A, B) = \max_{a\in A}\left\{\min_{b\in B} \operatorname{dist}(a, b)\right\} + \max_{b\in B}\left\{\min_{a\in A} \operatorname{dist}(b, a)\right\}$$

where dist(a, b) is the metric on K. Let A, A_i, $i \in I$, be elements of Comp(K). When we say that the A_i's converge to A we mean they converge in the Hausdorff metric.

We have the following Corollary to (1.2).

COROLLARY (1.3). *Let ρ, X, Q, Z, F and μ be as in* (1.2). *Let $\{q_n\}$ be a sequence in Q. If q_n converges to q in Q then $\mu(Z_{q_n})$ converges to $\mu(Z_q)$ in X.*

Proof. We claim that q_n converges to q in Q implies that Z_{q_n} converges to Z_q in the Hausdorff metric in Z, where $Z_{q_n} = f^{-1}(q_n)$, $Z_q = f^{-1}(q)$. Let z be an arbitrary point of Z_q, then any open neighborhood V of z must intersect Z_{q_n} for $n \gg 0$. Suppose not, since f: $Z \to Q$ is flat it is an open map and hence $f(V)$ is an open neighborhood of q. If Z_{q_n} does not intersect V then q_n would not be an element of $f(V)$ and therefore q_n would not converge to q. Thus we have that Z_{q_n} converges to Z_q and by the continuity of μ: $Z \to X$ that $\mu(Z_{q_n})$ converges to $\mu(Z_q)$. □

DEFINITION (1.4). Let ρ: $T \times X \to X$ be a meromorphic action of T on a normal compact analytic space. We say that ρ is a *locally linearizable action* if given any $x \in X$ there is a T-invariant neighborhood V of x and a proper T-equivariant holomorphic embedding of V into $\mathbf{C}^N$ with T acting linearly on $\mathbf{C}^N$.

PROPOSITION (1.5). *A holomorphic action ρ: $T \times X \to X$ on a normal irreducible compact complex space X is locally linearizable if either of the following is true*:

(a) *X is an algebraic variety and ρ is an algebraic action or*

(b) *$X^T \neq \varnothing$ and X can be equivariantly embedded in a compact Kaehler manifold Y with a holomorphic action $\tilde{\rho}$: $T \times Y \to Y$.*

Proof. (a) is due to Sumihiro [**Su**] and b) is due to Koras [**Kor$_2$**]. □

We shall also use extensively the following (cf. Corollary (0.2.4) of [**B − B + S**]).

PROPOSITION (1.6). *Let ρ: $T \times X \to X$ be a locally linearizable action of T on a compact analytic space X. Given any $q \in Q$ we can choose $\{x_1, \ldots, x_k\}$ in $\mu(Z_q) - \mu(Z_q)^T$ with*:

(a) $\phi^+(x_1) \in F_1$ *and* $\phi^-(x_k) \in F_r$

(b) $\phi^-(x_j) = \phi^+(x_{j+1})$ *for* $j = 1, \ldots, k-1$

(c) *if* $\phi^-(x_j) = \phi^+(x_i)$, *then* $i = j + 1$
(d) $\overline{T\{x_1,\ldots,x_k\}} = \mu(Z_q)$

Moreover, if X is normal, then $\mu(Z_q) \cap F_1 = \{x_1\}$, $\mu(Z_q) \cap F_r = \{x_k\}$. □

We note that the last statement of Proposition (1.6) may not hold if X is not normal, i.e., it is possible in such a case that $F_1 = F_r$, for example simply identify a point of F_1 with a point of F_r.

COROLLARY (1.7). *Let X and ρ be as in* (1.6). *For any connected component* F_i *of* X^T, $F_1 < F_i < F_r$. □

Let $\bar{\rho}$: $G \times Z \to Z$ be an action of a reductive group G on complex space Z. We can definie an equivalence relation on the points of Z by $x \sim y$ if and only if there is a sequence of points $x = x_0, x_1,\ldots,x_n = y$ in Z such that $\overline{Gx_i} \cap \overline{Gx_{i+1}} \neq \emptyset$, $i = 0,\ldots,n-1$. We define Z/G to be the set of equivalence classes under the above relation and define a map π: $Z \to Z/G$ by $\pi(x) = [x]$, where $[x]$ denotes the equivalence class containing x. Z/G is given the quotient topology, i.e. V is an open subset of Z/T if and only if $\pi^{-1}(V)$ is an open subset of Z. We call π: $Z \to Z/G$ the *categorical quotient* of Z by G.

We note that in general our definition of a categorical quotient does not coincide with the usual definition, in which the equivalence relation is defined by the invariant holomorphic functions. Our definition implies that fibers of π are connected and thus the quotient, X/G, need not be Hausdorff. When the quotient is assumed to be Hausdorff either definition will suffice.

DEFINITION (1.8). A categorical quotient π: $Z \to Z/G$ is a *semi-geometric quotient* if it is locally Stein, i.e. given any point $y \in Z/G$ there is a neighborhood W of y such that $\pi^{-1}(W)$ is Stein.

LEMMA (1.9). *Let* ρ: $T \times X \to X$ *be a meromorphic action of T on X a compact complex analytic space. Let U be a T-invariant open subset of X. If* π: $U \to U/T$ *is a semi-geometric quotient then each fiber contains at most one fixed point.*

Proof. Let $x, y \in U^T$ and suppose $x \sim y$. Then we can find a sequence of fixed points in U, $x = z_0, z_1,\ldots,z_n = y$ such that $z_i \in [x]$ and z_i is directly related to z_{i+1}, i.e. there is a point $z \in U$ with $\phi^+(z) = z_i$ and $\phi^-(z) = z_{i+1}$ or $\phi^+(z) = z_{i+1}$ and $\phi^-(z) = z_i$. Thus, if $x \neq y$ then

$\pi^{-1}([x])$ contains $\phi^+(z) \cup Tz \cup \phi^-(z)$ which is homeomorphic to $\mathbf{P}^1$ contradicting the assumption that π is locally Stein. □

COROLLARY (1.10). *Let* ρ, U, *and* π *be as in* (1.9). *Then* π *restricted to* U^T *is one to one onto* $\pi(U^T)$. □

The above allows us to identify U^T with a subset of U/T, namely $\pi(U^T)$.

LEMMA (1.11). *Let* ρ, U, *and* π *be as in* (1.9). *Then fibers of* π *are either orbits or* $x^+ \cup x^-$ *for some* $x \in U^T$, *where* $x^+ = \{z \in X \mid \phi^+(z) = x\}$ *and* $x^- = \{z \in X \mid \phi^-(z) = x\}$.

Proof. By (1.9) each fiber contains at most one fixed point. If the fiber does not contain a fixed point then it must consist of a single orbit since the intersection of the closures of two distinct orbits is either empty or contained in the set of fixed points. If the fiber contains a fixed point then since U is open and T-invariant, it follows that the fiber contains $x^+ \cup x^-$. For the fiber to contain anything else it must contain a second fixed point which is impossible. Thus the fibers are as stated. □

LEMMA (1.12). *Let* ρ: $T \times X \to X$ *be a meromorphic action of* T *on* X *a normal compact complex analytic space* X. *Let* U *be a* T-*invariant open subset of* X *such that* ρ: $U \to U/T$ *is a semi-geometric quotient. If* U/T *is Hausdorff it possess the structure of a complex analytic space and* π *is a holomorphic map*.

Proof. The definition of a semi-geometric quotient implies that we may cover U with π-saturated Stein sets, A_i. Each A_i/T is a complex Stein space such that π: $A_i \to A_i/T$ is holomorphic, [**Sn**]. Since the structure on A_i/T is induced by the invariant holomorphic functions on A_i it follows easily that the structure on the A_i/T's are compatable. Thus, if U/T is Hausdorff it is a complex analytic space and π is holomorphic. □

2. Semi-geometric quotients. Throughout this section we shall assume that ρ: $T \times X \to X$ is a locally linearizable action of $T = \mathbf{C}^*$ on an irreducible normal compact complex analytic space X with fixed point components $F_1, \ldots, F_r$.

We want to describe all T-invariant Zariski open subsets U of X whose quotient U/T is semi-geometric and a compact complex space. The following propositions enable us to partition all such U into these three disjoint classes:

Class I. U contains no fixed point components, i.e. $U \subset X - X^T$.

Class II. The only fixed point component U contains is either the source F_1 or the sink F_r.

Class III. U contains fixed point components F_i, $i \neq 1, r$, and if U contains F_i and F_j they are not directly related to each other.

PROPOSITION (2.1). *Let U be a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space. If $F_i \cap U \neq \varnothing$, then $X_i^+ \cup X_i^- \subset U$.*

Proof. Let $x \in F_i \cap U$. Since U is open and T-invariant the sets x^+ and x^- must both be contained in U. Since $X_i^+ = \bigcup_{x \in F_i} x^+$ and $X_i^- = \bigcup_{x \in F_i} x^-$ we must only show that if $F_i \cap U \neq \varnothing$, then $F_i \subset U$. Furthermore, since $F_i \cap U$ is open in F_i, this reduces to showing that $F_i \cap U$ is closed.

Let $x \in \overline{F_i \cap U} \subset F_i$ and let $\{x_n\}$ be a sequence of distinct points contained in F_i converging to x. Since $U \to U/T$ is a semi-geometric quotient each distinct x_n must have a distinct image in U/T, and so we may consider $\{x_n\}$ as a sequence in U/T. Now U/T is assumed to be compact and so, passing to a subsequence and renumbering if necessary, we have that x_n converges to some $y \in U/T$. The locally Stein condition of semi-geometric quotients implies that we can find a neighborhood $W_T \subset U/T$ of y and a Stein set $W = \pi^{-1}(W_T) \subset U$.

We can assume that $\{x_n\}$ is contained in $F_i \cap W$, which is a closed T-invariant subset of W. By Corollary 3.6 of [**Sn**] since W is Stein we have that $\pi(F_i \cap W) = F_i \cap W_T$ is closed in W_T (we note that in this case our definition of categorical quotient coincides with that of [**Sn**]). This implies that $y \in \pi(F_i \cap W)$ and thus by identification, cf (1.10), $y \in F_i \cap W \subset U$. The convergence of the $\{x_n\}$ yields $y = x$. □

PROPOSITION (2.2). *Let U be as in* (2.1). *Let F_i and F_j be two fixed point components and suppose that $F_i \subset U$. If F_j is directly related to F_i then $F_j \not\subset U$.*

Proof. Assume $F_i < F_j$. Suppose $F_j \subset U$ then we can find an $x \in U$ such that $\phi^+(x) \in F_i$ and $\phi^-(x) \in F_j$. (2.1) implies that $U \supset \phi^+(x) \cup Tx \cup \phi^-(x)$ which is biholomorphic to $\mathbf{P}^1$. This contradicts the local Steinness of the quotient since there can be no neighborhood of $\pi(x)$ in U/T whose inverse image in U is Stein. □

PROPOSITION (2.3). *Let U be as in* (2.1). *If U contains the source, F_1, or the sink, F_r, then U does not contain any other fixed point component.*

Proof. Assume $U \supset F_1$. Since F_r is directly related to F_1, $F_r \not\subset U$. Let $F_i \subset U$, $i \neq 1, r$. Let $x \in F_i$ and choose $q \in Q$ such that $x \in \mu(Z_q)$, where Q, Z_q and μ are as in (1.2). Let $\mathcal{U}$ also be as in (1.2) and choose a sequence $\{q_n\} \subset \mathcal{U}$ converging to q. $\mu(Z_{q_n})$ converges to $\mu(Z_q)$ by (1.3). Thus we can find a sequence of points $\{x_n\} \subset U$ such that $x_n \in \mu(Z_{q_n})$ and x_n converges to x. Let $\{y_n\}$, y be the image of $\{x_n\}$, x respectively in U/T. $\{y_n\} \subset$ image of F_1, which is identified with F_1, but since $x \notin F_1$, $y \notin F_1$. But every open neighborhood of y meets $\{y_n\}$, so y is in the closure of F_1 in U/T. Since F_1 is closed this contradiction proves the proposition. □

We now show our Main Theorem holds for each of the three Classes separately.

Assume U is of Class I, i.e. $U \subset X - X^T$. This case was done in [**B** − **B** + **S**]. Their description is given in terms of cross sections. However, by considering a cross section (A^-, A^+) as a semi-cross section (A^-, A^0, A^+) with $A^0 = \varnothing$, their result coincides with our Main Theorem.

We next assume U is of Class II, i.e. the only fixed point component U contains is either F_1 or F_r. To simplify things we assume $F_1 \subset U$.

PROPOSITION (2.4). *Let U be as in the preceding paragraph. If the quotient U/T is semi-geometric and a compact complex space then there exists a semi-cross section $A = (A^-, A^0, A^+)$ such that U is a semi-sectional set with respect to A.*

Proof. Since $F_1 \subset U$ we have by (2.1) that $X_1^+ \subset U$. The proof of Proposition (2.3) can in fact be used to show that U contains only X_1^+, i.e. $U = X_1^+$. Hence:

$$U = X_1^+ = X - \bigcup_{i=2}^{r} X_i^+ = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-$$

where $A^+ = \{2,\ldots,r\}$, $A^- = \varnothing$. Taking $A^0 = \{1\}$ we have the desired semi-cross section. □

PROPOSITION (2.5). *Suppose U is the semi-sectional set associated to the semi-cross section (A^-, A^0, A^+) where $A^- = \varnothing$, $A^0 = \{1\}$, $A^+\{2,\ldots,r\}$. Then U is a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space.*

Proof. By definition we have:

$$U = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-.$$

Using the facts that $A^- = \varnothing$ and that $X = \bigcup_{i=1}^r X_i^+$ which is a disjoint union we have that $U = X_1^+$. Thus we have that U is a T-invariant Zariski open subset of X.

Since $U/T = X_1^+/T = F_1$ we have that U/T is a compact complex analytic space. For each $x \in F_1$ let V_x be a T-invariant Stein neighborhood of x in U given by definition of the action being locally linearizable. Then we have covered U/T by sets whose inverse images in U are Stein. U/T is obviously a categorical quotient and so it is a semi-geometrical quotient. The proposition is proven. □

Combining (2.4) and (2.5) gives the Main Theorem for Class II sets.

From now on unless stated otherwise, we assume that U is of Class III, i.e. U contains fixed point components F_i, $i \neq 1, r$ and any two are not directly related.

LEMMA (2.6). *Let U be a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space. Then $X_1^+ \cap X_r^- \subset U$.*

Proof. Let $C = X_1^+ \cap X_r^-$, then C is a Zariski open subset of X. Since X is assumed to be irreducible C must also be irreducible. By Zariski openness and denseness C must intersect U. The same proof as that of Lemma (1.1.1) in [**B** − **B** + **S**] yields that C is contained in U. □

Let $\mathfrak{U}$ be the subset of Q from Theorem (1.2). The above Lemma allows us to identify $\mathfrak{U}$ with a dense open subset of U/T.

We have need of the following fact. Let A be a complex space and let B be a dense subset of A. Let $\{x_n\}$ be a sequence of points of A, then we can find a sequence of points contained in B, $\{y_n\}$, such that $\operatorname{dist}(x_n, y_n) < 1/n$ where dist is the metric on A. If $\{x_n\}$ diverges then $\{y_n\}$ diverges and if $\{x_n\}$ converges then $\{y_n\}$ converges to the same point. Thus if we have a sequence in U/T we can assume it is contained in $\mathfrak{U}$.

We shall make the following convention. Let $y \in U/T$, when we choose a point $x \in \pi^{-1}(y)$ we assume x is the unique fixed point if $\pi^{-1}(y)$ contains one, otherwise x may be any point of $\pi^{-1}(y)$.

LEMMA (2.7). *Let U be a T-invariant Zariski open subset of X. Then U/T is a semi-geometric quotient and a compact complex space if and only if given $q \in 0$, either*:

(a) *There exists a $y \in X - X^T$ such that $\mu(Z_q) \cap U = Ty$ or*

(b) *There exist $y_1, y_2 \in X - X^T$ with $\phi^-(y_1) = \phi^+(y_2)$ such that $\mu(Z_q) \cap U = Ty_1 \cup \phi^-(y_1) \cup Ty_2$, where Q, Z_q and μ are as given in* (1.2).

Proof. To prove the necessity of (a) or (b) we first shall show that $\mu(Z_q) \cap U \neq \varnothing$. Suppose not, we can find a sequence $\{q_n\} \subset \mathcal{U} \subset Q$, such that q_n converges to q and thus $\mu(Z_{q_n})$ converges to $\mu(Z_q)$ in the Hausdorff metric. We note this implies that any open neighborhood of $\mu(Z_q)$ contaiins $\mu(Z_{q_n})$, $n \gg 0$. By (2.6) we can consider $\{q_n\} \subset U/T$. By assumption U/T is compact and therefore, after passing to a subsequence and renumbering if necessary, q_n converges to an element y of U/T. Let $x \in \pi^{-1}(y)$. Let V_1 and V_2 be disjoint open subsets of X which contain $\mu(Z_q)$ and x respectively. We can assume that $V_2 \subset U$. $\pi(V)$ contains a dense open subset consisting of elements of $\mathcal{U}$. Since q_n converges to y and $y \in \pi(V)$ we can replace elements of $\{q_n\}$ for $n \gg 0$, with elements of $\pi(V) \cap \mathcal{U}$ without affecting convergence, so we may consider $q_n \in \pi(V)$ for $n \gg 0$. This implies $\pi^{-1}(q_n) \cap V_2 \neq \varnothing$. But $\pi^{-1}(q_n) = \mu(Z_{q_n}) \cap U \subset V_1$. This contradiction implies that $\mu(Z_q) \cap U \neq \varnothing$.

We now claim that $\mu(Z_q) \cap U$ is connected. Obviously this is true if $q \in \mathcal{U}$. Suppose q is not an element of $\mathcal{U}$ and that $\mu(Z_q) \cap U$ is not connected. Then we can find two disjoint closed invariant sets, S_1 and S_2, with $S_1 \cup S_2 = \mu(Z_{q_n}) \cap U$. Note $\pi(S_1) \neq \pi(S_2)$. As before we can find a sequence $\{q_n\}$ contained in $\mathcal{U}$, such that $\mu(Z_{q_n})$ converges to $\mu(Z_q)$. Let $x_n = \pi(\mu(Z_{q_n}))$, then by continuity we have that x_n converges to both $\pi(S_1)$ and $\pi(S_2)$ in U/T. This contradicts U/T being Hausdorff.

$\mu(Z_q) \cap U$ can contain at most one fixed point since if it contained two, connectivity would imply that it contains $\mathbf{P}^1$ and then U/T could not be a semi-geometric quotient. If $\mu(Z_q) \cap U$ contains no fixed point it has the form of (a), if it has a fixed point, x, since $x^+ \cup x^- \subset U$ by (2.1) it has the form of (b).

Suppose $\mu(Z_q) \cap U$ is of the form (a) or (b) for any $q \in Q$. We will first show that π: $U \to U/T$ is a semi-geometric quotient. The fiber over any point in U/T must either be a single orbit or $x^+ \cup x^-$ for some fixed point x. This can be seen by considering $\mu(Z_q) \cap U$. If it is just an orbit then it goes to a point in U/T and is the fiber over that point. If it contains a fixed point then every $\mu(Z_{q'}) \cap U$ which contains the x will go to the same point in U/T. Thus the fibers are as stated above and it is easily seen that this implies that U/T is a categorical quotient. For each $y \in U/T$ choose $x \in \pi^{-1}(y)$. For each x let V_x be the T-invariant Stein neighborhood of x in X given by the action being locally linearizable. Since U is T-invariant we can consider that $V_x \subset U$ and by the description of the fibers that the V_x are π-saturated. Thus we can cover U/T with sets whose inverse images are Stein and, so the quotient U/T is semi-geometric.

Assume U/T is not Hausdorff, then we can find $\{y_n\} \subset U/T$ with y_n converging to two distinct points z_1 and z_2. We may assume $\{y_n\} \subset \mathfrak{U}$ and so $\{y_n\} \subset Q$ which is compact. We may assume, after passing to a subsequence and renumbering if necessary that y_n converges to $q \in Q$. Let $x_i \in \pi^{-1}(z_i)$ and V_i be an open neighborhood of z_i in U/T. $V_i \supset y_n$ for $n \gg 0$ and so we can find a sequence of points $\{x_{i,n}\} \subset U$ with $x_{i,n} \in \mu(Z_{y_n}) \cap \pi^{-1}(V_i)$ such that $x_{i,n}$ converges to x_i. Since $\mu\colon Z \to X$ is continuous the above implies that $\mu(Z_q)$ contains both x_1 and x_2. But since x_1 and x_2 are both contained in U this implies that their image in U/T must be the same, i.e. $z_1 = z_2$. This contradiction implies U/T is Hausdorff. Applying Proposition (1.10) gives us that U/T is a complex analyltic space.

It remains to show that U/T is compact. Let $\{x_n\}$ be a sequence in U/T. We can assume it is contained in $\mathfrak{U}$ and therefore in Q which is compact and so we can find a convergent subsequence $\{x'_m\}$ with x'_m converging to $q \in Q$. Let $x \in \mu(Z_q) \cap U$. Since x'_m converges to q we have that $\mu(Z_{x'_m})$ contained in U with $z_m \in \mu(Z_{x'_m}) \cap U$ and such that the z_m converges to x. By the continuity of $U \to U/T$ we have that x'_m converges to y where y is the image of x in U/T. Thus we have shown that every sequence in U/T has a convergent subsequence which converges to a point in U/T. Therefore, U/T is compact.

This completes the proof of Lemma (2.7). □

THEOREM (2.8). *Let $A = (A^-, A^0, A^+)$ be a semi-cross section. If U is the semi-sectional set which corresponds to the semi-cross section then U is a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space.*

Proof. Recall

$$U = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-.$$

It is obvious that U is T-invariant. The proof that U is Zariski open is the same as that given in Theorem (1.3) of [**B** − **B** + **S**].

Let $q \in Q$. We claim that $\mu(Z_q) \cap U \neq \varnothing$. Suppose not, then $\{h \in \{1,\ldots,r\} \mid F_h \cap \mu(Z_q) \neq \varnothing\}$ would all lie in A^- or all in A^+. This follows from Proposition (1.6) since otherwise we could find an $x \in \mu(Z_q)$ and an i and j with $j \in A^-$ and $i \in A^+$, such that $\phi^+(x) \in F_j$ and $\phi^-(x) \in F_i$. By the above description of U we see that this implies that $x \in U$ and thus that $\mu(Z_q) \cap U \neq \varnothing$. Therefore the set of h with $F_h \cap \mu(Z_q) \neq \varnothing$ lies totally in either A^- or A^+. By (1.6) we would have that either $r \in A^-$ or $1 \in A^+$. The former implies that $A^- = \{1,\ldots,r\}$ and the latter that $A^+ = \{1,\ldots,r\}$. In either case this would imply that $U = \varnothing$. Thus for all $q \in 0$ we must have that $\mu(Z_q) \cap U \neq \varnothing$.

Let $q \in Q$ and let y_1 and y_2 be two points in $X - X^T$ such that $Ty_1 \cup Ty_2$ is contained in $\mu(Z_q) \cap U$. We claim that either $Ty_1 = Ty_2$ or either $\phi^+(y_1) = \phi^-(y_2)$ or $\phi^-(y_1) = \phi^+(y_2)$. Suppose $Ty_1 \neq Ty_2$. Under this condition assume also that $\phi^+(y_1) \neq \phi^-(y_2)$ and $\phi^-(y_1) \neq \phi^+(y_2)$. Then again applying (1.6) we can find a and b such that either $\phi^+(y_2) \in F_b$ and $\phi^-(y_1) \in F_a$ and $a < b$ or $\phi^+(y_1) \in F_b$ and $\phi^-(y_2) \in F_a$ and $a < b$. Either way we get a contradiction. In the former case if $a \in A^-$ then $y_1 \in F_a^-$ and is not in U, if $a \in A^0 \cup A^+$ then $b \in A^+$, (since (A^-, A^0, A^+) is a semi-cross section), and therefore $y_2 \in X_b^+$ and thus not in U. Likewise the latter case also implies that either y_1 or y_2 is not an element of U.

Assume that in fact $\phi^-(y_1) = \phi^+(y_2) = x$. Let $x \in F_k$, then $k \in A^0$, since otherwise if $k \in A^-$ this would mean y_1 is not an element of U and if $k \in A^+$ this would mean y_2 is not an element of U. Hence $x \in U$.

Therefore, we have shown that for every $q \in 0$ either $\mu(Z_q) \cap U = Ty$ for some $y \in X - X^T$ or $\mu(Z_q) \cap U = Ty_1 \cup \phi^-(y_1) \cup Ty_2$ for some y_1 and y_2 in $X - X^T$. Applying Lemma (2.7) finishes the proof. □

LEMMA (2.9). *Let U be a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space. Let $\{F_k\}$ be the set of fixed point components contained in U. Let $U' = U - \bigcup X_k^-$. Then U' is a Class* I *T-invariant Zariski open subset of X whose quotient U'/T is semi-geometric and a compact complex space.*

Proof. U' is obviously T-invariant and contained in $X - X^T$. Lemma (1.3.1) of [**B** − **B** + **S**] shows that $\bigcup X_k^-$ is a closed set and thus U' is an open constructible set and therefore is Zariski open.

For all $q \in Q$ we can consider $\mu(Z_q) \cap U'$ which is contained in $\mu(Z_q) \cap U$. If $\mu(Z_q) \cap U = Ty$ for some $y \in X - X^T$ then y is not an element of X_k for any F_k contained in U and so $\mu(Z_q) \cap U' = Ty$. If $\mu(Z_q) \cap U = Ty_1 \cup \phi^-(y_1) \cup Ty_2$, for some $y_1, y_2 \in X - X^T$, we have that $y_1 \in X_k^-$ for some F_k contained in U but that y_2 is not an element of X_k^- for any F_k contained in$U$ and thus that $\mu(Z_q) \cap U' = Ty_2$. Hence we have that for every $q \in Q$ there is an $y \in X - X^T$ sjuch that $\mu(Z_q) \cap U' = Ty$. Applying Lemma (1.2) of [**B** − **B** + **X**] gives the desired result. □

REMARK (2.10). In [**B** − **B** + **S**] it is shown that a Class I T-invariant Zariski open subset U of X has a compact complex space as quotient if and only if $(X - U)$ has two connected components, one which contains the source, F_1, and the other which contains the sink, F_r. We will use this fact in the next theorem.

THEOREM (2.11). *Let U be a T-invariant Zariski open subset of X whose quotient U/T is semi-geometric and a compact complex space. Then U is a semi-cross sectional set with respect to some semi-cross section (A^-, A^0, A^+).*

Proof. Given U let U' be the corresponding Class I set given by Lemma (2.9), i.e. $U' = U - \bigcup(X_{k_i}^-)$ where $\{F_{k_1},\ldots,F_{k_n}\}$ is the set of fixed point components contained in U. As noted in Remark (2.10), $(X - U')$ has two connected components, one containing F_1 and the other F_r. Since we assume that U does not contain either F_1 or F_r we must have that $\bigcup(X_{k_i}^-)$ does not contain them either. Therefore $(X - U') - \bigcup(X_{k_i}^-)$ must be disconnected, since F_1 and F_r are still in different components. But $(X - U') - \bigcup(X_{k_i}^-) = (X - U)$. Thus we have that $(X - U)$ is disconnected and that F_1 and F_r are in different components.

Let A_1 be the connected component of $X - U$ which contains F_1 and let A_2 be the connected component of $X - U$ which contains F_r. Assume there was another connected component of $X - U$ besides A_1 and A_2, call it A_3. Let $x \in A_3$ and choose a $q \in Q$ such that $x \in \mu(Z_q)$. By Lemma (2.7) we know that $\mu(Z_q) \cap U$ is either Ty for some $y \in X - X^T$ or $Ty_1 \cup \phi^-(y_1) \cup Ty_2$, for some $y_1, y_2 \in X - X^T$. In either case (1.6) implies that $\mu(Z_q) \cap (X - U)$ has two connected components, one which intersects F_1 and another which intersects F_r. Thus x must be in the same connected component of $X - U$ as F_1 or F_r, i.e. $A_3 = A_1$ or $A_3 = A_2$. Therefore, $X - U$ has exactly two connected components.

Let $\{F_1,\ldots,F_r\}$ be the set of connected components of X^T. Set $A^- = \{j\colon F_j$ is contained in $A_1\}$, $A^0 = \{k\colon F_k$ is contained in $U\}$ and $A^+ = \{i\colon F_i$ is contained in $A_2\}$. We claim that (A^-, A^0, A^+) forms a semi-cross section of $\{1,\ldots,r\}$. Let $j \in A^-$ and suppose j' is directly less than j. We can find $x_j \in F_j$, $x_{j'} \in F_{j'}$, and $x \in X$ such that $\phi^+(x) = x_{j'}$, and $\phi^{-}(x) = x_j$. Thus we can find a $q \in Q$ with $\mu(Z_q)$ containing $\{x_j, x_{j'}, x\}$. Looking at $\mu(Z_q) \cap (X - U)$ we see that $F_{j'}$ is contained in A_1, i.e. $j' \in A^-$. A finite application of the above step shows that $j' \in A^-$ for any $j' < j$. Now let $k \in A^0$ and suppose j is directly less than k. Proposition (2.2) shows that F_j is not contained in U and (1.6) implies that it must be contained in A_1. Thus $j \in A^-$ and therefore so is any $j' < k$. Now suppose there was a $k' \in A^0$ with $k < k'$, the above implies that $k \in A$. Therefore we see that if $k \in A^0$ and if k' is related to k then k' is not an element of A^0. It is also obvious by what we have shown that if $k \in A^0$ and if $k < i$ then $i \in A^+$. Hence (A^-, A^0, A^+) forms a semi-cross section of $\{1, \ldots, r\}$.

Let $x \in U$. Then either $x \in (X_k^+ \cup X_k^-)$, for some $k \in A^0$, or $x \in X_i^- \cap X_j^+$, for some $i \in A^+$ and $j \in A^-$. Therefore since U, A_1 and A_2 are T-invariant and the points of U satisfy the conditions stated above we have that U is given by:

$$U = X - \bigcup_{i \in A^+} X_i^+ - \bigcup_{j \in A^-} X_j^-.$$

Hence U is a semi-sectional set with respect to the semi-cross section (A^-, A^0, A^+).

Combining (2.8) and (2.11) yields the Main Theorem for Class III sets. □

3. An Example. Let T act on $\mathbf{P}^1 \times \mathbf{P}^1$ by $t([z_0 : z_1], [w_0 : w_1]) = ([z_0 : tz_1], [w_0 : tw_1])$. There are four fixed points of this section, $F_1 = ([1:0],[1:0])$, $F_2 = ([1:0],[0:1])$, $F_3 = ([0:1],[1:0])$ and $F_4 = ([0:1],[0:1])$. The plus decomposition is given by

$$X_1^+ = \mathbf{P}^1 \times \mathbf{P}^1 - \{z_0 = 0 \text{ or } w_0 = 0\},$$

$$X_2^+ = \{([z_0 : z_1], [0:1]) : z_0 \neq 0\},$$

$$X_3^+ = \{([0:1], [w_0 : w_1]) : w_0 \neq 0\} \text{ and}$$

$$X_4^+ = ([0:1],[0:1]).$$

The minus decomposition is given by

$$X_1^- = ([1:0],[1:0]),$$

$$X_2^- = \{([1:0], [w_0 : w_1]) : w_1 \neq 0\},$$

$$X_3^- = \{([z_0 : z_1], [0:1]) : z_1 \neq 0\} \text{ and}$$

$$X_4^- = \mathbf{P}^1 \times \mathbf{P}^1 + \{z_1 = 0 \text{ or } w_1 = 0\}.$$

Hence $([1:0],[1:0])$ is the source and $([0:1],[0:1])$ is the sink.

The following chart describes the possible T-invariant open subsets U of X whose quotient U/T is semi-geometric and a compact complex space:

Class	A^-	A^0	A^+	U	U/T
I	$\{1,2,3\}$	$\varnothing$	$\{4\}$	$\mathbf{C}^2 - 0$	$\mathbf{P}^1$
I	$\{1,2\}$	$\varnothing$	$\{3,4\}$	$\mathbf{C}^* \times \mathbf{P}^1$	$\mathbf{P}^1$
I	$\{1,3\}$	$\varnothing$	$\{2,4\}$	$\mathbf{C}^* \times \mathbf{P}^1$	$\mathbf{P}^1$
I	$\{1\}$	$\varnothing$	$\{2,3,4\}$	$\mathbf{C}^2 - 0$	$\mathbf{P}^1$
II	$\varnothing$	$\{1\}$	$\{2,3,4\}$	$\mathbf{C}^2$	point
II	$\{1,2,3\}$	$\{4\}$	$\varnothing$	$\mathbf{C}^2$	point
III	$\{1,2\}$	$\{3\}$	$\{4\}$	$\mathbf{P}^1 \times \mathbf{P}^1 - ([0:1],[0:1])$ $-\{([1:0],[w_0:w_1])\}$	$\mathbf{P}^1$
III	$\{1,3\}$	$\{2\}$	$\{4\}$	$\mathbf{P}^1 \times \mathbf{P}^1 - ([0:1],[0:1])$ $-\{([z_0:z_1],[1:0])\}$	$\mathbf{P}_1$
III	$\{1\}$	$\{2,3\}$	$\{4\}$	$\mathbf{P}^1 \times \mathbf{P}^1 - ([0:1],[0:1])$ $-\{([1:0],[1:0])\}$	$\mathbf{P}_1$
III	$\{1\}$	$\{3\}$	$\{2,4\}$	$\mathbf{P}^1 \times \mathbf{P}^2 - ([1:0],[1:0])$ $-\{([z_0:z_1],[0:1])\}$	$\mathbf{P}^1$
III	$\{1\}$	$\{2\}$	$\{3,4\}$	$\mathbf{P}^1 \times \mathbf{P}^1 - ([1:0],[1:0])$ $-\{([1:0],[w_0:w_1])\}$	$\mathbf{P}^1$

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