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A SPECTRAL DUALITY THEOREM FOR CLOSED OPERATORS

I. ERDELYI AND WANG SHENGWANG

The spectral duality theorem asserts that a densely defined closed operator T induces a spectral decomposition of the underlying Banach space X iff the conjugate T^* induces the same type of spectral decomposition of the dual space X^*. This theorem is known for bounded linear operators in terms of residual (S)-decomposability. In this paper we extend the spectral duality theorem to unbounded operators, under a general type of spectral decomposition. Our approach to the spectral duality leads through the successive conjugates T^*, T^{**} and T^{***} of T, under their domain-density assumptions.

1. Elements of local spectral theory for a closed operator. X is an abstract Banach space over the complex field $\mathbf{C}$. If S is a set, we write $\bar{S}$ for the closure, $\operatorname{Int} S$ for the interior, S^c for the complement, ∂S for the boundary, and $\operatorname{cov} S$ for the collection of all finite open covers of S. If S is a subset of $\mathbf{C}$, then the above mentioned topological constructs are referred to the topology of $\mathbf{C}$. Without loss of generality, we assume that for $S \subset \mathbf{C}$, each $\{G_i\}_{i=0}^n \in \operatorname{cov} S$ has, at most, one unbounded set G_0. An open $G \subset \mathbf{C}$ is said to be a neighborhood of ∞, in symbols $G \in V_\infty$, if for $r > 0$ sufficiently large, $\{\lambda \in \mathbf{C}: |\lambda| > r\} \subset G$. We write $S^\perp$ for the annihilator of $S \subset X$ in X^* (as well as that of $S \subset X^{**}$ in X^{***}) and ${}^\perp S$ for the preannihilator of $S \subset X^*$ in X (or that of $S \subset X^{***}$ in X^{**}). $B(X)$ denotes the Banach algebra of all bounded linear operators which map X into X. I stands for the identity operator.

For a linear operator T: D_T $(\subset X) \to X$, we use the following notations: spectrum $\sigma(T)$, resolvent set $\rho(T)$, and resolvent operator $R(\cdot\,; T)$.

If T has the single valued extension property (SVEP) then, for $x \in X$, $\sigma_T(x)$ is the local spectrum, $\rho_T(x)$ is the local resolvent set and $x(\cdot)$ is the local resolvent function. For $S \subset \mathbf{C}$, an extensive use will be made of the spectral manifold $X(T, S) = \{x \in X: \sigma_T(x) \subset S\}$.

$\operatorname{Inv} T$ represents the lattice of all invariant subspaces under T. For $Y \in \operatorname{Inv} T$, $T|Y$ is the restriction of T to Y and T/Y denotes the coinduced operator on the quotient space X/Y with domain $D_{T/Y} = \{\hat{x} \in X/Y: \hat{x} \cap D_T \neq \varnothing\}$.

If not mentioned otherwise, throughout this paper T is a densely defined unbounded closed operator with domain and range in X.

Given T, the following domain-density conditions will guarantee the existence of the successive conjugates:

($*$) T^* is densely defined;

($**$) T^* and T^{**} are densely defined;

($***$) T^*, T^{**} and T^{***} are densely defined.

With J and K, the natural embeddings of X into X^{**} and of X^* into X^{***}, respectively, we shall explore the direct sum decomposition

$$X^{***} = KX^* \oplus (JX)^{\perp}. \tag{1.1}$$

For completeness, we give a short proof of (1.1), (e.g. [**10**]). For every $x^{***} \in X^{***}$, one defines $x^* \in X^*$ by

$$\langle x, x^* \rangle = \langle Jx, x^{***} \rangle, \qquad x \in X.$$

Then, $\langle Jx, Kx^* \rangle = \langle x^*, Jx \rangle = \langle x, x^* \rangle = \langle Jx, x^{***} \rangle$ and hence $y^{***} = x^{***} - Kx^* \in (JX)^{\perp}$. This, together with $KX^* \cap (JX)^{\perp} = \{0\}$, establishes (1.1).

The spectral theoretic results will be expressed in terms of operators with the spectral decomposition property, decomposable operators and $\{\infty\}$-decomposable operators.

1.1. DEFINITION. T is said to have the spectral decomposition property (SDP) if, for any $\{G_i\}_{i=0}^n \in \operatorname{cov} \sigma(T)$ with $G_0 \in V_\infty$, there is a system $\{Y_i\}_{i=0}^n \subset \operatorname{Inv} T$ satisfying the following conditions:

(I) $Y_i \subset D_T$ if G_i is relatively compact in $\mathbf{C}$ $(1 \le i \le n)$;

(II) $X = \sum_{i=1}^n Y_i$ and $\sigma(T \mid Y_i) \subset G_i$, $0 \le i \le n$.

If we restrict n to $n = 1$ then T is said to have the 1-SDP.

The concept of spectral maximal space [**3**] has two distinct extensions to the case of unbounded operators.

1.2. DEFINITION. $Y \in \operatorname{Inv} T$ is called a spectral maximal space (SMS) of T if, for any $Z \in \operatorname{Inv} T$, the inclusion $\sigma(T \mid Z) \subset \sigma(T \mid Y)$ implies $Z \subset Y$.

1.3. DEFINITION. $Y \in \operatorname{Inv} T$ is said to be a T-bounded spectral maximal space (T-bounded SMS) if

(i) $Y \subset D_T$;

(ii) for every $Z \in \operatorname{Inv} T$, $Z \subset D_T$ and $\sigma(T \mid Z) \subset \sigma(T \mid Y)$ imply $Z \subset Y$.

This concept appears in [**8**] under the name of maximal invariant space. Clearly, every SMS of T is a T-bounded SMS. Conversely, however, not every T-bounded SMS is a SMS of T. In fact, if Y is a T-bounded

SMS and $Z \in \operatorname{Inv} T$ is not contained in D_T, then $\sigma(T|Z) \subset \sigma(T|Y)$ need not imply $Z \subset Y$. In the bounded case, the two concepts coincide.

The following properties of the spectral manifold $X(T, \cdot)$ for closed T are analogous, in statement and proof, to the one for a bounded operator [**3**, **5**].

1.4. PROPOSITION. *Let T have the SVEP. If, for closed $F \subset \mathbf{C}$, $X(T, F)$ is closed then $X(T, F)$ is a SMS of T and*

$$\sigma[T|X(T, F)] \subset F \cap \sigma(T). \tag{1.2}$$

Moreover, if T has the 1-*SDP then, for every closed $F \subset \mathbf{C}$, $X(T, F)$ is closed.*

1.5. PROPOSITION. *Given T, let $Y \in \operatorname{Inv} T$ be such that $\sigma(T|Y)$ is compact in $\mathbf{C}$. There exist Υ, $W \in \operatorname{Inv} T$ with the following properties*:

(I) $Y = \Upsilon \oplus W$, $\sigma(T|\Upsilon) = \sigma(T|Y)$, $\sigma(T|W) = \varnothing$;

(II) $\Upsilon \subset D_T$.

Proof. $\sigma_1 = \sigma(T|Y)$ and $\sigma_2 = \varnothing$ can be regarded as disjoint spectral sets of $\sigma(T|Y)$. Thus, the functional calculus produces (I). For a bounded Cauchy domain $\Delta \supset \sigma(T|Y)$, Υ and W can be expressed in terms of the spectral projection

$$Q = \frac{1}{2\pi i}\int_{\partial\Delta} R(\lambda; T|Y)\, d\lambda \tag{1.3}$$

(independent of the choice of Δ) as follows: $\Upsilon = QY$, $W = (I_Y - Q)Y$, where I_Y is the identity in Y. Since T is closed, it follows easily that Υ, $W \in \operatorname{Inv} T$ and $\Upsilon \subset D_T$. □

1.6. THEOREM. *Given T, let Y be a SMS of T with $\sigma(T|Y)$ compact in $\mathbf{C}$. Then Υ, as defined by Proposition* 1.5, *is a T-bounded SMS.*

Proof. Let $Z \in \operatorname{Inv} T$ be such that $Z \subset D_T$ and

$$\sigma(T|Z) \subset \sigma(T|\Upsilon). \tag{1.4}$$

By Proposition 1.5, (1.4) implies $\sigma(T|Z) \subset \sigma(T|Y)$ and since Y is a SMS of T, we have $Z \subset Y$. Then, for $x \in Z$, $\lambda \in \rho(T|Y)$, relation

$$R(\lambda; T|Z)x = R(\lambda; T|Y)x$$

implies

$$Qx = \frac{1}{2\pi i}\int_{\partial\Delta} R(\lambda;\, T|Y)x\,d\lambda = \frac{1}{2\pi i}\int_{\partial\Delta} R(\lambda;\, T|Z)x\,d\lambda = x,$$

where $\Delta \supset \sigma(T\,|\,Y)$ is a bounded Cauchy domain and Q is the projection (1.3). Thus, we have $x = Qx \in \Upsilon$ and hence $Z \subset \Upsilon$. □

The next theorem (partly adopted from **[11]**) gives some necessary and sufficient conditions for a T-bounded SMS to be a SMS of T.

1.7. THEOREM. *Given T, the following assertions are equivalent*:
(I) $\{0\}$ *is a SMS of T*;
(II) *for every $Y \in \operatorname{Inv} T$ with $\sigma(T\,|\,Y)$ compact in* $\mathbf{C}$, *we have $Y \subset D_T$*;
(III) *for every $Y \in \operatorname{Inv} T$, $Y \neq \{0\}$ implies that $\sigma(T\,|\,Y) \neq \varnothing$*;
(IV) *every T-bounded SMS is a SMS of T.*

Proof. (I) ⇒ (II). Given $Y \in \operatorname{Inv} T$ with $\sigma(T\,|\,Y)$ compact in $\mathbf{C}$, Proposition 1.5 gives $Y = \Upsilon \oplus W$, $\Upsilon \subset D_T$, $\sigma(T\,|\,\Upsilon) = \sigma(T\,|\,Y)$, $\sigma(T\,|\,W) = \varnothing$. Then, by hypothesis, $W = \{0\}$ and hence $Y = \Upsilon \subset D_T$.

(II) ⇒ (III). Let $Y \in \operatorname{Inv} T$ be such that $Y \neq \{0\}$ and suppose that $\sigma(T\,|\,Y) = \varnothing$. $\sigma(T\,|\,Y)$ being compact in $\mathbf{C}$, $Y \subset D_T$. Hence $T\,|\,Y$ is bounded and $Y \neq \{0\}$ implies that $\sigma(T\,|\,Y) \neq \varnothing$. This, however, contradicts the assumption on $\sigma(T\,|\,Y)$.

(III) ⇒ (IV): Let Z be a T-bounded SMS and let $Y \in \operatorname{Inv} T$ be such that

$$\sigma(T|Y) \subset \sigma(T|Z). \tag{1.5}$$

$\sigma(T\,|\,Z)$ being compact in $\mathbf{C}$, so is $\sigma(T\,|\,Y)$. It follows from Proposition 1.5 that $Y = \Upsilon \oplus W$ with $\Upsilon, W \in \operatorname{Inv} T$, $\Upsilon \subset D_T$, $\sigma(T\,|\,\Upsilon) = \sigma(T\,|\,Y)$ and

$$\sigma(T|W) = \varnothing. \tag{1.6}$$

By hypothesis, (1.6) implies that $W = \{0\}$ and hence

$$Y = \Upsilon \subset D_T. \tag{1.7}$$

It follows from (1.5), (1.7) that $Y \subset Z$ and hence Z is a SMS of T.

(IV) ⇒ (I). Evidently, $\{0\}$ is a T-bounded SMS and hence $\{0\}$ is a SMS of T, by hypothesis. □

1.8. LEMMA. *Let T have the SVEP. If, for $Y \in \operatorname{Inv} T$, $T\,|\,Y$ is bounded then $x \in Y$ and $\sigma_T(x) = \varnothing$ imply $x = 0$.*

Proof. By $\sigma_T(x) = \varnothing$, the local resolvent is an entire function. For $x \in Y$, the SVEP implies

$$x(\lambda) = R(\lambda;\, T|Y)x, \qquad |\lambda| > \|T|Y\|.$$

Consequently, for $\Gamma = \{\lambda \in \mathbf{C}\colon |\lambda| = \|T|Y\| + 1\}$, we have

$$x = \frac{1}{2\pi i}\int_\Gamma R(\lambda;\, T|Y)x\, d\lambda = \frac{1}{2\pi i}\int_\Gamma x(\lambda)\, d\lambda = 0. \qquad \square$$

1.9. THEOREM. *If T has the* 1-*SDP then, for every compact $F \subset \mathbf{C}$, there exists a T-bounded SMS $\Xi(T, F)$ with the following properties*:

(I) $X(T, F) = \Xi(T, F) \oplus X(T, \varnothing)$;

(II) $\sigma[T|\Xi(T, F)] = \sigma[T|X(T, F)]$.

Proof. Since $\sigma[T|X(T, F)] \subset F$ is compact in $\mathbf{C}$, for $Y = X(T, F)$ and $\Upsilon = \Xi(T, F)$, Proposition 1.5 gives

$$X(T, F) = \Xi(T, F) \oplus W,$$
$$\sigma\big[T|\Xi(T, F)\big] = \sigma\big[T|X(T, F)\big], \quad \sigma(T|W) = \varnothing.$$

$X(T, F)$ being a SMS of T, $\Xi(T, F)$ is a T-bounded SMS, by Theorem 1.6. Since $X(T, \varnothing)$ is a SMS of T, $\sigma(T|W) = \varnothing$ implies $W \subset X(T, \varnothing)$. Conversely, let $x \in X(T, \varnothing)$. Then $x \in X(T, F)$ and hence $Qx \in \Xi(T, F)$, where Q is the projection (1.3) for $Y = X(T, F)$. Since Q commutes with T, we have $\sigma_T(Qx) \subset \sigma_T(x) = \varnothing$ and hence $Qx = 0$, by Lemma 1.8. Thus, $X(T, \varnothing) \subset W$ and hence $W = X(T, \varnothing)$. $\square$

1.10. PROPOSITION. *Given T, every T-bounded SMS and every SMS of T is hyperinvariant under T.*

Proof. We confine the proof to a T-bounded SMS. Let $A \in B(X)$ commute with T and fix $\lambda \in \mathbf{C}$ with $|\lambda| > \|A\|$. Then $R(\lambda;\, A) = \sum_{n=0}^{\infty} \lambda^{-n-1}A^n$. For every $x \in D_T$ and positive integer k, we have

$$\sum_{n=0}^{k} \lambda^{-n-1}A^nTx = T\left(\sum_{n=0}^{k} \lambda^{-n-1}A^nx\right).$$

T being closed, $k \to \infty$ implies that $R(\lambda;\, A)x \in D_T$ and

$$R(\lambda;\, A)Tx = TR(\lambda;\, A)x. \tag{1.8}$$

Now, let Y be a T-bounded SMS and put $Y_\lambda = R(\lambda;\, A)Y$. Since $Y \subset D_T$, we have $Y_\lambda \subset D_T$ and $Y_\lambda \in \operatorname{Inv} T$. For $x \in Y$, (1.8) implies

$$R(\lambda; A)^{-1}(T|Y_\lambda)R(\lambda; A)x = (T|Y)x,$$

hence $T|\,Y_\lambda$ and $T|\,Y$ are similar. Thus, $\sigma(T|\,Y_\lambda) = \sigma(T|\,Y)$. Since $Y_\lambda \subset D_T$ and Y is a T-bounded SMS, $Y_\lambda \subset Y$, i.e. Y is invariant under $R(\lambda;\, A)$, for $|\lambda| > \|A\|$. It follows from the identity

$$A = \lim_{\lambda \to \infty} \lambda[\lambda R(\lambda; A) - I],$$

that Y is invariant under A. □

1.11. LEMMA. *Given T with the* 1-*SDP, let $F \subset \mathbf{C}$ be compact. Then $x \in \Xi(T, F)$ iff*

$$\text{(i)} \quad \sigma_T(x) \subset F \quad \textit{and} \quad \text{(ii)} \quad \lim_{\lambda \to \infty} x(\lambda) = 0.$$

Proof. (Only if): Let $x \in \Xi(T, F)$. We have

$$\sigma_T(x) \subset \sigma[T|\Xi(T, F)] = \sigma[T|X(T, F)] \subset F.$$

Since $T|\,\Xi(T, F)$ is bounded, it follows that

$$\lim_{\lambda \to \infty} x(\lambda) = \lim_{\lambda \to \infty} R[\lambda;\, T|\Xi(T, F)]x = 0.$$

(If): By (i), $x \in X(T, F)$. Since T is closed, it follows from (ii) and from the identity

$$\lambda x(\lambda) - x = Tx(\lambda),$$

that

$$\lim_{\lambda \to \infty} [\lambda x(\lambda) - x] = T \lim_{\lambda \to \infty} x(\lambda) = 0.$$

The function $f\colon V \to X$, defined by $f(\lambda) = \lambda x(\lambda) - x$ is analytic on a neighborhood V of ∞ and $f(\infty) = \lim_{\lambda \to \infty} f(\lambda) = 0$. Let $r > 0$ be sufficiently large, for

$$F \subset \{\lambda \in \mathbf{C}\colon |\lambda| < r\} \quad \text{and} \quad V \supset \{\lambda \in \mathbf{C}\colon |\lambda| \geq r\}.$$

We have

$$x(\lambda) - \frac{x}{\lambda} = \frac{f(\lambda)}{\lambda}$$

and note that ∞ is, at least, a double zero of $f(\lambda)/\lambda$. Consequently, we have

$$x = \frac{1}{2\pi i}\int_\Gamma \frac{x}{\lambda}\,d\lambda = \frac{1}{2\pi i}\int_\Gamma x(\lambda)\,d\lambda - \frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda}\,d\lambda$$

$$= \frac{1}{2\pi i}\int_\Gamma R[\lambda; T|X(T, F)]x\,d\lambda = Qx \in \Xi(T, F),$$

where

$$\Gamma = \{\lambda \in \mathbf{C}: |\lambda| = r\} \quad \text{and} \quad Q = \frac{1}{2\pi i}\int_\Gamma R[\lambda; T|X(T, F)]\,d\lambda. \quad \square$$

A direct sum decomposition property of $X(T, F)$ for a bounded decomposable operator [**1**, Lemma 2.3] admits the following generalization.

1.12. THEOREM. *Given T with the 1-SDP, let $F_1 \subset \mathbf{C}$ be closed and $F_2 \subset \mathbf{C}$ be compact. If F_1 and F_2 are disjoint, then*

$$X(T, F_1 \cup F_2) = X(T, F_1) \oplus \Xi(T, F_2).$$

Proof. By denoting $F = F_1 \cup F_2$, one obtain easily

$$X(T, F) \supset X(T, F_1) + \Xi(T, F_2). \tag{1.9}$$

On the other hand, by the functional calculus $X(T, F)$ admits a direct sum decomposition

$$X(T, F) = X_1 \oplus X_2,$$

with $\sigma(T|X_i) \subset F_i$ $(i = 1, 2)$ and $X_2 \subset D_T$. Then $X_i \subset X(T, F_i)$, $i = 1, 2$. Since $T|X_2$ is bounded, for every $x \in X_2$, we have $\sigma_T(x) \subset F_2$ and

$$\lim_{\lambda \to \infty} x(\lambda) = \lim_{\lambda \to \infty} R(\lambda; T|X_2)x = 0.$$

Lemma 1.11 implies that $x \in \Xi(T, F_2)$ and hence $X_2 \subset \Xi(T, F_2)$. Thus, the opposite of (1.9) is obtained and hence

$$X(T, F) = X(T, F_1) + \Xi(T, F_2). \tag{1.10}$$

To see that (1.10) is a direct sum, suppose that

$$x \in X(T, F_1) \cap \Xi(T, F_2) \subset X(T, F_1) \cap X(T, F_2) = X(T, \varnothing).$$

Then $\sigma_T(x) = \varnothing$ and hence $x(\cdot)$ is an entire function. It follows from $x \in \Xi(T, F_2)$ that $\lim_{\lambda \to \infty} x(\lambda) = 0$ and hence $x(\lambda) = 0$. Thus $x = 0$. $\square$

1.13. THEOREM. *Given T with the 1-SDP, if $F_1 \subset \mathbf{C}$ is closed and $F_2 \subset \mathbf{C}$ is compact then*

$$\Xi(T, F_1 \cap F_2) = X(T, F_1) \cap \Xi(T, F_2). \tag{1.11}$$

Proof. With the help of Lemma 1.11, inclusion

$$\Xi(T, F_1 \cap F_2) \subset X(T, F_1) \cap \Xi(T, F_2) \tag{1.12}$$

follows easily. Let $x \in X(T, F_1) \cap \Xi(T, F_2)$. Then $\sigma_T(x) \subset F_1 \cap F_2$ and hence $x \in X(T, F_1 \cap F_2)$. Since $x \in \Xi(T, F_2)$, Lemma 1.11 implies $\lim_{\lambda \to \infty} x(\lambda) = 0$. Quote again Lemma 1.11 and infer that $x \in \Xi(T, F_1 \cap F_2)$. Thus, the opposite of (1.12) follows and hence (1.11) is obtained. □

1.14. PROPOSITION. *Given T, if there is a decomposition*

$$X = X_1 + X_2 \quad \text{with } X_1, X_2 \in \operatorname{Inv} T \text{ and } X_1 \subset D_T$$

then $T|X_2$ is densely defined.

Proof. Recall that by our assumption, T is densely defined. For every $x \in X$, there is a representation

$$x = x_1 + x_2, \qquad x_i \in X_i, i = 1, 2,$$

and there is a number $M > 0$ (independent of x) such that $\|x_1\| + \|x_2\| \leq M\|x\|$. Let $x \in X_2$. There is a sequence $\{x_n\} \subset D_T$ converging to x. For every n, there is a representation

$$x - x_n = x_{n1} + x_{n2}, \qquad x_{ni} \in X_i \ (i = 1, 2)$$

with

$$\|x_{n1}\| + \|x_{n2}\| \leq M\|x - x_n\|.$$

Then $x_{ni} \to 0$ ($i = 1, 2$) as $n \to \infty$. By hypothesis,

$$y_n = x - x_{n2} = x_n + x_{n1} \in D_T \cap X_2$$

and hence

$$\|x - y_n\| = \|x_{n2}\| \to 0 \quad \text{as } n \to \infty$$

implies that $T|X_2$ is densely defined. □

1.15. LEMMA. *Given T and $Y \in \operatorname{Inv} T$, consider the following conditions*:

$$\sigma(T) \cup \sigma(T|Y) \neq \mathbf{C}; \tag{1.13}$$

(1.14) $\hat{T} = T/Y$ *is a closed operator on* X/Y.

Then (1.13) *implies* (1.14) *and either of them implies inclusions*

$$(1.15)\qquad \sigma(\hat{T}) \subset \sigma(T|Y) \cup \sigma(T); \quad \sigma(T|Y) \subset \sigma(T) \cup \sigma(\hat{T});$$
$$\sigma(T) \subset \sigma(\hat{T}) \cup \sigma(T|Y).$$

Proof. Assume (1.13) and let $\lambda \in \rho(T) \cap \rho(T|Y)$ be arbitrary. For every $x \in Y$, we have $R(\lambda;\, T)x = R(\lambda;\, T|Y)x \in Y$ and hence Y is invariant under $R(\lambda;\, T)$. Denote $R_\lambda = R(\lambda;\, T)$ and let $\hat{R}_\lambda$ be the coinduced operator by R_λ on the quotient space X/Y. The identities

$$(\lambda - T)R_\lambda x = x, \quad x \in X; \qquad R_\lambda(\lambda - T)x = x, \quad x \in D_T,$$

give rise to

$$(1.16)\quad (\lambda - \hat{T})\hat{R}_\lambda \hat{x} = \hat{x}, \quad \hat{x} \in X/Y; \qquad \hat{R}_\lambda(\lambda - \hat{T})\hat{x} = \hat{x}, \quad \hat{x} \in D_{\hat{T}}.$$

It follows from (1.16) that $\hat{R}_\lambda$ is the inverse of $\lambda - \hat{T}$. Since $\hat{R}_\lambda$ is bounded and defined on X/Y, it is closed and hence $\hat{T}$ is closed. By (1.16), $\lambda \in \rho(\hat{T})$ and this implies the first of (1.15). The remainder of the proof is routine and we omit it (see [**2**, Proposition 2.2]). □

1.16. LEMMA. *Given* T, *let* $X_0,\ X_1,\ Y \in \operatorname{Inv} T$ *satisfy the following conditions*:

$$(1.17)\qquad X = X_0 + X_1, \qquad X_1 \subset D_T \cap Y;$$

$$(1.18)\qquad \sigma(T|X_0) \subset F, \qquad \sigma(T|X_0 \cap Y) \subset F,$$

for some closed $F \subset \mathbf{C}$, $F \neq \mathbf{C}$.

Then $\hat{T} = T/Y$ *is closed on* X/Y. *Furthermore, if* $\tilde{T} = (T|X_0)/Y \cap X_0$, (*i.e.* $\tilde{T}$ *is the coinduced operator by* $T|X_0$ *on the quotient space* $X_0/Y \cap X_0$), *then*

$$(1.19)\qquad \sigma(\hat{T}) = \sigma(\tilde{T}).$$

Proof. The quotient spaces X/Y and $X_0/Y \cap X_0$ are topologically isomorphic. Since by (1.18), $\sigma(T|X_0) \cup \sigma(T|Y \cap X_0) \neq \mathbf{C}$, Lemma 1.15 implies that $\tilde{T}$ is closed.

Next, we show that $\tilde{T}$ and $\hat{T}$ are similar. In view of (1.17), every $x \in D_T$ has a representation

$$x = x_0 + x_1 \quad \text{with } x_i \in X_i, \qquad i = 0, 1.$$

Since $x_1 \in D_T$, we have $x_0 \in D_T$. Thus, $x_1 \in Y \cap D_T$ and $x_0 \in X_0 \cap D_T$. Let $A = X/Y \to X_0/Y \cap X_0$ be the topological isomorphism. For $x \in X$, let $\hat{x} = x + Y \in X/Y$ and, for $x \in X_0$, $\tilde{x} = x + Y \cap X_0 \in X_0/Y \cap X_0$. For every $\hat{x} \in D_{\hat{T}}$, there is $x \in \hat{x} \cap D_T$ and we have $A\hat{x} = A\hat{x}_0 = \tilde{x}_0 \in D_{\tilde{T}}$. Conversely, for every $\tilde{x} \in D_{\tilde{T}}$, there is $x \in \tilde{x} \cap (X_0 \cap D_T)$ and hence $\hat{x} \in D_{\hat{T}}$. Consequently, $AD_{\hat{T}} = D_{\tilde{T}}$. For every $\hat{x} \in D_{\hat{T}}$, we obtain successively

$$A\hat{T}\hat{x} = A(Tx)\hat{} = (Tx)\tilde{} = \tilde{T}\tilde{x} = \tilde{T}A\hat{x}$$

and hence $\hat{T}$ is similar to $\tilde{T}$. Therefore, $\hat{T}$ is closed and (1.19) holds. □

1.17. THEOREM. *Given T with the* 1-*SDP*, *let $G \subset \mathbf{C}$ be open and put*

$$Y = \begin{cases} X(T, \overline{G}), & \text{if } G \text{ is unbounded;} \\ \Xi(T, \overline{G}), & \text{if } G \text{ is bounded.} \end{cases}$$

Then $\hat{T} = T/Y$ is closed and

$$\sigma(\hat{T}) \subset G^c. \tag{1.20}$$

In particular, if $G \in V_\infty$, then $\hat{T}$ is bounded.

Proof. First, suppose that $G \notin V_\infty$. Let $\lambda \in G$ be arbitrary and let $\{G_0, G_1\} \in \operatorname{cov} \sigma(T)$ satisfy conditions: $G_0 \in V_\infty$, $\lambda \notin \overline{G}_0$, $\lambda \in G_1 \subset \overline{G}_1 \subset G$ and G_1 is relatively compact in $\mathbf{C}$. By the 1-SDP,

$$X = X(T, \overline{G}_0) + X(T, \overline{G}_1)$$

and since $X(T, \varnothing) \subset X(T, \overline{G}_0)$, Theorem 1.9 implies the spectral decomposition

$$X = X(T, \overline{G}_0) + \Xi(T, \overline{G}_1). \tag{1.21}$$

Put $X_0 = X(T, \overline{G}_0)$, $X_1 = \Xi(T, \overline{G}_1)$ and obtain

$$X_1 \subset D_T \cap Y. \tag{1.22}$$

Furthermore, we have

$$\sigma(T|X_0) = \sigma[T|X(T, \overline{G}_0)] \subset \overline{G}_0; \tag{1.23}$$

$$\sigma[T|X(T, \overline{G}_0) \cap X(T, \overline{G})] = \sigma[T|X(T, \overline{G}_0 \cap \overline{G})] \subset \overline{G}_0; \tag{1.24}$$

$$\sigma[T|X(T, \overline{G}_0) \cap \Xi(T, \overline{G})] = \sigma[T|\Xi(T, \overline{G}_0 \cap \overline{G})] \subset \overline{G}_0, \tag{1.25}$$

The last equality in (1.25) stems from Theorem 1.13. In view of the definition of Y, relations (1.21)–(1.25) fulfill all hypotheses of Lemma

1.16. Thus, $\hat{T}$ is closed and

$$\sigma(\hat{T}) = \sigma(\tilde{T}), \tag{1.26}$$

where $\tilde{T} = (T|X_0)/Y \cap X_0$. It follows from Lemma 1.15 and from (1.23)–(1.25) that

$$\sigma(\tilde{T}) \subset \sigma\big(T|X_0\big) \cup \sigma\big(T|X_0 \cap Y\big) \subset \overline{G}_0. \tag{1.27}$$

Since $\lambda \notin \overline{G}_0$, (1.26) and (1.27) imply that $\lambda \notin \sigma(\hat{T})$. Thus, (1.20) follows.

For $G \in V_\infty$, we may assume that $\mathbf{C} \neq \overline{G}$. Let $\lambda \in G$ be arbitrary. Choose G_1 relatively compact in $\mathbf{C}$ such that $\{G, G_1\} \in \operatorname{cov}\sigma(T)$ and $\lambda \notin \overline{G}_1$. Since $\hat{T}$ and $\tilde{T} = [T|\Xi(T, \overline{G}_1)]/\Xi(T, \overline{G}_1) \cap Y$ are similar and $\tilde{T}$ is bounded, $\sigma(\tilde{T}) = \sigma(\hat{T})$ and $\hat{T}$ is bounded. By Lemma 1.15,

$$\sigma(\hat{T}) = \sigma(\tilde{T}) \subset \sigma\big[T|\Xi\big(T, \overline{G}_1\big)\big] \cup \sigma\big[T|\Xi\big(T, \overline{G} \cap \overline{G}_1\big)\big] \subset \overline{G}_1$$

and hence $\lambda \in \rho(\hat{T})$. Thus, inclusion (1.20) follows. □

While the two-summand spectral decomposition property (1-SDP) of the given operator is a convenient mechanism in our spectral theoretic study, it does not confine its scope. Similarly to some other types of spectral decompositions (i.e. [**6, 7**]) it is shown that the 1-SDP and the general SDP are equivalent. Details of that proof will be included in another work. The two extensions of the spectral maximal space concept give rise to two generalizations of the decomposable operator concept.

1.18. Definition. T is said to be decomposable if, for any $\{G_i\}_{i=0}^n \in \operatorname{cov}\sigma(T)$ with $G_0 \in V_\infty$, there is a system $\{Y_i\}_{i=0}^n$ of SMS of T satisfying conditions (I) and (II) of Definition 1.1.

1.19. Definition. T is said to be $\{\infty\}$-decomposable if, for any $\{G_i\}_{i=0}^n \in \operatorname{cov}\sigma(T)$ with $G_0 \in V_\infty$, there is a SMS Y_0 of T and a system $\{Y_i\}_{i=1}^n$ of T-bounded SMS satisfying conditions (II) of Definition 1.1.

The case of $\{\infty\}$-decomposable operator fits into the theory of the residually decomposable operators [**8, 9**] with residuum $S = \{\infty\}$. If T is $\{\infty\}$-decomposable for n confined to $n = 1$, then T is said to be $(\{\infty\}, 1)$-decomposable. If T is $(\{\infty\}, 1)$-decomposable then its conjugate T^* is again $(\{\infty\}, 1)$-decomposable [**9**]. Moreover, for every open $G \subset \mathbf{C}$, the spectral manifold $X(T, \overline{G})$ is closed in X, as a fulfilment of condition γ [ibid.].

We conclude this section with some necessary and sufficient conditions which make the unbounded operators with the SDP and the unbounded decomposable operators equivalent.

1.20. THEOREM. *Given T, the following assertions are equivalent:*
(I) *T is decomposable;*
(II) *T has the SDP and $X(T, \varnothing) = \{0\}$, or*
T has the SDP and $\{0\}$ is a SMS of T;
(III) *T has the SDP and every T-bounded SMS is a SMS of T;*
(IV) *T has the SDP and $X(T, F) \subset D_T$ for some compact F in $\mathbf{C}$.*

Proof. The conclusion will be reached through the following sequel of implications: (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) $\Rightarrow$ (II) and (III) $\Rightarrow$ (I).

(I) $\Rightarrow$ (II). Clearly, T has the SDP. Let $\{G_0, G_1\} \in \operatorname{cov}\sigma(T)$ with $G_0 \in V_\infty$. There corresponds the spectral decomposition

$$X = X_0 + X_1 \quad \text{with } X_1 \subset D_T \text{ spectral maximal.}$$

Consequently, $\sigma[T|X(T, \varnothing)] = \varnothing \subset \sigma(T|X_1)$ implies $X(T, \varnothing) \subset X_1 \subset D_T$. Then $X(T, \varnothing) = \{0\}$, by Lemma 1.8.

(II) $\Rightarrow$ (III) follows from Theorem 1.7.

(III) $\Rightarrow$ (IV). Let $F \subset \mathbf{C}$ be compact. By hypothesis, $\Xi(T, F)$ is a SMS of T. Then $\sigma[T|X(T, F)] = \sigma[T|\Xi(T, F)]$ implies $X(T, F) \subset \Xi(T, F) \subset D_T$.

(IV) $\Rightarrow$ (II). If, for some compact $F \subset \mathbf{C}$, $X(T, F) \subset D_T$, then

$$X(T, F) = \Xi(T, F) \oplus X(T, \varnothing) \quad \text{and} \quad \Xi(T, F) \subset D_T$$

imply that $X(T, \varnothing) = \{0\}$.

(III) $\Rightarrow$ (I). By the SDP, for any $\{G_i\}_{i=0}^n \in \operatorname{cov}\sigma(T)$ with $G_0 \in V_\infty$, there exists $\{Y_i\}_{i=0}^n \subset \operatorname{Inv} T$ such that $\sigma(T|Y_i) \subset G_i$ $(0 \le i \le n)$ and

$$X = \sum_{i=0}^{n} Y_i \subset \sum_{i=0}^{n} X(T, \overline{G}_i) \subset X.$$

By Proposition 1.4, every $X(T, \overline{G}_i)$ is a SMS of T. Moreover,

$$X(T, \overline{G}_i) = \Xi(T, \overline{G}_i) \oplus X(T, \varnothing), \quad 1 \le i \le n, \qquad X(T, \varnothing) \subset X(T, \overline{G}_0)$$

imply

$$X = X(T, \overline{G}_0) + \sum_{i=1}^{n} \Xi(T, \overline{G}_i).$$

Since $X(T, \overline{G}_0)$ and, by hypothesis, every $\Xi(T, \overline{G}_i)$ $(1 \le i \le n)$ is a SMS of T, T is decomposable. □

(After this paper was accepted for publication, we noticed that Lemma 1.15 appeared explicitly in F.-H. Vasilescu, *Analytic Functional Calculus and Spectral Decompositions*, D. Reidel, Dordrecht: Holland, 1982.)

2. Elements of spectral duality theory. While this section prepares the main Theorem 3.1, some of the properties discussed here have intrinsic values. Various topologies are involved in the duality theory. If A and B are dual spaces, we use the notation $\tau(A, B)$ for the topology on A induced by B, under the given duality.

2.1. THEOREM. *If T has the SDP then T^* has the SDP.*

Proof. Let T have the SDP. Then, for closed $F \subset \mathbf{C}$, $X(T, F)$ is closed (Proposition 1.4). Let $\{G_0, G_1\} \in \operatorname{cov} \sigma(T)$ with $G_0 \in V_\infty$. The SDP implies the spectral decomposition (1.21)

$$X = X(T, \bar{G}_0) + \Xi(T, \bar{G}_1).$$

Since $X(T, \bar{G}_0)$ is a SMS of T and $\Xi(T, \bar{G}_1)$ is a T-bounded SMS, T is a $(\{\infty\}, 1)$-decomposable operator. By [**9**, Theorem 2.10] as mentioned in 1, T^* is $(\{\infty\}, 1)$-decomposable. Consequently, T^* has the 1-SDP and hence it has the SDP. □

2.2. LEMMA. *Given T, let $Y \in \operatorname{Inv} T$ be such that $Y \subset D_T$. If $T^* \mid Y^\perp$ is densely defined then T/Y is closable. Moreover, $(T/Y)^* = T^* \mid Y^\perp$.*

Proof. $Y^\perp$ can be viewed as the conjugate of X/Y, under the isometric isomorphism $(X/Y)^* \to Y^\perp$. For convenience, we make no distinction between $Y^\perp$ and $(X/Y)^*$ and denote by $\langle \hat{x}, x^* \rangle$ the linear functional $x^* \in Y^\perp$ on X/Y. For $\hat{x} \in D_{T/Y}$, $x \in \hat{x} \cap D_T$, $y \in Y$ and $x^* \in Y^\perp \cap D_{T^*}$, we have

$$(2.1) \quad \langle (T/Y)\hat{x}, x^* \rangle = \langle T(x+y), x^* \rangle = \langle Tx, x^* \rangle$$
$$= \langle x, T^*x^* \rangle = \langle x + y, T^*x^* \rangle = \langle \hat{x}, T^*x^* \rangle.$$

Hence T/Y and $T^* \mid Y^\perp$ are conjugates to each other. Since $T^* \mid Y^\perp$ is densely defined, T/Y is closable (e.g. [**4**, III. Theorem 5.28]). It remains to prove the second statement of the lemma. Let $\overline{T/Y}$ be the minimal closed extension of T/Y. It follows from $\bar{D}_T = X$ that $\bar{D}_{T/Y} = X/Y$ and hence $(\overline{T/Y})^* = (T/Y)^*$ exists. Then (2.1) implies

$$(2.2) \qquad G'(-T^*|Y^\perp) \subset [G(T/Y)]^\perp = G'[-(T/Y)^*].$$

It follows from (2.2) that $(T/Y)^* \supset T^* \mid Y^\perp$.

To prove the opposite inclusion, let $x^* \in D_{(T/Y)^*}$. For $x \in D_T$ and $y \in Y$, we obtain successively

$$(2.3) \quad \langle (\overline{T/Y})\hat{x}, x^* \rangle = \langle \hat{x}, (T/Y)^*x^* \rangle$$
$$= \langle x + y, (T/Y)^*x^* \rangle = \langle x, (T/Y)^*x^* \rangle.$$

Thus, for every $x^* \in D_{(T/Y)^*}$, $\langle(\overline{T/Y})\hat{x}, x^*\rangle$ is a bounded linear functional of x and hence $x^* \in D_{T^*}$. Furthermore, $x^* \in D_{(T/Y)^*} \subset Y^\perp$ and hence $x^* \in Y^\perp \cap D_{T^*}$.

On the other hand, since $x \in D_T$, for every $y \in Y$, we have

$$(2.4)\quad \langle(\overline{T/Y})\hat{x}, x^*\rangle = \langle(T/Y)\hat{x}, x^*\rangle = \langle T(x+y), x^*\rangle = \langle Tx, x^*\rangle = \langle x, T^*x^*\rangle.$$

It follows from (2.3) and (2.4) that $(T/Y)^* \subset T^* \mid Y^\perp$. □

2.3. Lemma. *Suppose that $Y^{***} \subset X^{***}$ is closed for $\tau(X^{***}, X^{**})$ and Y^{***} is invariant under the projection P of X^{***} onto KX^*, along $(JX)^\perp$. Then PY^{***} is closed for $\tau(KX^*, JX)$.*

Proof. Let S^{***} be the closed (for the metric topology of X^{***}) unit ball of PY^{***}. Let $\{x_\alpha^{***}\} \subset S^{***}$ be a net converging to $x_0^{***} \in KX^*$ for $\tau(KX^*, JX)$. Since $\{x_\alpha^{***}\}$ is bounded in X^{***}, there is a subnet $\{x_\beta^{***}\}$ of $\{x_\alpha^{***}\}$ such that $x_\beta^{***} \to x^{***} \in X^{***}$ for $\tau(X^{***}, X^{**})$. Since, by hypothesis, Y^{***} is closed for $\tau(X^{***}, X^{**})$, we have $x^{***} \in Y^{***}$. Let $Jx \in JX$. Then $\langle Jx, (I-P)x^{***}\rangle = 0$ and hence

$$(2.5)\qquad \lim_\beta \langle Jx, x_\beta^{***}\rangle = \langle Jx, x^{***}\rangle = \langle Jx, Px^{***}\rangle.$$

On the other hand, we have

$$(2.6)\qquad \lim_\beta \langle Jx, x_\beta^{***}\rangle = \lim_\alpha \langle Jx, x_\alpha^{***}\rangle = \langle Jx, x_0^{***}\rangle.$$

It follows from (2.5) and (2.6) that

$$\langle Jx, Px^{***}\rangle = \langle Jx, x_0^{***}\rangle.$$

Since both Px^{***} and x_0^{***} are elements of KX^*, we have

$$(2.7)\qquad x_0^{***} = Px^{***} \in PY^{***}.$$

Since, clearly $\|x_0^{***}\| \le 1$, (2.7) implies that $x_0^{***} \in S^{***}$ and hence S^{***} is closed for $\tau(KX^*, JX)$. By the Kreĭn-Šmul'jan theorem, PY^{***} is closed for $\tau(KX^*, JX)$. □

2.4. Theorem. *Given T, the following properties hold:*

(i) *If the density condition $(*)$ is satisfied then, for every $x \in D_T$, we have $Jx \in D_{T^{**}}$ and $T^{**}Jx = JTx$; likewise*

(i′) *if the density condition $(**)$ is satisfied then, for every $x^* \in D_{T^*}$, we have $Kx^* \in D_{T^{***}}$ and $T^{***}Kx^* = KT^*x^*$.*

(ii) *Suppose that the density condition $(**)$ is satisfied and $x^{***} \in KX^*$. If $\langle T^{**}Jx, x^{***}\rangle$ is a bounded linear functional of $Jx \in JD_T$, then $x^{***} \in KD_{T^*}$ and $T^{***}x^{***} = KT^*K^{-1}x^{***}$.*

Proof. (i): Since $G(t) = {}^{\perp}[G'(-T^*)]$ and $[G'(-T^*)]^{\perp} = G(T^{**})$, for every $(x, Tx) \in G(T)$ we have $(Jx, JTx) \in G(T^{**})$ or, equivalently, $Jx \in D_{T^{**}}$ and $T^{**}Jx = JTx$.

(i′) follows directly from (i), with the original space X^* and embedding K.

(ii): Let $x \in D_T$ and suppose that $\langle T^{**}Jx, x^{***}\rangle$ is a bounded linear functional of Jx. With the help of (i), we obtain

$$\begin{aligned}\langle Tx, K^{-1}x^{***}\rangle &= \langle K^{-1}x^{***}, JTx\rangle = \langle K^{-1}x^{***}, T^{**}Jx\rangle \\ &= \langle T^{**}Jx, x^{***}\rangle\end{aligned}$$

and hence $K^{-1}x^{***}$ is a bounded linear functional of x. Then $K^{-1}x^{***} \in D_{T^*}$ and by (i′), we obtain $x^{***} \in D_{T^{***}}$ and $T^{***}x^{***} = KT^*K^{-1}x^{***}$.□

2.5. COROLLARY. *Given T, suppose that $(**)$ holds. Then*

$$KD_{T^*} = KX^* \cap D_{T^{***}}.$$

Proof. It follows from Theorem 2.4 (i′) that $KD_{T^*} \subset KX^* \cap D_{T^{***}}$. To obtain the opposite inclusion, let $x^{***} \in KX^* \cap D_{T^{***}}$. Then

$$\langle T^{**}Jx, x^{***}\rangle = \langle Jx, T^{***}x^{***}\rangle$$

is a bounded linear functional of Jx. By Theorem 2.4 (ii), $x^{***} \in KD_{T^*}$ and hence $KX^* \cap D_{T^{***}} \subset KD_{T^*}$. □

2.6. LEMMA. *Given T, assume that $(**)$ holds. Then, the projection P of X^{***} onto KX^*, along $(JX)^{\perp}$ commutes with T^{***}.*

Proof. Let $x^{***} \in D_{T^{***}}$. Then $Px^{***} \in KX^*$. For $x \in D_T$, we have successively

$$\begin{aligned}\langle T^{**}Jx, Px^{***}\rangle &= \langle T^{**}Jx, Px^{***}\rangle + \langle T^{**}Jx, (I-P)x^{***}\rangle \\ &= \langle T^{**}Jx, x^{***}\rangle = \langle Jx, T^{***}x^{***}\rangle = \langle Jx, PT^{***}x^{***}\rangle\end{aligned}$$

and hence $\langle T^{**}Jx, Px^{***}\rangle$ is a bounded linear functional of $Jx \in JD_T$. Theorem 2.4 (ii) implies that $Px^{***} \in D_{T^{***}}$. Then

$$\langle T^{**}Jx, Px^{***}\rangle = \langle Jx, T^{***}Px^{***}\rangle$$

implies that

$$T^{***}Px^{***} = PT^{***}x^{***}.$$ □

2.7. THEOREM. *Given T, assume that $(***)$ is satisfied. Then*

$$JD_T = JX \cap D_{T^{**}}.$$

Moreover, for every $Jx \in JD_T$, we have

$$JTx = T^{**}Jx.$$

Proof. In view of Theorem 2.4 (i), $JD_T \subset JX \cap D_{T^{**}}$. Let $x^{***} \in (JX)^{\perp}$. There is a sequence $\{x_n^{***}\} \subset D_{T^{***}}$ such that $x_n^{***} \to x^{***}$ as $n \to \infty$. By Lemma 2.6, for every n, $Px_n^{***} \in D_{T^{***}}$ and hence $(I-P)x_n^{***} \in D_{T^{***}}$. Thus,

$$(I-P)x_n^{***} \to (I-P)x^{***} = x^{***}.$$

Consequently, $(JX)^{\perp} \cap D_{T^{***}}$ is dense in $(JX)^{\perp}$.

Now let $Jx \in JX \cap D_{T^{**}}$. For every $x^{***} \in (JX)^{\perp} \cap D_{T^{***}}$, we have

$$0 = \langle Jx, T^{***}x^{***} \rangle = \langle T^{**}Jx, x^{***} \rangle. \tag{2.8}$$

Since $(JX)^{\perp} \cap D_{T^{***}}$ is dense in $(JX)^{\perp}$, it follows from (2.8) that $T^{**}Jx \in JX$. For $x^* \in D_{T^*}$, Theorem 2.4 (i′) implies that

$$T^{***}Kx^* = KT^*x^*$$

and hence, we obtain successively

$$\begin{aligned}\langle x, T^*x^* \rangle &= \langle Jx, KT^*x^* \rangle = \langle Jx, T^{***}Kx^* \rangle = \langle T^{**}Jx, Kx^* \rangle \\ &= \langle x^*, T^{**}Jx \rangle = \langle J^{-1}T^{**}Jx, x^* \rangle.\end{aligned}$$

This means that the element $(x, J^{-1}T^{**}Jx) \in {}^{\perp}[G'(-T^*)] = G(T)$. Consequently, $x \in D_T$ and $J^{-1}T^{**}Jx = Tx$, i.e. $T^{**}Jx = JTx$. Thus, it follows that

$$JX \cap D_{T^{**}} \subset JD_T. \qquad \square$$

2.8. THEOREM. *Given T, assume that condition $(**)$ holds. If T^* has the SDP then*

(i) *for every closed $F \subset \mathbf{C}$, $X^*(T^*, F)$ is closed for $\tau(X^*, X)$;*

(ii) *for every compact $F \subset \mathbf{C}$, $\Xi^*(T^*, F)$ is closed for $\tau(X^*, X)$.*

Proof. We confine the proof to (ii), that of (i) is similar. Assuming that T^* has the SDP, it follows from Theorem 2.1 that both T^{**} and T^{***} have the SDP. Consequently, $\Xi^{***}(T^{***}, F)$ is a T^{***}-bounded SMS and it is closed for $\tau(KX^*, JX)$, by [**9**, Proposition 2.9]. It follows from Lemma 2.6, Proposition 1.10, and Lemma 2.3 that $P\Xi^{***}(T^{***}, F)$ is closed for $\tau(KX^*, JX)$.

Next, we prove the equality

$$K\Xi^*(T^*, F) = P\Xi^{***}(T^{***}, F). \tag{2.9}$$

Let $x^{***} \in K\Xi^*(T^*, F)$. Since T^* and $T^{***} \mid KX^*$ are similar (Theorem 2.4 (ii)), $T^{***} \mid KX^*$ has the SDP and $K\Xi^*(T^*, F)$ is a $T^{***} \mid KX^*$-bounded SMS. Consequently, $\sigma_{T^{***}}(x^{***}) \subset F$ and $\lim_{\lambda \to \infty} x^{***}(\lambda) = 0$. Lemma 1.11 implies that $x^{***} \in \Xi^{***}(T^{***}, F)$ and hence

$$K\Xi^*(T^*, F) = PK\Xi^*(T^*, F) \subset P\Xi^{***}(T^{***}, F). \tag{2.10}$$

Conversely, let $x^{***} \in \Xi^{***}(T^{***}, F)$. Then Lemma 1.11 implies that $\sigma_{T^{***}}(x^{***}) \subset F$ and $\lim_{\lambda \to \infty} x^{***}(\lambda) = 0$. Since, by Lemma 2.6 and Proposition 1.10, $P\Xi^{***}(T^{***}, F) \subset \Xi^{***}(T^{***}, F)$, we have

$$\sigma_{T^{***}}(Px^{***}) \subset F \quad \text{and} \quad \lim_{\lambda \to \infty} Px^{***}(\lambda) = 0.$$

Since $x^{***}(\lambda) \in D_{T^{***}}$, it follows from Lemma 2.6 that $Px^{***}(\lambda) \in D_{T^{***}}$. By Corollary 2.5, $Px^{***}(\lambda) \in KD_{T^*}$ and hence, for $\lambda \in \rho_{T^{***}}(x^{***})$, we obtain

$$\begin{aligned}(\lambda - T^*)K^{-1}Px^{***}(\lambda) &= K^{-1}(\lambda - T^{***})Px^{***}(\lambda) \\ &= K^{-1}P(\lambda - T^{***})x^{***}(\lambda) = K^{-1}Px^{***}.\end{aligned}$$

Thus, $\sigma_{T^*}(K^{-1}Px^{***}) \subset F$. Since $\lim_{\lambda \to \infty} K^{-1}Px^{***}(\lambda) = 0$, Lemma 1.11 implies that $K^{-1}Px^{***} \in \Xi^*(T^*, F)$, i.e. $Px^{***} \in K\Xi^*(T^*, F)$. Thus, we have

$$P\Xi^{***}(T^{***}, F) \subset K\Xi^*(T^*, F) \tag{2.11}$$

and hence (2.9) follows from (2.10) and (2.11). Now, by Lemma 2.3, $K\Xi^*(T^*, F)$ is closed for $\tau(KX^*, JX)$ and hence $\Xi^*(T^*, F)$ is closed for $\tau(X^*, X)$. □

2.9. THEOREM. *Given T, assume that condition $(***)$ holds and T^* has the SDP. Let $G \subset \mathbf{C}$ be open, $G \in V_\infty$ and $Y = {}^\perp X^*(T^*, \overline{G})$. Then*

(I) $Y \subset D_T$, $Y \in \operatorname{Inv} T$, $\sigma(T \mid Y) \subset G^c$;

(II) *T/Y is closable and, for its minimal closed extension $\overline{T/Y}$, we have $\sigma(\overline{T/Y}) \subset \overline{G}$.*

Proof. Since T^* has the SDP, $X^*(T^*, \overline{G})$ is closed for $\tau(X^*, X)$ and hence $Y^\perp = X^*(T^*, \overline{G})$. Clearly, $X^*(T^*, \overline{G})$ is invariant under T^*. Let H be a relatively compact open set such that $G^c \subset H$. Then $\{G, H\} \in \operatorname{cov} \mathbf{C}$ with $G \in V_\infty$ and the SDP of T^* gives rise to the decomposition

$$X^* = X^*(T^*, \overline{G}) + \Xi^*(T^*, \overline{H}). \tag{2.12}$$

Since $\Xi^*(T^*, \overline{H}) \subset D_{T^*}$, it follows from Proposition 1.14 that

$$T^* \mid X^*(T^*, \overline{G}) = T^* \mid Y^\perp$$

is densely defined. It can be easily shown that $Y \in \operatorname{Inv} T$.

Next, we show that $Y \subset D_T$. In view of (2.12), for every $x^* \in X^*$, there is a representation

$$x^* = x_1^* + x_2^*, \qquad x_1^* \in X^*(T^*, \overline{G}), x_2^* \in \Xi^*(T^*, \overline{H})$$

with $\|x_1^*\| + \|x_2^*\| \le M\|x^*\|$, where the number M is independent of x^*. Let $x^* \in D_{T^*}$. Then $x_2^* \in \Xi^*(T^*, \overline{H}) \subset D_{T^*}$ implies that $x_1^* \in D_{T^*}$ and hence, for every $x \in Y$, we have successively

$$\begin{aligned} |\langle T^*x^*, Jx\rangle| &= |\langle x, T^*x^*\rangle| = |\langle x, T^*(x_1^* + x_2^*)\rangle| = |\langle x, T^*x_2^*\rangle| \\ &\le \|T^*|\Xi^*(T^*, \overline{H})\| \cdot \|x\| \cdot \|x_2^*\| \le M\|T^*|\Xi^*(T^*, \overline{H})\| \cdot \|x\| \cdot \|x^*\|. \end{aligned}$$

Thus, $\langle T^*x^*, Jx\rangle$ is a bounded linear functional of x^* and hence $Jx \in D_{T^{**}}$. By Theorem 2.7, $Jx \in JD_T$, i.e. $x \in D_T$. Consequently, $Y \subset D_T$. Since Y satisfies all hypotheses of Lemma 2.2, T/Y is closable and

$$(2.13) \qquad (T/Y)^* \cong T^*|Y^\perp = T^*|X^*(T^*, \overline{G}).$$

Thus, it follows that

$$\sigma(\overline{T/Y}) = \sigma[(T/Y)^*] = \sigma[T^*|X^*(T^*, \overline{G})] \subset \overline{G} \cap \sigma(T).$$

It remains to show that $\sigma(T|Y) \subset G^c$. By applying Theorem 1.17 to T^*, we infer that $T^*/X^*(T^*, \overline{G})$ is bounded and

$$\sigma[T^*/X^*(T^*, \overline{G})] \subset G^c.$$

Now, it follows from the unitary equivalence

$$(T|Y)^* \cong T^*/X^*(T^*, \overline{G}),$$

that

$$\sigma(T|Y) = \sigma[(T|Y)^*] = \sigma[T^*/X^*(T^*, \overline{G})] \subset G^c. \qquad \square$$

3. The duality theorem.

3.1. THEOREM. *Given T, assume that $(***)$ holds. If T^* has the SDP then T has the SDP.*

Proof. Given $\{G_0, G_1\} \in \operatorname{cov}\sigma(T)$ with $G_0 \in V_\infty$ and G_1 relatively compact, let $F_0, F_1 \subset \mathbf{C}$ be closed such that $F_0 \in V_\infty$, $F_0 \subset G_0$, $F_1 \subset G_1$

and $\{\text{Int } F_0, \text{ Int } F_1\} \in \text{cov}\,\sigma(T)$. Then $H_0 = F_1^c \in V_\infty$, $H_1 = F_0^c$ is compact. Put

$$Y = {}^\perp[X^*(T^*, \overline{H}_0)], \qquad Z = {}^\perp[\Xi^*(T^*, \overline{H}_1)], \tag{3.1}$$

use Theorem 2.9 and obtain

$$Y \subset D_T, \quad Y \in \text{Inv}\, T, \qquad \sigma(T|Y) \subset H_0^c = F_1 \subset G_1. \tag{3.2}$$

Our next objective is to obtain the decomposition

$$X = Y + Z. \tag{3.3}$$

Since, by Theorem 2.8, $X^*(T^*, \overline{H}_0)$ and $\Xi^*(T^*, \overline{H}_1)$ are closed for $\tau(X^*, X)$, we have

$$Y^\perp = X^*(T^*, \overline{H}_0), \qquad Z^\perp = \Xi^*(T^*, \overline{H}_1). \tag{3.4}$$

It follows from

$$X^*(T^*, \overline{H}_0) = X^*[T^*, \overline{H}_0 \cap \sigma(T)],$$
$$\Xi^*(T^*, \overline{H}_1) = \Xi^*[T^*, \overline{H}_1 \cap \sigma(T)],$$

$[\overline{H}_0 \cap \sigma(T)] \cap [\overline{H}_1 \cap \sigma(T)] = \varnothing$, and Theorem 1.12, that $X^*(T^*, \overline{H}_0) + \Xi^*(T^*, \overline{H}_1)$ is a direct sum and

$$X^*(T^*, \overline{H}_0 \cup \overline{H}_1) = X^*(T^*, \overline{H}_0) \oplus \Xi^*(T^*, \overline{H}_1). \tag{3.5}$$

Apply [**4**, IV. Theorem 4.8] to Y and Z, as defined by (3.1), and infer that $Y + Z$ is closed. On the other hand,

$$(Y + Z)^\perp = Y^\perp \cap Z^\perp = X^*(T^*, \overline{H}_0) \cap \Xi^*(T^*, \overline{H}_1) = \{0\}$$

implies (3.3).

It remains to show that

$$\text{(a)} \quad Z \in \text{Inv}\, T, \qquad \text{(b)} \quad \sigma(T|Z) \subset G_0. \tag{3.6}$$

For $x \in Z \cap D_T$, $x^* \in \Xi^*(T^*, \overline{H}_1) = Z^\perp$, we clearly have

$$\langle Tx, x^* \rangle = \langle x, T^*x^* \rangle = 0$$

and hence $Tx \in Z$. This implies (3.6, a). By (3.3) and Proposition 1.14, $T|Z$ is densely defined and hence $(T|Z)^*$ exists. Next, we shall obtain

$$(T|Z)^* = T^*/\Xi^*(T^*, \overline{H}_1) = (T^*)\hat{}. \tag{3.7}$$

For $x \in Z \cap D_T$, $x^* \in D_{T^*}$ and

$$(x^*)\hat{} = x^* + \Xi^*(T^*, \overline{H}_1) \in X^*/\Xi^*(T^*, \overline{H}_1),$$

we have

$$\langle (T|Z)x, (x^*)^\wedge \rangle = \langle (T|Z)x, x^* \rangle = \langle x, T^*x^* \rangle = \langle x, (T^*)^\wedge (x^*)^\wedge \rangle.$$

Consequently,

$$(T|Z)^* \supset (T^*)^\wedge. \tag{3.8}$$

To obtain the opposite of (3.8), let $x \in D_T$ and $(x^*)^\wedge \in D_{(T|Z)^*}$. In view of (3.3), there is a number $M > 0$ and a representation

$$x = x_1 + x_2, \quad x_1 \in Y, x_2 \in Z \qquad \text{and} \qquad \|x_1\| + \|x_2\| \leq M\|x\|.$$

Then, for every $x^* \in (x^*)^\wedge$ ($\in D_{(T|Z)^*}$), we have successively:

$$\begin{aligned} |\langle Tx, x^* \rangle| &\leq |\langle Tx_1, x^* \rangle| + |\langle Tx_2, x^* \rangle| \\ &= |\langle (T|Y)x_1, x^* \rangle| + |\langle (T|Z)x_2, (x^*)^\wedge \rangle| \\ &\leq \|T|Y\| \cdot \|x_1\| \cdot \|x^*\| + \|x_2\| \cdot \|(T|Z)^*(x^*)^\wedge\| \\ &\leq M\{\|T|Y\| \cdot \|x^*\| + \|(T|Z)^*(x^*)^\wedge\|\} \|x\|. \end{aligned}$$

Thus, $\langle Tx, x^* \rangle$ is a bounded linear functional of x. Consequently, $x^* \in D_{T^*}$ and hence $(x^*)^\wedge \in D_{(T^*)^\wedge}$. In view of (3.8), (3.7) is obtained. Now Theorem 1.17 applied to T^*, gives

$$\sigma[(T^*)^\wedge] \subset H_1^c$$

and hence (3.6, b) follows by

$$\sigma(T|Z) = \sigma[(T|Z)^*] = \sigma[(T^*)^\wedge] \subset H_1^c = F_0 \subset G_0.$$

By (3.3), (3.2) and (3.6), T has the SDP. □

The combination of Theorems 2.1 and 3.1 gives

3.2. COROLLARY. *Given T, assume that condition* (***) *holds. Then T has the SDP iff T^* has the SDP.*

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TEMPLE UNIVERSITY
PHILADELPHIA, PA 19122

AND

NANJING UNIVERSITY
NANJING, CHINA