# COBORDISM OBSTRUCTIONS TO FIBERING MANIFOLDS OVER SPHERES

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We consider here the problem introduced by Conner and Floyd of determining necessary and sufficient conditions for a manifold $M$ to be cobordant to a bundle over a given sphere $S^k$. Two recent studies by D. F. X. O'Reilly [7] and A. Didierjean [4] presented obstructions to fibering manifolds over spheres in terms of the "top" Stiefel-Whitney classes of $M$. While these conditions were shown by Conner and Floyd [3] and R. L. W. Brown [2] to be sufficient when restricted to the cases of fiberings over $S^1$ and $S^2$, they are not at all sufficient for guaranteeing the fibering of a cobordism class over a sphere of any higher dimension. This is shown in O'Reilly's study of fiberings over the 4-sphere.

In this paper we exhibit an obstruction to fibering a manifold over a sphere that extends the obstructions mentioned above. We then essentially answer all open questions but one regarding the problem of which cobordism classes can be represented by a bundle over $S^4$.

**1. Introduction.** In [3], Conner and Floyd introduced the problem of determining which cobordism classes in $\mathfrak{N}_*$ could be represented by a manifold fibered (smoothly) over a given sphere. They showed that if $w_i(M)$ denotes the $i$th Stiefel-Whitney class of a manifold $M$, then a class $\omega \in \mathfrak{N}_n$ fibers over $S^1$ (i.e. $\omega$ contains a representative that fibers over $S^1$) if and only if the Stiefel-Whitney number $w_n(\omega) = 0$. Extending the study, R. L. W. Brown [2] showed that a class $\omega \in \mathfrak{N}_n$ fibers over $S^2$ if and only if the Stiefel-Whitney number $w_n(\omega) = 0$ if $n$ is even and $w_2 w_{n-2}(\omega) = 0$ if $n$ is odd. Subsequent investigations have considered fiberings over a variety of manifolds, yielding complete solutions in a number of cases. But R. E. Stong [8] observed that spheres, aside from having a natural importance, actually play a key role here. He showed

(1.1) If a class $\omega \in \mathfrak{N}_n$ fibers over $S^k$, then $\omega$ fibers over any manifold $N^q$ with $q \leq k$.

At present, for fiberings over spheres of any higher dimension than two, there are only partial results. In solving the $S^2$ problem, Brown exhibited a general necessary condition [2, Prop. 2.1] for a manifold to fiber over a sphere $S^k$ that is based on the Brown-Peterson relations among characteristic classes. Using it, he derived the Stiefel-Whitney number obstructions mentioned above. D. F. X. O'Reilly [7] (see also A.

Didierjean [4]) generalized those obstructions as follows:

(1.2) If $M^n$ fibers over $S^{2k}$, then all Stiefel-Whitney numbers divisible by

$$\begin{cases} w_n, w_{n-1}, \ldots, w_{n-2k+1} & \text{for } n \text{ even} \\ w_n, w_{n-1}, \ldots, w_{n-2k} & \text{for } n \text{ odd} \end{cases}$$

are zero.

What is significant, is that although these conditions turn out to be sufficient for a cobordism class to fiber over $S^2$, they are far from sufficient for guaranteeing the fibering of a class over a sphere of any higher dimension. This is made clear in O'Reilly's study of fiberings over the 4-sphere. The conditions given by (1.2) are shown to leave the fibering status of a good number of classes unaccounted for.

The purpose of this paper is to somewhat expand Brown's condition (involving the Brown-Peterson relations) and then more importantly, demonstrate its use; not as a device for obtaining obstructions in terms of Stiefel-Whitney numbers but actually as an obstruction itself to fibering cobordism classes over spheres. With this point of view we are able to extend the results of O'Reilly and Didierjean, essentially answering all open questions but one, regarding the problem of which cobordism classes fiber over $S^4$.

Throughout this paper we will use no notation to differentiate between a manifold and its cobordism class. Also all cohomology will be with $\mathbf{Z}_2$ coefficients.

**2. Main results.** Given a (smooth, closed) manifold $M^n$, let $\tau_M \colon M^n \to BO$ classify the stable tangent bundle of $M^n$ and let the (mod 2) Steenrod algebra $\mathcal{C}$ act on the right of $H^*(BO)$ and $H^*(M)$ as in [1]. Before stating our results we briefly recall the Brown-Peterson relations for reference [1].

(2.1) (Brown-Peterson): $\bigcap_{M^n} \ker \tau_M^* = \Sigma H^s(BO)\mathrm{Sq}^t, \ 2t > n - s.$

Now, using Didierjean's notation let

$$I_0^k(M^n) = \left\{ \alpha \in H^k(BO) \,\middle|\, \text{if } k \le n, \tau_M^*(\alpha \cdot \beta) = 0 \ \forall \beta \in H^{n-k}(BO) \right\}.$$

Noting that $I_0^*(M^n)$ depends only on the cobordism class $\omega$ of $M^n$ let

$$A_0^*(\omega) = H^*(BO)/I_0^*(\omega).$$

THEOREM 1. *If* $\omega \in \mathfrak{N}_n$ *fibers over* $S^k$, *then* $[\mathrm{Sq}^r(H^s(BO)\mathrm{Sq}^t)] = 0$ $\in A_0^*(\omega)$ *and* $[(\Sigma H^s(BO)\mathrm{Sq}^t)^2] = 0 \in A_0^*(\omega)$ *for* $2t > n - k - s$ *and* $r > s + t - k$.

*Proof.* Let $p: M^n \to S^k$ be a fibration with fiber $F^{n-k}$ and with $M^n$ representing the cobordism class $\omega$. That $[\mathrm{Sq}^r(H^s(BO)\mathrm{Sq}^t)] = 0$ is essentially Brown's condition [2, Prop. 2.1]. That $[(\Sigma H^s(BO)\mathrm{Sq}^t)^2] = 0$ will follow from the same reasoning. As in [2], we consider the exact sequence

$$(2.2) \qquad \to H^*(M, F) \xrightarrow{j^*} H^*(M) \xrightarrow{i^*} H^*(F) \to$$

and note that by excision

$$H^*(M, F) = H^*(M, F \times D^k) = H^*(F \times D^k, F \times S^{k-1})$$

where $D^k$ is the $k$-ball. Then

$$(2.3) \qquad H^*(M, F) = H^{*-k}(F) \otimes H^k(D^k, S^{k-1}).$$

Now

$$i^* \tau_M^* H^s(BO)\mathrm{Sq}^t = \tau_F^* H^s(BO)\mathrm{Sq}^t = 0 \quad \text{for } 2t > n - k - s$$

by (2.1). Let $u, v \in \Sigma H^s(BO)\mathrm{Sq}^t$, $2t > n - k - s$. Then by (2.2) and (2.3), $\tau_M^* u = j^*(x \otimes a)$ and $\tau_M^* v = j^*(y \otimes a)$ for some $x, y \in H^*(F)$, $a \in H^k(D^k, S^{k-1})$ being the generator. Hence $\tau_M^*(uv) = \tau_M^* u \cdot \tau_M^* v = 0$ since $a^2 = 0$ and so $[(\Sigma H^s(BO)\mathrm{Sq}^t)^2] = 0 \in A_0^*(\omega)$.  $\square$

Interestingly, having added to Brown's original condition, we now show that in practice Theorem 1 can actually be simplified back again.

PROPOSITION 2.4. *For any class* $\omega \in \mathfrak{N}_n$ *and* $k \leq n$,

$$\left[\mathrm{Sq}^r(H^s(BO)\mathrm{Sq}^t)\right] = 0 \in A_0^*(\omega) \Leftrightarrow \left[\left(\sum H^s(BO)\mathrm{Sq}^t\right)^2\right] = 0 \in A_0^*(\omega)$$

*for* $2t > n - k - s$ *and* $r > s + t - k$.

*Proof.* Let $M^n$ represent $\omega$.

($\Leftarrow$) Let $u \in H^s(BO)$. By definition of the right action of $\mathcal{Q}$ on $H^*(M)$, we have for any $\beta \in H^{n-r-s-t}(BO)$

$$(2.5) \quad \left\langle \tau_M^*(\beta \cdot \mathrm{Sq}^r(u\,\mathrm{Sq}^t)), [M^n] \right\rangle = \left\langle \tau_M^*(\beta\,\mathrm{Sq}^r \cdot u\,\mathrm{Sq}^t), [M^n] \right\rangle.$$

Now since $r > s + t - k$, we get that $2r > n - k - (n - r - s - t)$ and so $\beta\,\mathrm{Sq}^r \in \Sigma H^s(BO)\mathrm{Sq}^t$ for $2t > n - k - s$. Therefore by (2.5),

$$\left[\left(\sum H^s(BO)\mathrm{Sq}^t\right)^2\right] = 0 \Rightarrow \left[\mathrm{Sq}^r(H^s(BO)\mathrm{Sq}^t)\right] = 0.$$

($\Rightarrow$) Now suppose $[\mathrm{Sq}^r(H^s(BO)\mathrm{Sq}^t)] = 0$. Let $x = u\mathrm{Sq}^j \in H^i(BO)\mathrm{Sq}^j$ and $y = v\mathrm{Sq}^m \in H^l(BO)\mathrm{Sq}^m$ with $2j > n - k - i$ and $2m > n - k - l$ (i.e. $x, y \in \Sigma H^s(BO)\mathrm{Sq}^t, 2t > n - k - s$). To show that $[x \cdot y] = 0 \in A_0^*(\omega)$, let $\gamma \in H^{n-(i+j+l+m)}(BO)$. Since $\Sigma H^s(BO)\mathrm{Sq}^t, 2t > n - k - s$ is an ideal (see (2.1)), we have that $\gamma \cdot v\mathrm{Sq}^m \in \Sigma H^s(BO)\mathrm{Sq}^t, 2t > n - k - s$. Thus $\tau_M^*(\gamma \cdot v\mathrm{Sq}^m \cdot u\mathrm{Sq}^j) = \tau_M^*(\beta\,\mathrm{Sq}^r \cdot u\mathrm{Sq}^j)$ for some $\beta\,\mathrm{Sq}^r \in H^{n-r-i-j}(BO)\mathrm{Sq}^r$ with $2r > n - k - (n - r - i - j)$. By (2.5)

$$\left\langle \tau_M^*(\gamma \cdot v\mathrm{Sq}^m \cdot u\mathrm{Sq}^j), [M] \right\rangle = \left\langle \tau_M^*(\beta \cdot \mathrm{Sq}^r(u\,\mathrm{Sq}^j)), [M] \right\rangle$$

and since $r > i + j - k$ the result follows.                          $\square$

The fact that the conditions given by Theorem 1 prove to be sufficient for a class in $\mathfrak{N}_*$ to fiber over $S^1$ or $S^2$ can be expressed as follows. All characteristic number obstructions to fibering a manifold over $S^1$ or $S^2$ come from those characteristic classes of the total space that by the Brown-Peterson relations restrict to zero on the fiber. In [5] however, it was observed that all obstructions to fibering a complex manifold over $S^3$ (or $S^4$), with the exception of a signature condition, actually arise simply from Wu classes $v_i$ pulling back to zero on the fiber. Indeed, this is really the case with the Conner-Floyd and Brown results as well. Noting that $w_{2n}(M^{2n}) = \mathrm{Sq}^n v_n(M^{2n})$ and $w_{2n-1}(M^{2n+1}) = \mathrm{Sq}^{n-1}v_n(M^{2n+1})$ by the Wu formula, we see that consideration of the full range of Brown-Peterson relations is unnecessary in the $S^1$ and $S^2$ cases. The Wu classes alone produce all sufficient fibering conditions. Conceivably, for any $\omega \in \mathfrak{N}_n$ and $k \leq n$, $[\mathrm{Sq}^r(H^s(BO)\mathrm{Sq}^t)] = 0 \in A_0^*(\omega)$ for $2t > n - k - s$, $r > s + t - k$ if and only if $[\mathrm{Sq}^r v_t] = 0 \in A_0^*(\omega)$ for $2t > n - k$, $r > t - k$. (Note that $v_t = (1)\mathrm{Sq}^t \in H^0(BO)\mathrm{Sq}^t$.)

Turning to the problem of fiberings over $S^4$, the key role is again played by the Wu classes.

Let $X_4$, $X_5$ and $X_6$ be indecomposables in $\mathfrak{N}_*$ of dimensions 4, 5 and 6 respectively.

THEOREM 2. (i) *A class* $\omega \in \mathfrak{N}_{2n+1}$ *fibers over* $S^4$ *if and only if*

$$[\mathrm{Sq}^r v_t] = 0 \in A_0^*(\omega) \quad \text{for } t = n, n - 1, r > t - 4.$$

(ii) *If* $\omega \in \mathfrak{N}_{2n}$ *is such that* $[\mathrm{Sq}^r v_t] = 0 \in A_0^*(\omega)$ *for* $t = n, n - 1$, $r > t - 4$, *then* $\omega = Y + a\mathbf{R}P(2)^{n-5}(X_4X_6 + X_5^2)$ *where* $Y$ *fibers over* $S^4$ *and* $a \in \mathbf{Z}_2$.

REMARK. For $\omega = X_4X_6 + X_5^2 \in \mathfrak{N}_{10}$, $[\mathrm{Sq}^r v_t] = 0 \in A_0^*(\omega)$ for $t = 4, 5$, $r > t - 4$. In fact, $[\mathrm{Sq}^r(H^s(BO))\mathrm{Sq}^t] = 0 \in A_0^*(\omega)$. It is still

unknown however whether or not there exists a representative of $\omega$ that fibers over $S^4$.

The proof of Theorem 2 essentially takes up the remainder of the paper. The key to the proof is in §3 where Theorem 1 is applied. Specific obstructions, going beyond those given in [7] and [4], are exhibited and are shown to account for a number of families of classes in $\mathfrak{N}_*$ not fibering over $S^4$. In §4 we tie up the loose ends through actual construction of fibrations and formally complete the proof. In §5 we present a result about fiberings over $S^3$.

**3. Obstructions.** We begin this section by recalling results of Brown and O'Reilly.

PROPOSITION 3.1 (*Brown*).

(i) *There exist indecomposables* $X_i \in \mathfrak{N}_i$ *fibered over* $S^2$, $i \neq 2^s - 1$, *and* $i \neq 2, 5$.

(ii) *If* $\omega \in \mathfrak{N}_n$ *fibers over* $S^2$, *then* $\omega^2$ *fibers over* $S^4$.

PROPOSITION 3.2 (*O'Reilly*).

(i) *There exist indecomposables* $X_i \in \mathfrak{N}_i$ *fibered over* $S^4$, $i \neq 2^s - 1$, *and* $i \neq 2, 4, 5, 6, 11$.

(ii) *The class* $X_4 X_5 X_6 + X_5^3$ *fibers over* $S^4$.

Throughout the rest of this paper we will let $X_i \in \mathfrak{N}_i$ denote O'Reilly's indecomposables for $i \neq 2, 4, 5, 6, 11$ while denoting Brown's indecomposables (fibered over $S^2$) for $i = 4, 6$ and $11$.

O'Reilly showed that no indecomposable $X_{11}$ could fiber over $S^4$ and proved the same thing for the classes $X_4 X_6$ and $X_5^2$. He conjectured that $X_2 X_{11}$, $X_2^2 X_{11}$ and $X_5^3$ also do not fiber over $S^4$ even though all obstructions given by (1.2) vanish on these classes. The following two propositions prove these conjectures while extending O'Reilly's results in general.

PROPOSITION 3.3. *For any non-zero linear combination*

$$\omega = a_1 X_2^{n-2} X^5 + a_2 X_2^{n-4} X_4 X_5 + a_3 X_2^{n-5} X_5 X_6 + a_4 X_2^{n-5} X_{11}$$
$$+ a_5 X_2^{n-7} X_4 X_5 X_6 + a_6 X_2^{n-7} X_5^3, \qquad a_i \in \mathbf{Z}_2,$$

*in* $\mathfrak{N}_{2n+1}$, *with the exception of* $\omega = X_2^{n-7}(X_4 X_5 X_6 + X_5^3)$, $[\mathrm{Sq}^r v_t]$ *for* $t = n$, $n - 1$, $r > t - 4$ *is non-zero in* $A_0^*(\omega)$. *And no such combination fibers over* $S^4$.

*Proof.* If $a_1 = 1$, Brown's $S^2$ result shows that $w_2 w_{2n-1}(\omega) \neq 0$. Since $w_{2n-1}(M^{2n+1}) = \mathrm{Sq}^{n-1} v_n(M^{2n+1})$, the result follows. Thus we need only look at the case where $a_1 = 0$. Consider the following chart obtained via the product formulas for the Stiefel-Whitney and Wu classes:

|  | $X_2^{n-4}X_4X_5$ | $X_2^{n-5}X_5X_6$ | $X_2^{n-5}X_{11}$ | $X_2^{n-7}X_5^3$ |
|---|---|---|---|---|
| $w_2^3 \mathrm{Sq}^{n-4} v_{n-1}$ | $n$ | $n+1$ | $1$ | $0$ |
| $w_6 \mathrm{Sq}^{n-4} v_{n-1}$ | $n$ | $\binom{n-2}{2}$ | $0$ | $1$ |
| $w_1 w_3 \mathrm{Sq}^{n-3} v_n$ | $1$ | $1$ | $0$ | $0$ |
| $w_4 \mathrm{Sq}^{n-3} v_n$ | $1$ | $0$ | $0$ | $0$ |

Clearly the above matrix is non-singular. Furthermore, since $X_4 X_5 X_6 + X_5^3$ fibers over $S^4$, Theorem 1 implies that

$$[\mathrm{Sq}^r v_t] = 0 \in A_0^*\big(X_2^{n-7}(X_4 X_5 X_6 + X_5^3)\big)$$

and so we need not consider a column for $X_2^{n-7}X_4 X_5 X_6$. The result follows.                                                                    □

PROPOSITION 3.4. *For any non-zero linear combination*

$$\omega = a_1 X_2^n + a_2 X_2^{n-2}X_4 + a_3 X_2^{n-3}X_6 + a_4 X_2^{n-5}X_4 X_6$$
$$+ a_5 X_2^{n-5}X_5^2, \qquad a_i \in \mathbf{Z}_2,$$

*in* $\mathfrak{N}_{2n}$, *with the exception of* $\omega = X_2^{n-5}(X_4 X_6 + X_5^2)$, $[\mathrm{Sq}^r v_t]$, $t = n, n-1$, $r > t - 4$ *is non-zero in* $A_0^*(\omega)$. *And no such combination fibers over* $S^4$.

*Proof.* Stong [8, Prop. 7.2] showed that the classes $X_2^n$, $X_2^{n-2}X_4$ and $X_2^{n-3}X_6$ are distinguished by characteristic numbers involving the Stiefel-Whitney classes $w_{2n}$, $w_{2n-1}$, $w_{2n-2}$ and $w_{2n-3}$. By the Wu formula,

$$w_{2n}(M^{2n}) = \mathrm{Sq}^n v_n(M^{2n}), \qquad w_{2n-1}(M^{2n}) = \mathrm{Sq}^{n-1} v_n(M^{2n}),$$

$$w_{2n-2}(M^{2n}) = \mathrm{Sq}^{n-2} v_n + \mathrm{Sq}^{n-1} v_{n-1}(M^{2n})$$

and

$$w_{2n-3}(M^{2n}) = \mathrm{Sq}^{n-3} v_n + \mathrm{Sq}^{n-2} v_{n-1}(M^{2n}).$$

Thus for any class $\omega$, if $[\mathrm{Sq}^r v_t] = 0$ for $t = n, n-1$, $r > t - 4$, then $[w_{2n}] = [w_{2n-1}] = [w_{2n-2}] = [w_{2n-3}] = 0 \in A_0^*(\omega)$. Since all numbers involving $w_{2n}$, $w_{2n-1}$, $w_{2n-2}$ and $w_{2n-3}$ vanish for both $X_2^{n-5}X_4 X_6$ and $X_2^{n-5}X_5^2$ we see that the only combination we must account for is

$\omega = a_4 X_2^{n-5} X_4 X_6 + a_5 X_2^{n-5} X_5^2$. An easy calculation using the product formula for the Stiefel-Whitney and Wu classes shows that

$$(3.5) \qquad w_4 \operatorname{Sq}^{n-3} v_{n-1}\left(X_2^{n-5} X_4 X_6\right) = w_4 \operatorname{Sq}^{n-3} v_{n-1}\left(X_2^{n-5} X_5^2\right) \neq 0$$

and so if $a_4 \neq a_5$, then $[\operatorname{Sq}^r v_t]$ $t = n, n-1$, $r > t - 4$ is non-zero in $A_0^*(a_4 X_2^{n-5} X_4 X_6 + a_5 X_2^{n-5} X_5^2)$. The result follows.     $\square$

**4. Some fibrations over $S^4$.** Since the classes in $\mathfrak{N}_*$ which fiber over $S^4$ form an ideal, the results of §3 combined with Propositions 3.1 and 3.2 leave only a few classes still unaccounted for. In this section we show that the classes $X_4 X_5^2$, $X_4 X_{11}$, $X_5^2 X_6$, $X_5 X_{11}$ and $X_6 X_{11}$ all fiber over $S^4$. First some preliminaries.

Let $\xi$ be an $n$-dimensional vector bundle over a manifold $B$. The real projective bundle $\mathbf{R}P(\xi)$ is the manifold consisting of all lines in the fibers of $\xi$. $\mathbf{R}P(\xi)$ fibers over $B$, say $p: \mathbf{R}P(\xi) \to B$ (with fiber $\mathbf{R}P(n-1)$), and supports a canonical line bundle $\eta$.

Let $c = w_1(\eta) \in H^1(\mathbf{R}P(\xi))$. Then $H^*(\mathbf{R}P(\xi))$ is a free $H^*(B)$ module on generators $1, c, \ldots, c^{n-1}$ with the relation

$$(4.1) \qquad \qquad \sum_{i=0}^{n} c^i p^*\left(w_{n-i}(\xi)\right) = 0.$$

The tangent bundle of $\mathbf{R}P(\xi)$ is given by

$$(4.2) \qquad \qquad \tau(\mathbf{R}P(\xi)) \oplus \theta^1 = p^* \tau(B) \oplus (p^* \xi \otimes \eta)$$

where $\theta^1$ denotes the trivial line bundle.

For convenience, we will from now on omit the "$p^*$" from our notation and will let $\eta$ once and for all denote the appropriate canonical line bundle over whatever projective bundle we may be dealing with. We let $\gamma$ denote the quaternionic line bundle over $S^4 = \mathbf{H}P(1)$ ($\gamma$ is a real 4-bundle) and we let $\theta^n$ denote the trivial $n$-bundle.

Finally, we recall some facts (see [6]) about the so-called $s$ numbers of a manifold. If the total Stiefel-Whitney class $w$ of a bundle $E$ over $M$ is given by $w(E) = (1 + t_1)(1 + t_2) \cdots (1 + t_k)$ for $t_i \in H^i(M)$, then for any partition $I = i_1, i_2, \ldots, i_r$ of $k$, the polynomial $s_I$ in the Stiefel-Whitney classes of $E$ is defined by

$$(4.3) \qquad \qquad s_I(w_1, \ldots, w_k) = \sum t_1^{i_1} \cdots t_r^{i_r}$$

the sum ranging over all permutations of $t_1, \ldots, t_k$. For any manifold $M$, $s_I(M) = s_I(\tau(M))$. We also note

$$(4.4) \qquad \qquad s_I(E_1 \oplus E_2) = \sum_{JK=I} s_J(E_1) s_K(E_2)$$

and for any indecomposables $X_i \in \mathfrak{N}_i$, $i \neq 2^s - 1$,

$(4.5)$   $s_I(X_{i_1} X_{i_2} \cdots X_{i_r}) = 0$   if $I$ is not a refinement of $J = i_1, \ldots, i_r$.

End of preliminaries.

The following lemmas are needed for future calculations.

LEMMA 4.6. *For Brown's indecomposables* $X_4 = \mathbf{R}P(\lambda \oplus \theta^1)$ *over* $S^2$ *and* $X_6 = \mathbf{R}P(\lambda \oplus \theta^3)$ *over* $S^2$, $\lambda$ *being the canonical complex line bundle over* $S^2 = \mathbf{C}P(1)$, $s_{2,2}(X_4) = 1$ *and* $s_{2,4}(X_6) = s_{2,2,2}(X_6) = 0$.

*Proof.* By (4.2), $w(X_4) = ((1 + c)^2 + a)(1 + c)$ while $w(X_6) = ((1 + c)^2 + a)(1 + c)^3$ where $a \in H^2(S^2)$ is the generator and $c = w_1(\eta) \in H^1(\mathbf{R}P(\lambda \oplus \theta^1))$ (resp. $H^1(\mathbf{R}P(\lambda \oplus \theta^3)))$. Now, $s_{2,2}(X_4) = w_2^2(X_4) = 1$ (using (4.1)). Similarly,

$$s_{2,4}(X_6) = s_{1,2}^2(X_6) = (w_1 w_2 + w_3)^2(X_6) = 0$$

while

$$s_{2,2,2}(X_6) = s_{1,1,1}^2(X_6) = w_3^2(X_6) = 0. \qquad \square$$

LEMMA 4.7. *For the indecomposable* $X_{10} = \mathbf{R}P(\gamma \oplus \theta^3)$ *over* $S^4$, $s_{5,5}(X_{10}) = 1$.

*Proof.* By (4.2), $w(X_{10}) = ((1 + c)^4 + a)(1 + c^3)$ where $c = w_1(\eta)$ and $a \in H^4(S^4)$ is the generator. Since the only non-zero Stiefel-Whitney class of $\gamma \otimes \eta$ is $w_4(\gamma \otimes \eta) = c^4 + a$, any $s$-class of $\gamma \otimes \eta$ with dimension not a multiple of 4 vanishes. By (4.4) then, $s_{5,5}(X_{10}) = s_{5,5}(\theta^3 \otimes \eta)$. But $s_{5,5}(\theta^3 \otimes \eta) = c_1^5 c_2^5 + c_1^5 c_3^5 + c_2^5 c_3^5$. The result follows from (4.1).   $\square$

LEMMA 4.8. *For the indecomposable* $X_{11} = \mathbf{R}P(\lambda \oplus \tau \mathbf{R}P(4))$ *over* $S^2 \times \mathbf{R}P(4)$, $\lambda$ *being the canonical complex line bundle over* $S^2 = \mathbf{C}P(1)$, $s_{5,6}(X_{11}) = s_{2,9}(X_{11}) = s_{5,2,4}(X_{11}) = s_{5,2,2,2}(X_{11}) = 0$.

*Proof.* By (4.2), $w(X_{11}) = (1 + \alpha)^5((1 + c)^2 + a)(1 + c + \alpha)^5$ where $\alpha$ and $a$ are the pullbacks of the generators of $H^1(\mathbf{R}P(4))$ and $H^2(S^2)$ respectively and $c = w_1(\eta)$. Write $((1 + c)^2 + a) = (1 + t_1)(1 + t_2)$ and note that $t_1 = t_2$ and $t_1 t_2 = c^2 + a$. The rest of the proof is a routine calculation using (4.3) and (4.1).   $\square$

LEMMA 4.9. *For the indecomposable* $X_{12} = \mathbf{R}P(\gamma \oplus \theta^5)$ *over* $S^4$, $s_{10,2}(X_{12}) = s_{5,5,2}(X_{12}) = 0$.

*Proof.* By (4.2), $w(X_{12}) = ((1 + c)^4 + a)(1 + c)^5$. Since the only non-zero Stiefel-Whitney class of $\gamma \otimes \eta$ is $w_4(\gamma \otimes \eta) = c^4 + a$, any $s$-class with dimension not a multiple of 4 vanishes. Furthermore $w_4^3(\gamma \otimes \eta) = c^{12} + c^8 a = 0$ by (4.1), so any $s$-class of dimension 12 vanishes as well. Then by (4.4), $s_{10,2}(X_{12}) = s_{10,2}(\theta^5 \otimes \eta)$ and $s_{5,5,2}(X_{12}) = s_{5,5,2}(\theta^5 \otimes \eta)$. The result follows from (4.3) and (4.1). □

We now turn to the main results of the section.

PROPOSITION 4.10. *The class* $X_4 X_5^2$ *fibers over* $S^4$.

*Proof.* Let

$$M^{14} = \mathbf{R}P(\eta_1 \oplus \theta^5)$$
$$\downarrow$$
$$\mathbf{R}P(\gamma \oplus \theta^2)$$
$$\downarrow$$
$$S^4$$

Since $M^{14}$ fibers over $S^4$, Theorem 1 guarantees that all characteristic numbers of $M$ divisible by $Sq^r v_t$, $t = 6, 7$, $r > t - 4$ are zero in $A_0^*(X_4 X_5^2)$. Therefore, applying Propositions 3.1, 3.2 and 3.4, we have that the cobordism class of $M^{14}$ is given by

$$M^{14} = a_1 X_4 X_5^2 + a_2 X_2^2 (X_4 X_6 + X_5^2) + Y, \quad a_k \in \mathbf{Z}_2,$$

where $Y$ fibers over $S^4$. Now by (4.2),

$$w(M^{14}) = ((1 + c)^4 + a)(1 + c)^2(1 + d + c)(1 + d)^5$$

where $c = w_1(\eta_1)$ and $d = w_1(\eta_2)$ ($\eta_2$ being the canonical line bundle over $\mathbf{R}P(\eta_1 \oplus \theta^5)$). As in the proof of Lemma 4.9, any $s$-class of $\gamma \otimes \eta_1$ with dimension not a multiple of 4 or with dimension 12 vanishes. In addition, for any bundle $E$, $s_4(E) = w_1^4(E)$ and $s_{2,2}(E) = w_2^2(E)$, so $s_4(\gamma \otimes \eta_1)$ and $s_{2,2}(\gamma \otimes \eta_1)$ vanish as well. Applying (4.4) and (4.3) as in the previous proofs, we arrive at the following $s$-numbers for $M$: $s_{14} = 0$, $s_{10,4} = 1$, $s_{9,5} = 0$, $s_{5,5,4} = 0$, $s_{10,2,2} = 1$ and $s_{5,5,2,2} = 0$. By (4.5) and Lemmas 4.6, 4.7 and 4.9, it must be that

$$M^{14} = 1 \cdot X_4 X_5^2 + 0 \cdot X_2^2 (X_4 X_6 + X_5^2) + Y$$

and so by adding $Y$ to both sides we get that $X_4 X_5^2$ fibers over $S^4$. □

PROPOSITION 4.11. *The class* $X_4 X_{11}$ *fibers over* $S^4$.

*Proof.* Let

$$M^{15} = \mathbf{R}P(\eta_1 \oplus \theta^5)$$
$$\downarrow$$
$$\mathbf{R}P(\gamma \oplus \lambda \oplus \theta^1)$$
$$\downarrow$$
$$S^4 \times S^1$$

where $\lambda$ is the pullback of the canonical line bundle over $S^1$. Noting that $[\mathrm{Sq}^r v_t] = 0 \in A_0^*(X_4 X_{11})$ (for $t = 6, 7$ $r > t - 4$), we invoke the results of section 3, as in the proof of Proposition 4.10, to see that $M^{15} = aX_4 X_{11} + Y$ where $Y$ fibers over $S^4$. Now by (4.2)

$$w(M^{15}) = \left((1 + c)^4 + a\right)(1 + c + \alpha)(1 + c)(1 + d + c)(1 + d)^5$$

where $\alpha$ is the pullback of the generator of $H^1(S^1)$. Again as in the previous proofs we can ignore any contribution from $\gamma \otimes \eta_1$, obtaining $s_{15}(M^{15}) = 0$ and $s_{4,11}(M^{15}) = 1$. Then $M^{15} = 1 \cdot X_4 X_{11} + Y$ and so $X_4 X_{11}$ fibers over $S^4$.                                               $\square$

PROPOSITION 4.12. *The classes $X_5^2 X_6$ and $X_5 X_{11}$ fiber over $S^4$.*

*Proof.* Let

$$M_1^{16} = \mathbf{R}P(\eta_1 \oplus \theta^3) \qquad \text{and let} \qquad M_2^{16} = \mathbf{R}P(\eta_1 \oplus \theta^5)$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$
$$\mathbf{R}P(\gamma \oplus \lambda \oplus \theta^3) \qquad\qquad\qquad \mathbf{R}P(\gamma \oplus \lambda \oplus \theta^1)$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$
$$S^4 \times \mathbf{R}P(2) \qquad\qquad\qquad\qquad S^4 \times \mathbf{R}P(2)$$

where $\lambda$ is the pullback of the canonical line bundle over $\mathbf{R}P(2)$. As before, we see that

$$M_i^{16} = a_{i1} X_5^2 X_6 + a_{i2} X_5 X_{11} + a_{i3} X_2^3 (X_4 X_6 + X_5^2) + Y_i, \qquad i = 1, 2,$$

where $Y_i$ fibers over $S^4$ and $a_{ij} \in \mathbf{Z}_2$. By (4.2)

$$w(M_1^{16}) = (1 + \alpha)^3 \left((1 + c)^4 + a\right)(1 + c + \alpha)(1 + c)^3$$
$$\times (1 + d + c)(1 + d)^3,$$
$$w(M_2^{16}) = (1 + \alpha)^3 \left((1 + c)^4 + a\right)(1 + c + \alpha)(1 + c)$$
$$\times (1 + d + c)(1 + d)^5$$

where $\alpha$ is the pullback of the generator of $H^1(\mathbf{R}P(2))$. Now by techniques demonstrated earlier, we obtain the following chart of $s$-numbers:

|  | $s_{16}$ | $s_{5,11}$ | $s_{10,6}$ | $s_{2,14}$ | $s_{5,5,6}$ | $s_{2,5,9}$ | $s_{2,4,10}$ | $s_{5,5,2,4}$ | $s_{10,2,2,2}$ | $s_{5,5,2,2,2}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $M_1^{16}$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $M_2^{16}$ | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Then by (4.5) and Lemmas 4.6–4.9, $M_1^{16} = X_5^2 X_6 + Y_1$ and $M_2^{16} = X_5^2 X_6 + X_5 X_{11} + Y_2$. The result follows by addition. $\qquad\square$

PROPOSITION 4.13. *The class $X_6 X_{11}$ fibers over $S^4$.*

*Proof.* Let

$$M^{17} = \mathbf{R}P(\eta_2 \oplus \theta^7)$$
$$\downarrow$$
$$\mathbf{R}P(\eta_1 \oplus \theta^3)$$
$$\downarrow$$
$$\mathbf{R}P(\gamma)$$
$$\downarrow$$
$$S^4$$

Routine calculation shows that $s_{17}(M^{17}) = 1$ and $s_{6,11}(M^{17}) = 0$, while for the indecomposable $X_{17}$, $s_{17}(X_{17}) = 1$ and $s_{6,11}(X_{17}) = 1$. As in the previous proofs, we see that $M^{17} = X_6 X_{11} + Y$ where $Y$ fibers over $S^4$. Thus $X_6 X_{11}$ fibers over $S^4$. $\qquad\square$

Combining the above results with those of §3 reduces the proof of Theorem 2 to the following:

*Proof of Theorem 2.*
(i) ($\Rightarrow$) Trivial by Theorem 1.
($\Leftarrow$) By Propositions 3.1, 3.2 and 4.10–4.13, (and the fact that the classes fibered over $S^4$ form an ideal), any class in $\mathfrak{N}_{2n+1}$ is of the form $\omega = \alpha + \beta$ where $\alpha$ fibers over $S^4$ and $\beta$ is of the form given in Proposition 3.3. Now suppose $[\mathrm{Sq}^r v_t] = 0 \in A_0^*(\omega)$ for $t = n, n - 1, r > t - 4$. By Theorem 1 $[\mathrm{Sq}^r v_t] = 0 \in A_0^*(\alpha)$ for $t = n, n - 1, r > t - 4$ as well. Thus $[\mathrm{Sq}^r v_t] = 0 \in A_0^*(\beta)$ for $t = n, n - 1, r > t - 4$ and hence by Proposition 3.3 $\beta = 0$. Therefore $\omega$ fibers over $S^4$.

(ii) By Propositions 3.1, 3.2 and 4.10–4.13, any class in $\mathfrak{N}_{2n}$ is of the form $\omega = \alpha + \beta$ where $\alpha$ fibers over $S^4$ and $\beta$ is of the form given in

Proposition 3.4. As in (i) we get that $[\mathrm{Sq}^r v_t] = 0 \in A_0^*(\beta)$ for $t = n, n - 1$, $r > t - 4$ and hence by Proposition 3.4 and the remark following it, $\beta = aX^{n-5}(X_4 X_6 + X_5^2)$. The result follows.                □

## 5. A note on fiberings over $S^3$.

In [5], it was shown that a complex cobordism class $\omega \in \Omega_*^u$ fibers over $S^3$ if and only if it fibers over $S^4$. Here we offer the following

*Conjecture* 5.1. A class $\omega \in \mathfrak{N}_{2n}$ fibers over $S^3$ if and only if it fibers over $S^4$.

We note that by Stong's result (1.1), one direction above is trivial. With regards to the other direction, we have

THEOREM 3. *If* $\omega \in \mathfrak{N}_{2n}$ *fibers over* $S^3$, *then*

$$\omega = Y + a\mathbf{R}P(2)^{n-5}(X_4 X_6 + X_5^2)$$

*where* $Y$ *fibers over* $S^4$ *and* $a \in \mathbf{Z}_2$.

*Proof.* By Propositions 3.1, 3.2 and 4.10–4.13, any class in $\mathfrak{N}_{2n}$ is of the form

$$(5.2) \quad \omega = Y + a_1 X_2^n + a_2 X_2^{n-2} X_4 + a_3 X_2^{n-3} X_6 + a_4 X_2^{n-5} X_4 X_6$$
$$+ a_5 X_2^{n-5} X_5^2, \qquad a_i \in \mathbf{Z}_2,$$

where $Y$ fibers over $S^4$. Now, if $\omega$ fibers over $S^3$, then by Theorem 1 and the Wu formula we have that all the Stiefel-Whitney numbers of $\omega$ divisible by $w_{2n}$, $w_{2n-1}$ and $w_{2n-2}$ must be zero.

*Claim.* All numbers divisible by $w_{2n-3}$ must be zero as well.

*Proof.*
(i)

$$
\begin{aligned}
w_1^3 w_{2n-3}(\omega) &= w_1^3 \big(\mathrm{Sq}^{n-3} v_n + \mathrm{Sq}^{n-2} v_{n-1}\big)(\omega) && \text{by the Wu formula} \\
&= w_1^3 \mathrm{Sq}^{n-3} v_n(\omega) && \text{by Theorem 1} \\
&= w_1^2 \cdot \mathrm{Sq}^1 \mathrm{Sq}^{n-3} v_n(\omega) && \text{since } w_1^3 = w_1^2 \mathrm{Sq}^1 \\
&= 0 && \text{by the Adem relations} \\
& && \text{and Theorem 1.}
\end{aligned}
$$

(ii) Similarly,

$$w_3 w_{2n-3}(\omega) = 0 \quad \text{since } w_3 = w_2 \mathrm{Sq}^1.$$

(iii)

$$w_1 w_2 w_{2n-3}(\omega) = w_1 w_2 \operatorname{Sq}^{n-3} v_n(\omega) \qquad \text{as in (i)}$$

$$= \operatorname{Sq}^3 \operatorname{Sq}^{n-3} v_n(\omega) \qquad \text{since } w_1 w_2 = v_3$$

$$= \begin{cases} 0 & n = 4k + 1, \\ \operatorname{Sq}^n v_n(\omega), & n = 4k + 3, \\ \operatorname{Sq}^{n-1} \operatorname{Sq}^1 v_n(\omega), & n \text{ even}, \end{cases} \quad \text{by the Adem relations}$$

$$= 0 \qquad \text{by Theorem 1.}$$

This completes the proof of the claim.

Now as in the proof of Proposition 3.4, we apply Stong's result and get that in (5.2) $a_1 = a_2 = a_3 = 0$. Finally we note that if $\omega$ fibers over $S^3$ or $S^4$ then $w_4 \operatorname{Sq}^{n-3} v_{n-1}(\omega) = 0$ by Theorem 1. Invoking (3.5), we get that $a_4 = a_5$ and the result follows.     □

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