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WEAK FACTORIZATION OF DISTRIBUTIONS IN H^p SPACES

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The weak factorization theorem for real Hardy spaces $H^p(\mathbf{R}^n)$, previously obtained by Coifman, Rochberg and Weiss, and by Uchiyama for the case $p > n/(n+1)$, is extended to the case $p \leq n/(n+1)$.

1. Introduction. The purpose of this paper is to give an extension of the following

THEOREM A. (*Coifman-Rochberg-Weiss* [**3**; *Theorem* II], *Uchiyama* [**7**; *Corollary to Theorem* 1], [**8**].) *Let K be a homogeneous singular integral operator of Calderón-Zygmund type on $\mathbf{R}^n$ and K' its conjugate. Suppose $p, q, r > 0$ satisfy $1/p = 1/q + 1/r < 1 + 1/n$.* (i) *If $h \in L^2 \cap H^q(\mathbf{R}^n)$, $g \in L^2 \cap H^r(\mathbf{R}^n)$ and*

$$f = hKg - gK'h,$$

then $f \in H^p(\mathbf{R}^n)$ and

$$\|f\|_{H^p} \leq C_1 \|h\|_{H^q} \|g\|_{H^r}.$$

(ii) *Conversely, if, furthermore, K is not a constant multiple of the identity operator and $p \leq 1$, every $f \in H^p(\mathbf{R}^n)$ can be written as*

$$f = \sum_{j=1}^{\infty} \lambda_j \left(h_j K g_j - g_j K' h_j \right),$$

where λ_j are complex numbers, $h_j \in L^2 \cap H^q(\mathbf{R}^n)$, $g_j \in L^2 \cap H^r(\mathbf{R}^n)$ and

$$\|h_j\|_{H^q} \|g_j\|_{H^r} \leq C_2, \qquad \left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C_3.$$

The constants C_1, C_2 and C_3 depend only on p, q, r, K and n.

As for the definition of $H^p(\mathbf{R}^n)$, see Fefferman-Stein [**4**]; as for the operators K and K', see the definitions given in the next section.

An extension of part (i) to the case $1/p \geq 1 + 1/n$ is given in the following

THEOREM B. (*Miyachi* [**6**].) *Let $K_1, \ldots, K_N$ be homogeneous singular integral operators of Calderón-Zygmund type on $\mathbf{R}^n$ and K_j' their conjugates.*

Set, for $h \in L^2 \cap H^q(\mathbf{R}^n)$ and $g \in L^2 \cap H^r(\mathbf{R}^n)$,

$$P(K_1,\dots,K_N; h, g) = \sum_J (-1)^{|J|}\left\{\left(\prod_{j\in J} K_j'\right)h\right\}\left\{\left(\prod_{j\in J^c} K_j\right)g\right\},$$

where the summation ranges over all subsets J of $\{1,\dots,N\}$, $|J|$ denotes the number of elements of J, J^c is the complement of J, and $\prod$ is the product of operators; if J or J^c is the empty set, the corresponding product $\prod$ means the identity operator. Then, if $p, q, r > 0$ satisfy $1/p = 1/q + 1/r < 1 + N/n$, there is a constant C depending only on $K_1,\dots,K_N$, p, q, r and n such that, for all $h \in L^2 \cap H^q(\mathbf{R}^n)$ and all $g \in L^2 \cap H^r(\mathbf{R}^n)$,

$$\|P(K_1,\dots,K_N; h, g)\|_{H^p} \leq C\|h\|_{H^q}\|g\|_{H^r}.$$

In this paper, we shall extend part (ii) of Theorem A to the case $1/p \geq 1 + 1/n$ by using the "product" given in Theorem B.

Throughout this paper, we use the following

NOTATION. For $x \in \mathbf{R}^n$ and $r > 0$, $B(x, r)$ denotes the ball with respect to the usual metric with center x and radius r. If $\alpha_1,\dots,\alpha_n$ are nonnegative integers and $\alpha = (\alpha_1,\dots,\alpha_n)$, the differential operator ∂^α is defined by

$$\partial^\alpha f(x) = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f(x), \qquad x \in \mathbf{R}^n,$$

and $|\alpha|$ by $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We shall also use the notation

$$(\partial/\partial x)^\alpha f(x) = \partial^\alpha f(x).$$

If s is a real number, $[s]$ denotes the largest integer not greater than s. $\mathfrak{F}$ denotes the Fourier transform.

2. The result. Before we state our theorem, we shall explain the singular integral operators considered in this paper.

DEFINITION 1. We say that K is a *homogeneous singular integral operator of Calderón-Zygmund type* if it is defined by

$$Kf = \mathfrak{F}^{-1}(m\mathfrak{F}f) \tag{1}$$

with a bounded function m smooth in $\mathbf{R}^n\backslash\{0\}$ and homogeneous of degree zero, i.e. satisfying

$$m(t\xi) = m(\xi), \qquad t > 0, \xi \neq 0.$$

We shall call m the *multiplier* corresponding to K.

DEFINITION 2. If K is a homogeneous singular integral operator of Calderón-Zygmund type defined by (1), the *conjugate operator* K' is defined by

$$K'f = \mathcal{F}^{-1}(\check{m}\mathcal{F}f),$$

where $\check{m}(\xi) = m(-\xi)$.

By using the Fourier transform, the "product" of Theorem B can be redefined by

$$\begin{aligned}&\mathcal{F}P(K_1,\dots,K_N; h, g)(\xi)\\ &\quad = \int \mathcal{F}h(\eta)\mathcal{F}g(\xi-\eta)\prod_{j=1}^{N}\left(m_j(\xi-\eta) - m_j(-\eta)\right)d\eta,\end{aligned}$$

where m_j is the multiplier corresponding to K_j.

The theorem of this paper reads as follows.

THEOREM. *Let $K_1,\dots,K_N$ be homogeneous singular integral operators of Calderón-Zygmund type and m_j the multipliers corresponding to K_j. Suppose p, q, $r > 0$ satisfy $1 \le 1/p = 1/q + 1/r < 1 + N/n$ and the multipliers m_j satisfy the following condition: for any $\xi \neq 0$, there exists an $\eta \neq 0$ such that*

$$\prod_{j=1}^{N}\left(m_j(\xi) - m_j(\eta)\right) \neq 0.$$

Then every $f \in H^p(\mathbf{R}^n)$ can be decomposed as

$$f = \sum_{j=1}^{\infty}\lambda_j P(K_1,\dots,K_N; h_j, g_j),$$

where λ_j are complex numbers, $h_j \in L^2 \cap H^q(\mathbf{R}^n)$, $g_j \in L^2 \cap H^r(\mathbf{R}^n)$ and

$$\|h_j\|_{H^q}\|g_j\|_{H^r} \le C, \qquad \left(\sum_{j=1}^{\infty}|\lambda_j|^p\right)^{1/p} \le C\|f\|_{H^p}$$

with a constant C depending only on $K_1,\dots,K_N$, p, q, r and n.

The rest of the paper will be devoted to the proof of this theorem.

3. Proof of Theorem. The proof will be based on the following

LEMMA 1. *If* $0 < p \leq 1$, *every* $f \in H^p(\mathbf{R}^n)$ *can be decomposed as follows*:

$$f = \sum_{j=1}^{\infty} \lambda_j f_j,$$

where λ_j *are complex numbers*, f_j *are functions satisfying, for some balls* $B(x_j, \rho_j)$,

$$(2) \qquad \begin{cases} \operatorname{support}(f_j) \subset B(x_j, \rho_j), \\ \|f_j\|_{L^\infty} \leq \rho_j^{-n/p}, \\ \int f_j(x) x^\alpha \, dx = 0 \quad \text{for } |\alpha| \leq [n/p - n] \end{cases}$$

and

$$\left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \leq A \|f\|_{H^p}.$$

The constant A depends only on p and n.

This lemma is given by Latter [5].

We shall introduce a class of functions: for $p, t > 0$ and a nonnegative integer M, we denote by $\mathcal{Q}_{p,M}(t)$ the set of all functions $f \in L^2(\mathbf{R}^n)$ such that

$$\mathcal{F}f(\xi) = 0 \quad \text{for } |\xi| \leq 1/t$$

and

$$\|\partial^\alpha \mathcal{F} f\|_{L^2} \leq t^{|\alpha| - n/p + n/2} \quad \text{for } |\alpha| \leq M.$$

LEMMA 2. *If* $0 < p \leq 2$ *and* $M > n/p - n/2$, *then* $\mathcal{Q}_{p,M}(t) \subset H^p(\mathbf{R}^n)$ *and there is a constant C depending only on n and p such that*

$$\|f\|_{H^p} \leq C \quad \textit{for all } f \in \mathcal{Q}_{p,M}(t), \qquad t > 0.$$

Proof. We may assume $M = [n/p - n/2] + 1$. We shall prove that

$$\|\mathcal{F}^{-1}(m \mathcal{F} f)\|_{L^p} \leq C \quad \text{for all } f \in \mathcal{Q}_{p,M}(t), \qquad t > 0,$$

whenever m is a bounded function satisfying

$$|\partial^\alpha m(\xi)| \leq |\xi|^{-|\alpha|} \quad \text{for } |\alpha| \leq M.$$

This will prove the lemma by the singular integral characterization of $H^p(\mathbf{R}^n)$ (see Fefferman-Stein [**4**; §8] or Coifman-Dahlberg [**2**]).

Now suppose $f \in \mathcal{Q}_{p,M}(t)$, $t > 0$, and m is as above; we set $g = \mathcal{F}^{-1}(m\mathcal{F}f)$. Then

$$\|\partial^\alpha \mathcal{F}g\|_{L^2} \leq Ct^{|\alpha|-n/p+n/2}, \qquad |\alpha| \leq M,$$

and hence, by Plancherel's theorem,

$$\big\| |x|^k g(x)\big\|_{L^2} \leq Ct^{k-n/p+n/2}, \qquad k = 0, 1, \ldots, M.$$

From this we can derive the desired estimate by using Hölder's inequality. In fact, if $0 < p \leq 2$ and $1/p = 1/2 + 1/q$, we have

$$\left(\int_{|x|<t} |g(x)|^p\, dx\right)^{1/p} \leq \|g\|_{L^2}\left(\int_{|x|<t} dx\right)^{1/q} \leq C$$

and

$$\left(\int_{|x|>t} |g(x)|^p\, dx\right)^{1/p} \leq \big\| |x|^M g(x)\big\|_{L^2}\left(\int_{|x|>t} |x|^{-Mq}\, dx\right)^{1/q} \leq C,$$

where we used the fact that $Mq > n$; thus $\|g\|_{L^p} \leq C$. This completes the proof.

LEMMA 3. *If $0 < p \leq 1$ and $M > n/p - n/2$, every $f \in H^p(\mathbf{R}^n)$ can be decomposed as follows*:

$$f = \sum_{j=1}^{\infty} \lambda_j f_j(\cdot - x_j),$$

where λ_j are complex numbers, $f_j \in \mathcal{Q}_{p,M}(t_j)$ with some $t_j > 0$, $x_j \in \mathbf{R}^n$ and

$$\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} \leq A' \|f\|_{H^p}$$

with a constant A' depending only on M, p and n.

Proof. We shall prove that if f satisfies

$$(3) \qquad \begin{cases} \text{support}(f) \subset B(x_0, \rho), \\ \|f\|_{L^\infty} \leq \rho^{-n/p}, \\ \int f(x)x^\alpha\, dx = 0 \quad \text{for } |\alpha| \leq [n/p - n], \end{cases}$$

then we can take a constant A'' depending only on M, p and n and a function $g \in \mathcal{Q}_{p,M}(t)$, $t > 0$, such that

$$\|f - A''g(\cdot - x_0)\|_{H^p} \le 1/2A, \tag{4}$$

where A is the constant in Lemma 1.

For the moment we assume the approximation (3)–(4) and derive Lemma 3 from Lemma 1. Let f be an arbitrary element of $H^p(\mathbf{R}^n)$. Apply Lemma 1 to f to obtain

$$f = \sum_{j=1}^{\infty} \lambda_j f_j$$

with f_j satisfying (2) and λ_j satisfying

$$\left(\sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \le A\|f\|_{H^p};$$

then apply the approximation (3)–(4) to each f_j to obtain

$$f = \sum_{j=1}^{\infty} \lambda_j A'' g_j(\cdot - x_j) + f_{(1)}$$

with $g_j \in \mathcal{Q}_{p,M}(t_j)$, $t_j > 0$, and

$$\|f_{(1)}\|_{H^p} \le 2^{-1}\|f\|_{H^p}.$$

Next apply the same process to $f_{(1)}$ to obtain a smaller error $f_{(2)}$, and then again apply the same process to $f_{(2)}$ to obtain $f_{(3)}, \ldots$; repeating this process, we obtain, for each N,

$$f = \sum_{k=0}^{N} \sum_{j=1}^{\infty} \lambda_j^k A'' g_j^k(\cdot - x_j^k) + f_{(N+1)},$$

where $g_j^k \in \mathcal{Q}_{p,M}(t_j^k)$, $t_j^k > 0$, and

$$\left(\sum_{j=1}^{\infty} |\lambda_j^k|^p \right)^{1/p} \le 2^{-k} A\|f\|_{H^p},$$

$$\|f_{(N+1)}\|_{H^p} \le 2^{-N-1}\|f\|_{H^p}.$$

Now the decomposition of Lemma 3 can be obtained by letting $N \to \infty$ since

$$\left(\sum_{k=0}^{\infty} \sum_{j=1}^{\infty} |\lambda_j^k A''|^p \right)^{1/p} \le \left(\sum_{k=0}^{\infty} 2^{-kp} \right)^{1/p} A''A\|f\|_{H^p} = A'\|f\|_{H^p}.$$

Now we shall prove the approximation (3)–(4). We may assume $x_0 = 0$; suppose f satisfies (3) with $x_0 = 0$.

First observe that the Fourier transform of f has the following estimates:

$$\|\partial^\alpha \mathfrak{F} f\|_{L^2} \leq C_\alpha \rho^{|\alpha| - n/p + n/2}, \tag{5}$$

$$|\partial^\alpha \mathfrak{F} f(\xi)| \leq C_\alpha \rho^{[n/p]+1-n/p} |\xi|^{[n/p]-n-|\alpha|+1} \quad \text{if } |\xi| \leq \rho^{-1}, \tag{6}$$

where the constant C_α depends only on p, n and α. Estimate (5) follows from

$$\|x^\alpha f(x)\|_{L^2} \leq C_\alpha \rho^{|\alpha| - n/p + n/2}$$

via Plancherel's theorem. Estimate (6) follows, if $|\alpha| \leq [n/p - n]$, from the estimates

$$\partial^\beta \partial^\alpha \mathfrak{F} f(0) = 0 \quad \text{for } |\beta| \leq [n/p - n] - |\alpha|,$$

$$\|\partial^\beta \partial^\alpha \mathfrak{F} f\|_{L^\infty} \leq C \rho^{[n/p]+1-n/p} \quad \text{for } |\beta| = [n/p - n] - |\alpha| + 1$$

via Taylor's formula; if $|\alpha| > [n/p - n]$, (6) is a consequence of the stronger estimate

$$\|\partial^\alpha \mathfrak{F} f\|_{L^\infty} \leq C_\alpha \rho^{|\alpha| - n/p + n}.$$

For $T > 2$, consider the function

$$h_T = \mathfrak{F}^{-1}(\psi(T\rho\, \cdot)\mathfrak{F} f(\cdot)),$$

where ψ is a fixed smooth function on $\mathbf{R}^n$ such that $\psi(\xi) = 1$ for $|\xi| \geq 2$ and $\psi(\xi) = 0$ for $|\xi| \leq 1$. From (5) and (6) we shall derive the estimates

$$\|\partial^\alpha \mathfrak{F} h_T\|_{L^2} \leq C'_\alpha T^{|\alpha|} \rho^{|\alpha| - n/p + n/2}, \tag{7}$$

$$\|f - h_T\|_{H^p} \leq C T^{-[n/p]-1+n/p}, \tag{8}$$

where C'_α and C do not depend on f, ρ and T. Once these estimates are proved, the approximation (4) can be obtained by setting

$$g = A''^{-1} h_T \in \mathfrak{A}_{p,M}(T\rho)$$

with A'' and T sufficiently large; A'' and T can be taken depending only on M, p and n.

Thus the proof is reduced to that of (7) and (8). (7) follows directly from (5). In order to prove (8), decompose $f - h_T$ as

$$f - h_T = \sum_{j=0}^{\infty} \mathfrak{F}^{-1}(\chi(2^j T\rho\, \cdot)\mathfrak{F} f(\cdot)) = \sum_{j=0}^{\infty} f_j,$$

where $\chi(\xi) = \psi(2\xi) - \psi(\xi)$. As for f_j, we have

$$\text{support}(\mathcal{F}f_j) \subset \{\xi;\, 2^{-1} \leq 2^j T\rho\,|\xi| \leq 2\},$$

and, from (6),

$$\|\partial^\alpha \mathcal{F}f_j\|_{L^2} \leq C_\alpha (2^j T)^{-[n/p]-1+n/p} (2^j T\rho)^{|\alpha|-n/p+n/2},$$

and, hence, by Lemma 2,

$$\|f_j\|_{H^p} \leq C(2^j T)^{-[n/p]-1+n/p}.$$

Thus

$$\|f - h_T\|_{H^p} \leq \left(\sum_{j=0}^{\infty} \|f_j\|_{H^p}^p\right)^{1/p} \leq CT^{-[n/p]-1+n/p}.$$

This proves (8) and completes the proof of Lemma 3.

Proof of Theorem. Since $1/p = 1/q + 1/r \geq 1$, either q or r is less than or equal to 2; we assume $r \leq 2$.

We shall prove that, for any $f \in \mathcal{Q}_{p,M}(t)$, $t > 0$, $M = [n/p - n/2] + 2$, we can take $h_j \in L^2 \cap H^q(\mathbf{R}^n)$, $g_j \in L^2 \cap H^r(\mathbf{R}^n)$ and complex numbers λ_j so that we have

$$\left\| f - \sum_{j=1}^{\infty} \lambda_j P(K_1, \ldots, K_N;\, h_j, g_j) \right\|_{H^p} \leq \frac{1}{2A'},$$

$$\|h_j\|_{H^q} \|g_j\|_{H^r} \leq C, \qquad \left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} \leq C,$$

where A' is the constant in Lemma 3 corresponding to $M = [n/p - n/2] + 2$ and C is a constant depending only on $K_1, \ldots, K_N$, p, q, r and n. Once this is proved, the Theorem is derived from Lemma 3 by the same argument as Lemma 3 was derived from Lemma 1.

Firstly, observe that our assumption on the multipliers means, via a compactness argument, that there exist a finite open covering $\{V_k;\, k = 1, 2, \ldots, m\}$ of $S^{n-1} = \{\xi \in \mathbf{R}^n;\, |\xi| = 1\}$, points $\{\eta_k;\, k = 1, 2, \ldots, m\} \subset S^{n-1}$, and a positive number c such that, for each k,

$$\inf_{\xi \in V_k} \left| \prod_{j=1}^{N} \left(m_j(\xi) - m_j(-\eta_k)\right) \right| \geq c. \tag{9}$$

Let $\{\varphi_k;\ k = 1,2,\ldots,m\}$ be a smooth partition of unity on S^{n-1} subordinate to the covering $\{V_k;\ k = 1,2,\ldots,m\}$. Take an arbitrary $f \in \mathcal{C}_{p,M}(t)$, $t > 0$, $M = [n/p - n/2] + 2$. Decompose f as

$$f = \sum_{k=1}^{m} f_k, \qquad f_k = \mathcal{F}^{-1}(\tilde{\varphi}_k \mathcal{F} f),$$

where $\tilde{\varphi}_k(\xi) = \varphi_k(\xi/|\xi|)$. It is sufficient to show that for each k we can take $h_k \in L^2 \cap H^q(\mathbf{R}^n)$ and $g_k \in L^2 \cap H^r(\mathbf{R}^n)$ such that

$$(10) \qquad \begin{cases} \|f_k - P(K_1,\ldots,K_N;\, h_k,\, g_k)\|_{H^p} \leq m^{-1/p}(2A')^{-1}, \\ \|h_k\|_{H^q}\|g_k\|_{H^r} \leq C. \end{cases}$$

In order to prove (10), we set

$$g_k = \mathcal{F}^{-1}\left(\left(\prod_{j=1}^{N}\left(m_j(\cdot) - m_j(-\eta_k)\right)\right)^{-1} \mathcal{F} f_k\right).$$

As a candidate for h_k, we consider the following function. Take a smooth function θ satisfying support$(\theta) \subset B(0,1)$ and $\int \theta(x)\,dx = 1$, and set

$$h_{k,\delta,\varepsilon} = \mathcal{F}^{-1}\Big((\varepsilon^{-1}t)^n \theta\big(\varepsilon^{-1}t(\cdot - \delta t^{-1}\eta_k)\big)\Big),$$

where δ and ε are small positive numbers satisfying $\varepsilon < \delta/2$ and $\delta + \varepsilon < 1/2$. We shall prove the following estimates:

$$(11) \qquad \|g_k\|_{H^r} \leq C t^{-n/p+n/r},$$

$$(12) \qquad \|h_{k,\delta,\varepsilon}\|_{H^q} \leq C(\varepsilon^{-1}t)^{n/q},$$

$$(13) \qquad \|f_k - P(K_1,\ldots,K_N;\, h_{k,\delta,\varepsilon},\, g_k)\|_{H^p} \leq C(\delta + \delta^{-1}\varepsilon),$$

where C is a constant depending only on $K_1,\ldots,K_N$, p, q, r and n. If these estimates are established, (10) can be obtained by taking $h_k = h_{k,\delta,\varepsilon}$ with δ and ε sufficiently small; δ and ε can be taken depending only on $K_1,\ldots,K_N$, p, q, r and n.

Proof of (11). By (9) and by the homogeneity of m_j, the function

$$G(\xi) = \left(\prod_{j=1}^{N}\left(m_j(\xi) - m_j(-\eta_k)\right)\right)^{-1}$$

satisfies

$$|\partial^\alpha G(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}$$

in an appropriate neighborhood of support($\mathfrak{F}f_k$). Hence the well-known multiplier theorem for H^p spaces (see [**4**; Theorem 12] or [**1**; Theorems 4.6 and 4.7]) gives

$$\|g_k\|_{H^r} \leq C\|f_k\|_{H^r} \leq C\|f\|_{H^r} \leq Ct^{-n/p+n/r},$$

where the last inequality is due to Lemma 2.

Proof of (12). If $q > 2$, we have

$$\begin{aligned}\|h_{k,\delta,\varepsilon}\|_{H^q} &\approx \|h_{k,\delta,\varepsilon}\|_{L^q} \\ &= \|\mathfrak{F}^{-1}\theta(\varepsilon t^{-1}\,\cdot)\|_{L^q} = C(\varepsilon t^{-1})^{-n/q};\end{aligned}$$

if $q \leq 2$, then (12) is obtained by using Lemma 2 since

$$\|\partial^\alpha \mathfrak{F}h_{k,\delta,\varepsilon}\|_{L^2} \leq C_\alpha(\varepsilon^{-1}t)^{|\alpha|+n/2}$$

and $\mathfrak{F}h_{k,\delta,\varepsilon}(\xi) = 0$ for $|\xi| < \varepsilon t^{-1}$.

Proof of (13). We shall again appeal to Lemma 2. We have

$$\begin{aligned}&\mathfrak{F}(f_k - P(K_1,\ldots,K_N; h_{k,\delta,\varepsilon}, g_k))(\xi) \\ &= \int \mathfrak{F}h_{k,\delta,\varepsilon}(\eta)(\mathfrak{F}f_k(\xi) - \mathfrak{F}f_k(\xi-\eta))\,d\eta \\ &\quad + \int \mathfrak{F}h_{k,\delta,\varepsilon}(\eta)\mathfrak{F}f_k(\xi-\eta) \\ &\quad \times \left(1 - \prod_{j=1}^{N} \frac{m_j(\xi-\eta) - m_j(-\eta)}{m_j(\xi-\eta) - m_j(-\eta_k)}\right) d\eta \\ &= \mathrm{I}(\xi) + \mathrm{II}(\xi).\end{aligned}$$

Supports of the functions I and II are contained in

$$\{\xi \in \mathbf{R}^n;\ \mathrm{dist}(\xi, \mathrm{support}(\mathfrak{F}f_k)) \leq (\delta+\varepsilon)t^{-1}\}$$

and, hence, in $\{|\xi| > (2t)^{-1}\}$. As for the function I, we have, if $|\alpha| \leq M - 1 = [n/p - n/2] + 1$,

$$\begin{aligned}\|\partial^\alpha \mathrm{I}\|_{L^2} &\leq \|\mathrm{grad}\,\partial^\alpha \mathfrak{F}f_k\|_{L^2} \int |\mathfrak{F}h_{k,\delta,\varepsilon}(\eta)|\,|\eta|\,d\eta \\ &\leq C\delta t^{|\alpha|-n/p+n/2}.\end{aligned}$$

In order to estimate II, observe the following inequalities: if $\xi - \eta \in$ support($\mathfrak{F}f_k$) and $\zeta \in B(\delta t^{-1}\eta_k, \varepsilon t^{-1})$,

$$\left|\frac{\partial}{\partial\zeta_i}\left(\frac{\partial}{\partial\xi}\right)^\alpha \prod_{j=1}^{N} \frac{m_j(\xi-\eta) - m_j(-\zeta)}{m_j(\xi-\eta) - m_j(-\eta_k)}\right| \leq C_\alpha \delta^{-1} t|\xi-\eta|^{-|\alpha|},$$

and, hence, if $\xi - \eta \in \text{support}(\mathfrak{F}f_k)$ and $\eta \in \text{support}(\mathfrak{F}h_{k,\delta,\varepsilon})$,

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(1 - \prod_{j=1}^{N}\frac{m_j(\xi-\eta) - m_j(-\eta)}{m_j(\xi-\eta) - m_j(-\eta_k)}\right)\right|$$

$$= \left|\left[\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\prod_{j=1}^{N}\frac{m_j(\xi-\eta) - m_j(-\zeta)}{m_j(\xi-\eta) - m_j(-\eta_k)}\right]_{\zeta=\delta t^{-1}\eta_k} - [\cdots]_{\zeta=\eta}\right|$$

$$\leq C'_\alpha \delta^{-1}\varepsilon|\xi - \eta|^{-|\alpha|} \leq C'_\alpha\delta^{-1}\varepsilon t^{|\alpha|}.$$

Using this inequality, we obtain, for $|\alpha| \leq M$,

$$\|\partial^\alpha \text{II}\|_{L^2} \leq C\delta^{-1}\varepsilon t^{|\alpha| - n/p + n/2}.$$

Now we can utilize Lemma 2 to obtain

$$\|\mathfrak{F}^{-1}\text{I}\|_{H^p} + \|\mathfrak{F}^{-1}\text{II}\|_{H^p} \leq C\delta + C\delta^{-1}\varepsilon,$$

which implies (13).

This completes the proof of the Theorem.

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