PACIFIC JOURNAL OF MATHEMATICS
Vol 115, No. 2, 1984

DENSITY OF A FINAL SEGMENT OF THE TRUTH-TABLE DEGREES

JEANLEAH MOHRHERR

This work answers two questions from the topic of degrees of unsolvability, which is part of recursive function theory. We give a simple and explicit example of elementary inequivalence of the Turing degrees to the truth-table degrees. In constructing this example, we show that every truth-table degree above that of the halting problem is the jump of another truth-table degree.

The theory of degrees of unsolvability grew directly from the study of questions about decidability. In 1939, Turing presented the concept of one theory being decidable with respect to another theory [**T**]. Later, based on Turing's work with decidability, Kleene and Post defined degrees of unsolvability [**K1P**]. Refinements of degees of unsolvability were defined in [**P**].

$\mathscr{D}_{\mathrm{T}}$ is the structure of all Turing (T) degrees with the induced partial ordering; $\mathscr{D}_{\mathrm{tt}}$, the structure of all truth-table (tt) degrees with the induced partial ordering. For a discussion of Turing degrees and truth-table degrees see Rogers [**R**] or Odifreddi [**02**]. As is standard we use boldface lower case letters to denote degrees of unsolvability. The superscript denotes Turing jump and $^{(n)}$, the Turing jump applied n times.

In their paper on reducibility orderings [**NS**], A. Nerode and R. Shore have the result that for each automorphism Φ of $\mathscr{D}_{\mathrm{T}}$ there is an element $\mathbf{d} \in \mathscr{D}_{\mathrm{T}}$ such that Φ is fixed on the cone of $\mathbf{b} \in \mathscr{D}_{\mathrm{T}}$ for which $\mathbf{b} \geq \mathbf{d}$. They make the point that the proof of this result relies on the completeness property of $\mathscr{D}_{\mathrm{T}}$

$$(\forall \mathbf{a} \in \mathscr{D}_{\mathrm{T}})\big(\mathbf{a} \geq \mathbf{0}^{(n)} \to (\exists \mathbf{b} \in \mathscr{D}_{\mathrm{T}})(\mathbf{b} \cup \mathbf{0}^{(n)} = \mathbf{b}^{(n)} = \mathbf{a})\big)$$

for every $n > 0$; and consequently an analogous theorem for the (for instance) truth-table reducibility ordering would give a proof that each automorphism of $\langle \mathscr{D}_{\mathrm{tt}}, \leq \rangle$ is fixed on a cone.

Later Shore conjectured that the following jump property (a weaker form of completeness) holds for $\mathscr{D}_{\mathrm{tt}}$:

$$(\forall \mathbf{a} \in \mathscr{D}_{\mathrm{tt}})\big(\mathbf{a} \geq \mathbf{0}' \to (\exists \mathbf{b} \in \mathscr{D}_{\mathrm{tt}})(\mathbf{b}' = \mathbf{a})\big).$$

We prove that $\mathscr{D}_{\mathrm{tt}}$ does have this jump property and that moreover

$$(\forall \mathbf{a} \in \mathscr{D}_{\mathrm{tt}})\big(\mathbf{a} \geq \mathbf{0}^{(n)} \to (\exists \mathbf{b} \in \mathscr{D}_{\mathrm{tt}})(\mathbf{b}^{(n)} = \mathbf{a})\big),$$

for every $n > 0$. Still, we do not know whether there is a $\mathbf{d} \in \mathscr{D}_{\mathrm{tt}}$ such that

$$(\forall \mathbf{a} \in \mathscr{D}_{\mathrm{tt}})\big(\mathbf{a} > \mathbf{d} \to (\exists \mathbf{b} \in \mathscr{D}_{\mathrm{tt}})(\mathbf{b} < \mathbf{a} \text{ and } \mathbf{a} = \mathbf{b} \cup \mathbf{d})\big).$$

We also do not know whether every automorphism of $\langle \mathscr{D}_{\mathrm{tt}}, \leq \rangle$ is fixed on a cone. However, we make use of the jump theorem for $\mathscr{D}_{\mathrm{tt}}$ to answer another question about $\mathscr{D}_{\mathrm{tt}}$.

In **[01]**, P. Odifreddi asked if the orderings of T-degrees and tt-degrees are elementary inequivalent. This follows his discussion of r.e. tt-degrees where he reviews the result that there are minimal r.e. tt-degrees **[Ko]**. Shore answers this question in the affirmative **[Sh]** but notes that his method of proof is quite indirect and suggests that a natural sentence might be found to distinguish $\mathscr{D}_{\mathrm{T}}$ from $\mathscr{D}_{\mathrm{tt}}$. We give such a sentence in this paper.

Through the jump theorem for $\mathscr{D}_{\mathrm{tt}}$ we prove that in the tt reducibility ordering the degree $\mathbf{0}'$ has no minimal cover. $\mathbf{a}$ is a *minimal cover* for $\mathbf{b}$ if $\mathbf{a} > \mathbf{b}$ and, for every $\mathbf{c}$, $\mathbf{a} \geq \mathbf{c} \geq \mathbf{b} \to \mathbf{c} = \mathbf{a}$ or $\mathbf{c} = \mathbf{b}$. That every degree in $\mathscr{D}_{\mathrm{T}}$ has a minimal cover is a fundamental result proved by Spector **[Sp]**. Therefore the sentence

$$(\forall \mathbf{a})(\exists \mathbf{b})(\mathbf{b} > \mathbf{a} \text{ and } (\forall \mathbf{c})(\mathbf{b} \geq \mathbf{c} \geq \mathbf{a} \to \mathbf{c} = \mathbf{b} \text{ or } \mathbf{c} = \mathbf{a}))$$

is true for $\mathscr{D}_{\mathrm{T}}$ but false for $\mathscr{D}_{\mathrm{tt}}$. Actually a stronger result holds. In $\mathscr{D}_{\mathrm{tt}}$ no degree above $\mathbf{0}'$ is a minimal cover. By **[JS]**, in $\mathscr{D}_{\mathrm{T}}$ every degree above $\mathbf{0}^{(\omega)}$ is a minimal cover where $0^{(\omega)} = \{\langle n, i\rangle \mid i \in 0^{(n)}\}$.

Other related properties of $\mathscr{D}_{\mathrm{tt}}$ are investigated.

This work with the exception of the corollary to the jump theorem formed part of the author's Ph.D. thesis at the University of Illinois at Chicago written with Carl Jockusch. We thank Carl Jockusch for suggesting that we work on these problems, for his constant example of an open-minded approach, and for his promptness and thoroughness in reading numerous drafts, particularly of the jump theorem for tt-degrees. We thank Richard Shore for pointing out a new direction for this work: In the first draft we had only used our result of density upward holding for complete truth-table degrees to observe that complete truth-table degrees have no minimal covers; he suggested that, since density in the substructure of complete degrees is immediate, we investigate basic properties of dense upper semilattices such as, for instance, the existence of minimal pairs.

A jump theorem for tt-degrees. The Friedberg Completeness Theorem states that for $\mathbf{a} \geq \mathbf{0}'$ $\exists \mathbf{b}$ $(\mathbf{b}' = \mathbf{b} \cup \mathbf{0}' = \mathbf{a})$. So it is a combined cupping up and jump theorem. Here we prove a jump theorem for $\mathscr{D}_{\mathrm{tt}}$.

R. Shore conjectured that the Turing jump operator is onto the complete truth-table degrees. Although he observed that if $A \geq_{\mathrm{tt}} 0'$ and B is the set Friedberg constructs to prove his completeness theorem then $B' \leq_{\mathrm{tt}} A$; Shore did not know if the other direction, $A \leq_{\mathrm{tt}} B'$ holds. Certainly if $A \equiv_{\mathrm{tt}} 0'$ then $A \leq_{\mathrm{tt}} B'$. We give a proof for both $B' \leq_{\mathrm{tt}} A$ and $A \leq_{\mathrm{tt}} B'$; we give the details for $B' \leq_{\mathrm{tt}} A$ as a warm-up for the reduction of A to B' (by truth-tables).

We are breaking from the tradition of degree theory which is to prove theorems by constructing sets. We begin with a set known in the literature and prove the theorem by constructing recursive functions. However, we can cite a precedent. It is an exercise in Rogers [**Ro**] to show that $X \geq_{\mathrm{tt}} 0'$ and $Y \leq_{\mathrm{wtt}} X$ imply $Y \leq_{\mathrm{tt}} X$. Carl Jockusch related a proof of this that employs a technique we use in the jump theorem. (According to our memory) his proof goes this way:

X is an oracle for r.e. questions by the hypothesis $X \geq_{\mathrm{tt}} 0'$. In a truth-table reduction of Y to X, I can ask X any number of r.e. questions as long as I decide recursively which questions I want to ask. Fix a reduction of $Y \leq_{\mathrm{wtt}} X$ and consider the computation for the number n. I have a recursive bound k on the use of the oracle X in the computation. There are 2^k possibilities for the subset of X bounded by k. When I choose one, F say, it is an r.e. question to ask if the computation for n using F puts n into Y. I ask two questions of the oracle for each of the 2^k subsets of ω bounded by k: Is n computed to be in Y and is this subset the true subset of X?

We comment that the disjunction of this pair of questions over 2^k is recusively determined and it a correct truth table for Y using the oracle X. The choice of questions depends on the reduction that was fixed for $Y \leq_{\mathrm{wtt}} X$ and the procedure to find these questions still converges if an oracle other than X is used.

THEOREM 1 (*jump theorem*). *For every $A \geq_{\mathrm{tt}} 0'$, there is a B such that $B' \equiv_{\mathrm{tt}} A$.*

Proof. Let B be the set Friedberg constructed to prove his completeness theorem. We will show that this set satisfies our theorem after we review his construction.

Friedberg's construction.

Let a string be a sequence of zeros and ones. We will use Greek letters σ, δ to represent strings. For a given string σ, $|\sigma|$ means the length of σ and

$\sigma(n)$ refers to the value of the nth position. We say that string σ is contained in set C, i.e. $\sigma \subseteq C$, when, for every $n < |\sigma|$, $\sigma(n) = 0$ if and only if $n \in C$. We use $\Phi_e(C; x)$ to denote the partial recursive function having Gödel number e and using the set C as an oracle. For $|\sigma| = s$, $\Phi_e(\sigma; x)\downarrow$ means using the values coded by σ, the computation refers to the oracle for numbers $< s$ only and the computation converges in s steps. In other words, for every $C \supseteq \sigma$, $\Phi_{e,s}(C; x)\downarrow$. Now given e, x and C it is always the case that when $\Phi_e(C; x)\downarrow$ we can determine this by using finitary information about C. Consequently, given e, x, and finite σ it is the case that when there is a string σ' extending σ for which $\Phi_e(\sigma'; x)\downarrow$, we can find σ' recursively.

For a fixed $A \geq_{tt} 0'$ we construct, in stages, an infinite binary sequence σ so that $B = \{n|\sigma(n) = 0\}$. Beginning with state 0, σ_0 is the empty string; at stage s, σ_{s+1} is a string that extends σ_s.

At even stages $s = 2e$:
If there is an x such that there is a binary string δ of length x which extends σ_s and $\Phi_{e,x}(\delta; e)\downarrow$, we set σ_{s+1} equal to the least such δ. Otherwise, $\sigma_{s+1} = \sigma_s$.

At odd stages $s = 2e + 1$:
We set $\sigma_{s+1} = \sigma_s^*0$ if $e \in A$. Otherwise, $\sigma_{s+1} = \sigma_s^*1$.
End of the construction.

First direction: $B' \leq_{tt} A$.

Let us assume we have completed a truth-table for numbers less than n. So given u as a canonical index of a finite set, we can compute a table to decide if D_u is the set of all $m < n$ with $m \in D'$.

Our table for n is the disjunction of all statements of the following form, where $D_u \subseteq \{m|m < n\}$ and δ is a string of length n: "D_u is the set of all $m < n$ in B', and δ is the binary string of length n that codes an initial segment of A and comp(n, u, δ) holds." comp(n, u, δ) holds if there are strings α_s, for all $s \leq 2n$ such that:

$\alpha_{s+1} = \alpha_s$ if $s = 2j$ and $j \notin D_u$;
α_{s+1} is the first extension of α_s that puts j into the jump if $s = 2j$ and ($j \in D_u$ or $j = n$);
$\alpha_{s+1} = \alpha_s^*\delta(j)$ if $s = 2j + 1$.

What we are doing with comp(n, u, δ) is analyzing the construction of B. It should be noted that comp(n, u, δ) is an r.e. relation and hence, since $0' \leq_{tt} A$, comp(n, u, δ) can be replaced by an equivalent truth-table about A which can be effectively found. Our use function for A is recursive because we need only $A \restriction n$ plus a number of $0'(A)$ questions that can be determined uniformly for each n. We also need to know which $m < n$ have been put into B' before the jump for n was described, but we have that by our induction hypothesis. No information about A is needed to construct the table. We have effectively constructed truth-table conditions with this property: If X is an arbitrary set and $X \geq_{\mathrm{tt}} 0'$, then $\{n | X$ satisfies the truth-table condition for $n\}$ = the jump of the Friedberg set constructed for X.

Other direction: $A \leq_{\mathrm{tt}} B'$.

Since one might be more skeptical about this direction, we first show what the table is for deciding $0 \in A$. Here, let "use $\Phi_0(B; 0)$" be defined only if $0 \in B'$ and be equal to the actual bound on information in B used for the computation.

$$0 \in A \leftrightarrow \text{“}(0 \notin B' \text{ and } 0 \in B) \quad \text{or} \quad (0 \in B' \text{ and use } \Phi_0(B; 0) \in B)\text{”}.$$

The first disjunct is co-r.e. in B and the second disjunct is r.e. in B. For the general case, we ask whether, for some $j \leq n$, $j \in B'$ and use $\Phi_j(B; j) \geq |\sigma_{2j}|$. If not, $n \in A$ iff $n \in B$. Otherwise letting m be the greatest such j, $n \in A$ iff (use $\Phi_m(B; m) + n - m) \in B$. We need to know $A \restriction n$ in order to compute $|\sigma_{2j}|$, for $j \leq n$. However by induction, we can always assume we know this much. We will define a relation r.e. in B that will tell us if $m \leq n$ is such a j. Let $D_u \subseteq \{m | m \leq n\}$ and δ be a string of length n: growth$^B(n, u, \delta, m)$ holds if there are strings α_s for all $s \leq 2m$ such that:

$\alpha_{s+1} = \alpha_s$ if $s < 2m$, $s = 2i$ and $i \notin D_j$;

α_{s+1} is the first extension of α_s that puts i into the jump if $s < 2m$, $s = 2i$ and $i \in D_j$;

$\alpha_{s+1} = \alpha_s^* \delta(i)$ if $s = 2i + 1$; and $|\alpha_{2m}| \leq$ use $\Phi_m(B; m)$.

Now we can show what the table is for deciding $n \in A$.

$$\begin{aligned} n \in A \leftrightarrow \text{“}\bigvee_{u,\delta} \{ & D_u = \{j | j \leq n \text{ and } j \in B'\} \text{ and } \delta \text{ codes } A \restriction n \\ & \text{and } \big\{(n \in B \text{ and } \forall j \leq n \text{ growth}^B(n, u, \delta, j) \text{ is false}) \\ & \text{or } \bigvee_{j \leq n} \big((\text{use } \Phi_j(B; j) + n - j) \in B \text{ and growth}^B(n, u, \delta, j) \\ & \qquad \text{and } \forall m > j \text{ growth}^B(n, u, \delta, m) \text{ is false})\big\}\}\text{”}. \end{aligned}$$

This is a boolean combination of statements r.e. or co-r.e. in B and these statements can be computed effectively for n. □

An n-generic set B is like this: If a number e is not in $B^{(n)}$, the nth jump of B; then B has an initial segment, say $B \upharpoonright s$, such that e is never in $C^{(n)}$ for any set C, $C \upharpoonright s = B \upharpoonright s$. $0'$ is an example of a set that is not n-generic.

The set we use in the proof above is a special case of a 1-generic set. It is special because step $2n$ decides whether $n \in B'$ and therefore the order of the steps is effective.

DEFINITION. $B^{(\omega)} = \{\langle n, i\rangle | i \in B^{(n)}\}$.

COROLLARY (*to the proof*). *For every $k > 0$, for every $A \geq_{\mathrm{tt}} 0^{(k)}$, there is a B such that $B^{(k)} \equiv_{\mathrm{tt}} A$ and, for every $A \geq_{\mathrm{tt}} 0^{(\omega)}$, there is a B such that $B^{(\omega)} \equiv_{\mathrm{tt}} A$.*

Proof. For every $k \geq 2$, let $Q^k\bar{y}$ abreviate the alternation of $k - 1$ quantifiers beginning with $\forall y_2$. For example, $Q^4\bar{y}$ abreviates $\forall y_2 \exists y_3 \forall y_4$. Just as

$$X' = \{e | \exists \delta \subseteq X\, \Phi_e(\delta; e)\downarrow\}$$

and every question of the form

$$\exists \delta \supseteq \sigma\, \Phi_e(\delta; e)\downarrow$$

can be effectively coded as a question concerning membership of $0'$,

$$X^{(k)} = \{e | \exists \delta \subseteq X Q^k\bar{y}\, \Phi_e(\delta; \bar{y}, e)\downarrow\}$$

and every question

$$\exists \delta \supseteq \sigma\; Q^k\bar{y}\, \Phi_e(\delta; \bar{y}, e)$$

can be effectively coded as a question concerning membership of $0^{(k)}$.

For $A \geq_{\mathrm{tt}} 0^{(k)}$, we construct B as in the proof above but with $Q^k\bar{y}\, \Phi_e(\delta; \bar{y}, e)$ in place of $\Phi_e(\delta; e)$ so that we obtain a set B with the property that for every e there is a string $\sigma \subseteq B$ that forces $Q^k\bar{y}\, \Phi_e(\sigma; \bar{y}, e)$. σ forces $Q^k\bar{y}\, \Phi_e(\sigma; \bar{y}, e)$ if $Q^k\bar{y}\, \Phi_e(\sigma; \bar{y}, e)\downarrow$ or, for every δ extending σ, $Q^k\bar{y}\, \Phi_e(\delta; \bar{y}, e)\uparrow$. $B^{(k)} \equiv_{\mathrm{tt}} A$ follows from the reduction procedure for $B' \equiv_{\mathrm{tt}} A$ but asking $0^{(k)}$ questions instead of $0'$ questions.

Now for the case $A \geq_{\mathrm{tt}} 0^{(\omega)}$ and $B^{(\omega)} \equiv_{\mathrm{tt}} A$ is desired, we use what is essentially a forcing argument; satisfaction (or refutation) of every sentence arithmetic in B is determined by a finite amount of information about B. We construct B in the following manner. Let $\langle n, m\rangle$ be the standard enumeration of $\omega \times \omega$.

Construction.

At stages $s = \langle k, e \rangle$ where $k > 0$:
If there is an x such that there is a string δ of length x which extends σ_s and $Q^k \bar{y}\, \Phi_e(\delta;\, \bar{y}, e)$, we set σ_{s+1} equal to the least such δ. Otherwise, $\sigma_{s+1} = \sigma_s$.

At states $s = \langle 0, e \rangle$:
We set $\sigma_{s+1} = \sigma_s^*0$ if $e \in A$. Otherwise $\sigma_{s+1} = \sigma_s^*1$.
End of the construction.

Now $A \equiv_{\text{tt}} B^{(\omega)}$ is proved by induction as $B' \equiv_{\text{tt}} A$ is proved above. □

Application of the jump theorem.

DEFINITIONS. For $A >_{\text{tt}} D$, *A splits over D* if there are sets A_1 and A_2 strictly in between A and D such that $A \leq_{\text{tt}} A_1 \oplus A_2$. For $D <_{\text{tt}} A$, *D cups to A* if there is a set $A_0 <_{\text{tt}} A$ and $A \leq_{\text{tt}} D \oplus A_0$.

THEOREM 2. *If $A \geq_{\text{tt}} 0'$ and $D <_{\text{tt}} A$, then either A splits over D or D cups to A.*

Proof. Choose B so that $B' \equiv_{\text{tt}} A$. Do any Sacks splitting of B' relative to B; say W_i^B and W_j^B. We have $W_i^B,\, W_j^B \leq_m B'$ and $B' \leq_{\text{tt}} W_i^B \oplus W_j^B$.

Now

$$D \leq_{\text{tt}} \begin{array}{c} W_i^B \oplus D \\ W_j^B \oplus D \end{array} \leq_{\text{tt}} B'.$$

If

$$D <_{\text{tt}} \begin{array}{c} W_i^B \oplus D \\ W_j^B \oplus D \end{array} <_{\text{tt}} B',$$

then A splits over D. Otherwise D cups up to C. □

Kallibekov [**Ka**] proved that the r.e. truth-table degrees are dense upwards. We observe that the truth-table degrees below $\mathbf{0}'$ are dense upwards and relativize this result.

THEOREM 3. *Let $A <_{\text{tt}} C'$. There is a set U such that $A <_{\text{tt}} U <_{\text{tt}} C'$.*

Proof. Let $A <_{\text{tt}} C'$. We build a set U such that $A <_{\text{tt}} U <_{\text{tt}} C'$.

Preliminaries. We fix an approximation of A that is recursive in C and has the property that, for every initial segment of A, there is an n such that every approximation after the nth is correct on the initial segment. Also we fix an enumeration of C' that is recursive in C. U, the set we are going to build, is partitioned into uniformly recursive rows. One row is set aside to copy A and the other rows are numbered.

Strategy. We prevent $U \leq_{\mathrm{tt}} A$ by copying C' into U whenever we think there is a reduction. We prevent $C' \leq_{\mathrm{tt}} U$ by withholding all information but A from U when we think there is a reduction. In the end, $U \leq_{\mathrm{tt}} C'$ because U is the recursive join of A with a set *r.e.* in C.

Notation. A^s, U^s, $(C')^s$ are the approximation at stage s. $l(s, i, U, A)$ is the largest $n \leq s$ such that for all $m \leq n$ $\Phi_{i,s}(m)\downarrow$ and $m \in U^s$ iff A^s satisfies the truth-table condition indexed by $\Phi_{i,s}(m)$. Define $l(s, i, C', U)$ similarly.

Construction. Stage s.

We see if $l(s, i, U, A)$ has grown for any $i \leq s$. If so, pick the least such i and copy $(C')^s$ into row i. Next, if $l(s, i, C', U)$ has grown for some $i \leq s$, then for any j, $i < j \leq s$, put $\{0, \ldots, s\}$ into row j. (This is covering up information.)

LEMMA 1. *If Φ_i does not tt-reduce U to A, there is a stage s_0 and a witness w such that, for all stages $s \geq s_0$, $l(s, i, U, A) \leq w$..*

Proof. This is a property of our approximation of A.

LEMMA 2. *If Φ_i does not tt-reduce C' to U; there is a stage s_0 and a witness w such that, for all stages $s \geq s_0$, $l(s, i, C', U) \leq w$.* □

LEMMA 3. *$A \leq_{\mathrm{tt}} U \leq_{\mathrm{tt}} C'$ and U is strictly in between A and C'.*

Proof. U is a recursive join of A and a set r.e. in C. Therefore, $A \leq_{\mathrm{tt}} U \leq_{\mathrm{tt}} C'$. We prove that U is strictly in between by contradiction.

Suppose $U \leq_{\mathrm{tt}} A$. Then we know by Lemma 2 that for every i, $l(s, i, C', U)$ as a function of s is bounded. Let e be the index of a partial recursive function that reduces $U \leq_{\mathrm{tt}} A$. Then by the construction $\{e\} \times C' \subseteq U$ and for some m, $s > m$ implies $\langle e, s\rangle \in C'$ iff $\langle e, s\rangle \in U$. However, this contradicts $U \leq_{\mathrm{tt}} A$.

Now suppose $C' \leq_{tt} U$ and let e be an index for which Φ_e: $C' \leq_{tt} U$. It is clear from the construction that in this case $(N \setminus \{0, \dots, e\} \times N \subseteq U)$. Now add to this that, $l(s, i, U, A)$ as a function of s is bounded, for $i \leq e$, by Lemma 1 and we see that U must be tt-equivalent to A. $U \leq_{tt} A$, however, contradicts $C' \leq_{tt} U$. Thus we have proved $A <_{tt} U <_{tt} C'$. □

Corollary to the proof. If $A <_{tt} C'$, then there is an independent set of U_i, $i \in N$, strictly in between A and C'.

Proof. For every i, we require $U_i \nleq_{tt} \oplus_{j \neq i} U_j$ and therefore, we have the requirement P_e^i: not Φ_e: $U_i \leq_{tt} \oplus_{j \neq i} U_j$. As before we copy C' into row e of U_i when Φ_e looks like it is going to truth-table reduce U_i to $\oplus_{j \neq i} U_j$. Here in addition, we copy N into row k of U_j, for $\langle j, k \rangle > \langle i, e \rangle$.

Now assume that some P_e^i is not satisfied and choose i and e so that, for all j and k such that P_k^j fails, $\langle i, e \rangle \leq \langle j, k \rangle$. Then $C' \leq_{tt} U_i$ and $\oplus_{j \neq i} U_j \leq_m A$ which is a contradiction. □

As usual this implies that any countable partial ordering can be embedded in the truth-table degrees in the interval between A and C' if $A <_{tt} C'$.

THEOREM 4 (*interpolation theorem*).

$$(\exists A)(\forall B \geq_{tt} A)(\forall C <_{tt} B)(\exists D)(C <_{tt} D <_{tt} B).$$

Proof. Let $A = 0'$. By Theorem 1 and by Theorem 3 above, the sentence is true. □

This interpolation theorem goes counter to what might be expected of the complete truth-table degrees. There are minimal r.e. truth-table degrees [**Ko**] and of course the construction relativizes to $0'$ to produce a set M that has degree minimal over $0'$ with respect to truth-table reducibility via functions recursive in $0'$. Note that M is over $0'$ with respect to a function recursive in $0'$.

The tt-degrees above $0'$ are dense. The sentence of Theorem 4 is glaringly false for $\mathscr{D}_T$. This interpolation property for $\mathscr{D}_{tt}$ shows that $\mathscr{D}_{tt}$ is dense above truth-table degeee $\mathbf{0'}$. If $\mathbf{a}$, $\mathbf{b}$ are truth-table degrees above $\mathbf{0'}$ and $\mathbf{0} < \mathbf{b}$ then there is a tt-degree $\mathbf{c}$ for which $\mathbf{a} < \mathbf{c} < \mathbf{b}$ by Theorems 1 and 3.

A partial ordering is an upper semilattice if finite supremums are always defined. An upper semilattice $\mathscr{L}$ is homogeneous if any embedding into $\mathscr{L}$ of a substructure of a finite upper semilattice $\mathscr{F}$ can be extended to an embedding of $\mathscr{F}$. Since density is one of the properties of a homogeneous structure, it is natural to ask if the upper semilattice of the tt-degrees above $\mathbf{0}'$ is homogeneous. It is well known that the Turing degrees and the r.e. degrees are not homogeneous. The r.e. truth-table degrees and the r.e. weak-truth-table degrees have also been shown not to be homogeneous. S. Schwartz gives some evidence that the quotient of the r.e. degrees by a definable ideal may be homogeneous [**Sc**].

A commonly used procedure for showing that upper semilattices with a least element are not homogeneous is producing a minimal pair. Two nonzero elements form a minimal pair if they have an infimum and infimum is equal to 0. The tt-degrees above $\mathbf{0}'$ fail this test because every truth-table degree is the infimum of two strictly greater truth-table degrees.

THEOREM 5. *Let A and B be given with $A <_{\mathrm{tt}} B$. For some set C, $C >_{\mathrm{tt}} A$ and the truth-table degree of A is the infimum of those of B and C.*

Proof. We will construct an infinite binary string σ to represent the set, S, of all n such that $\sigma(n) = 0$. The plan is that $C = S \oplus A$. We construct σ by making a sequence of finite extensions, working always with a finite initial segment σ_s, the initial segment defined by stage s.

For each partial recursive function Φ_i we have the requirement

$$R_i\colon \text{Not } \Phi_i\colon S \leq_{\mathrm{tt}} A.$$

We use $\{i\}^A$ to denote $\{n \mid A$ satisfies the truth-table condition coded by $\Phi_i(n)\}$. $\{i\}^{\sigma_s \oplus A}$ refers to that initial segment of $\{i\}^{S\oplus A}$ whose use function is less than the length of σ_s. For each pair $\{i\}^{S\oplus A}$ and $\{j\}^B$, we have the requirement

$$Q_{i,j}\colon \Phi_i \text{ and } \Phi_j \text{ total and } \{i\}^{S\oplus A} = \{j\}^B \to \{j\}^B \leq_{\mathrm{tt}} A.$$

Construction.

Stages $s = 2i$: We can satisfy R_i once and for all at this stage because σ_s has a finite extension that satisfies the requirement.

Stages $s = 2(\langle i, j\rangle) + 1$: We deal with $Q_{i,j}$; possibly, we may not satisfy this directly. We look for a finite extension σ_{s+1} of σ_s such that, for some x, $\{i\}^{\sigma_{s+1}\oplus A}(x)$ and $\{j\}^B(x)$ are defined and distinct. If none such exists let $\sigma_{s+1} = \sigma_s$.

End of the constuction.

Set $C = S \oplus A$. $A <_{\text{tt}} C$ because we satisfy not Φ_i: $S \leq_{\text{tt}} A$ at stage $2i$. We also succeed in making A the infimum of B and C: If the set $X \leq_{\text{tt}} B, C$, then for some i and j, $X = \{i\}^C$ and $X = \{j\}^B$. This can only happen (since Φ_i, Φ_j are total) if $\{i\}^{T \oplus A} = \{j\}^B$ for every set T extending σ_s where $s = 2(\langle i, j \rangle) + 1$. Let T be recursive with T extending σ_s so that $X = \{j\}^B \leq_{\text{tt}} T \oplus A \leq_{\text{tt}} A$. □

REFERENCES

[F] Richard M. Friedberg, *A criterion for completeness of degrees of unsolvability*, J. SL. **22** (1958), 159–160.

[JS] Carl G. Jockusch, Jr. and Richard A. Shore, *Pseudo jump operators II: The transfinite case and the minimal cover problem*, to appear.

[Ka] S. Kallibekov, *On degrees of recursively enumerable sets*, Siberian Math. J., **14–2** (1973), 290–293.

[K1P] S. C. Kleene and Emil L. Post, *The upper semilattice of degrees of unsolvability*, Ann. of Math., (2) **59** (19954), 379–407.

[Ko] G. N. Kobzev, *On tt-degrees of recursively enumerable Turing degrees*, Math. USSR Sbornik, **35–2** (1979), 173–180.

[NS] Anil Nerode and Richard A. Shore, *Reducibility orderings: theories, definability, and automorphisms*, Ann. Math. Logic, **18** (1980), 61–89.

[01] Piergiorgio Odifreddi, *Strong reducibilities*, Bull. Amer. Math. Soc., **4** (1981), 37–85.

[02] ______, *Classical Recursion Theory*, Springer-Verlag, to appear.

[P] Emil L. Post, *Recursively enumerable sets of positive integers and their decision problems*, Bull. Amer. Math. Soc., **50** (1944), 284–316.

[R] Hartley Rogers, Jr., *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York (1967).

[Sa] G. E. Sacks, *Degrees of Solvability*, Annals of Math. Studies, **55**, Princeton University Press (1963).

[Sc] Steven Thomas Schwartz, Ph.D. thesis, 1982, University of Chicago, Illinois.

[Sh] Richard A. Shore, *The Turing and Truth-Table-Degrees are not Elementarily Equivalent*, Logic Colloquim'80, D. van Dolen ed., North-Holland, Amsterdam.

[Sp] C. Spector, *On degrees of rerecursive unsolvability*, Ann. of Math., (2) **64** (1965), 581–92.

[T] A. M. Turing, *Systems of logic based on ordinals*, Proc. London Math. Soc., **45** (1939), 161–228.

Received February 7, 1983.

DEPARTMENT OF COMPUTER SCIENCE
NORTHERN ILLINOIS UNIVERSITY
DEKALB, IL 60115