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COBORDISM OF MANIFOLDS WITH STRONG ALMOST TANGENT STRUCTURES

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This paper studies the unoriented cobordism classes of closed smooth manifolds whose tangent bundles admit nilpotent bundle endomorphisms.

1. Introduction. An almost tangent manifold is a smooth (differentiable of class C^∞) manifold M^{2n} for which the structure group of its tangent bundle $\tau(M)$ reduces to the group of matrices of the form $\left(\begin{smallmatrix} A & 0 \\ B & A \end{smallmatrix}\right)$, $A \in \mathrm{GL}_n(R)$, $B \in \mathrm{End}(R^n)$. The study of these manifolds is motivated by the observation that tangent manifolds have this property, i.e., if N^n is a smooth manifold and E^{2n} is the total space of $\tau(N)$, then E^{2n} (a tangent manifold) is an almost tangent manifold.

Note that if M^{2n} is almost tangent, then the matrix

$$\begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix} \in \mathrm{End}(R^{2n}),$$

where I_n is the $n \times n$ identity matrix, induces a bundle endomorphism J: $\tau(M) \to \tau(M)$ having constant rank n and satisfying $J^2 = 0$. As J has constant rank, $\tau(M)$ is isomorphic to the Whitney sum $\operatorname{im} J \oplus \ker J$, and $\operatorname{im} J = \ker J$, so one has $\tau(M) \cong 2(\operatorname{im} J)$. Thus M^{2n} has an almost complex structure and, for M^{2n} closed, one is led to the well-known result that M is cobordant to a square $N^n \times N^n$. In particular, the cobordism class of M is decomposable in $\mathfrak{N}_{2n}$, the unoriented cobordism group.

In [4] Epiopoulous defines an almost tangent structure of order r on a smooth manifold M^n to be a bundle endomorphism J: $\tau(M) \to \tau(M)$ of constant rank which satisfies $J^r \neq 0$ and $J^{r+1} = 0$ (thus, an almost tangent manifold has an almost tangent structure of order 1). Motivated by the splitting property enjoyed by almost tangent manifolds, one says that an almost tangent structure of order r, J, on M^n is strong and of type k provided J^r: $\tau(M) \to \tau(M)$ has constant rank k.

PROPOSITION 1.1. *Let $r, k \geq 1$, and $n \geq (r+1)k$. A compact smooth manifold M^n has a strnog almost tangent structure of order r and type k iff*

$$\tau(M) \cong (r+1)\xi^k \oplus \eta^{n-(r+1)k}$$

for some vector bundles ξ^k, $\eta^{n-(r+1)k} \to M^n$. (*Note*: $(r+1)\xi^k$ *denotes the* $(r+1)$*-fold Whitney sum of* ξ^k.)

Proof. If J is such a structure, then for each fiber TM_x of $\tau(M)$, one has $TM_x \cong (r+1)(\text{im } J_x^r) \oplus V_x$, where $J_x = J \mid TM_x$, $\oplus$ is the direct sum of vector spaces, and V_x is a complementary vector subspace of TM_x. Since J^r has constant rank k, necessity follows by setting $\xi^k = \text{im } J^r$. For sufficiency, note that if $\tau(M) \cong (r+1)\xi^k \oplus \eta$, then J: $\tau(M) \to \tau(M)$, defined by $J(x_1,\ldots,x_{r+1}, y) = (0, x_1,\ldots,x_r, 0)$, has the desired properties. □

This splitting characterization will be exploited to study the unoriented cobordism classes of closed smooth manifolds having strong almost tangent structures. Preliminary results are presented in §2. Manifolds with structures of type 1 are studied in §3, and manifolds with structures of type k, $k > 1$, are treated briefly in §4. Most of the results of this paper are contained in the author's dissertation at the University of Virginia.

2. Preliminaries. In [**8**] Stong shows that if $\tau(M)$ contains an odd-dimensional subbundle, then $w_n(M)[M] = 0$. The following lemma, which is used throughout the paper, is a direct consequence.

LEMMA 2.1. *If* M^n *has a strong almost tangent structure of type* k, *and* k *is odd, then* $w_n(M)[M] = 0$.

The extreme case of a strong almost tangent structure on M^n, $n \geq 2$, is a structure of order $(n-1)$ and type 1, i.e., $\tau(M^n) \cong n\xi^1$. If $w(\xi) = 1 + a$, then $w(M) = (1+a)^n$ and $w_i(M) = \binom{n}{i}a^i$. By 2.1, $a^n[M] = w_n(M)[M] = 0$. Since all Stiefel-Whitney numbers of M are multiples of $a^n[M] = 0$, it follows that M^n bounds.

The converse of this result is false, for in [**6**] Rosendo and Gadea show that for $n \neq 3$ or 7, the sphere S^n does not admit an almost tangent structure of order $(n-1)$. However, if M^n bounds, then M is cobordant to an n-fold product of circles which has tangent bundle isomorphic to an n-fold Whitney sum of trivial line bundles. Thus, one has

PROPOSITION 2.2. *A class* $\alpha \in \mathfrak{N}_n$, $n \geq 2$, *is represented by a manifold with a strong almost tangent structure of order* $(n-1)$ *and type* 1 *iff* $\alpha = 0$.

THEOREM 2.3. *Every 3-manifold* M^3 *has* $\tau(M) \cong 3\xi^1$ *for some* $\xi^1 \to M^3$ (*hence, every* M^3 *has a strong almost tangent structure of order* 2 *and type* 1).

Proof. By restricting attention to components, one may assume M^3 connected. If $v = 1 + v_1$ is the Wu class, then

$$w(M) = \mathrm{Sq}(1 + v_1) = 1 + v_1 + v_1^2 = 1 + w_1 + w_1^2.$$

Let ξ^1 be the determinant of $\tau(M^3)$. Then

$$w(\xi \otimes \tau(M)) = (1 + w_1(\xi))^3 + w_1(1 + w_1(\xi))^2 + w_1^2(1 + w_1(\xi)),$$

and since $w_1(\xi) = w_1$, it follows that $w(\xi \otimes \tau(M)) = 1$, so $\xi \otimes \tau(M)$ is a Spin_3 bundle, and there is a lift:

$$\begin{array}{ccc} & & B\,\mathrm{Spin}_3 \cong HP(\infty) \\ & \overset{f}{\nearrow} & \downarrow \\ M^3 & \xrightarrow[\xi \otimes \tau(M)]{} & BO_3 \end{array}$$

Since $HP(\infty)$ is 3-connected, f is null homotopic, so $\xi \otimes \tau(M)$ is trivial. Therefore,

$$\tau(M) \cong 1 \otimes \tau(M) \cong (\xi \otimes \xi) \otimes \tau(M) \cong \xi \otimes 3 \cong 3\xi^1. \qquad \square$$

REMARKS. (1) Clark and Goel [2] prove this for $M^3 = S^3$.

(2) This is a modification of the well-known argument that an orientable 3-manifold is parallelizable.

Fix integers $r \geq 2$, $k \geq 1$, and, for $n \geq rk$, define $A_n(r, k)$ to be the subset of classes in $\mathfrak{N}_n$ represented by manifolds having strong almost tangent structures of order $(r - 1)$ and type k. In terms of 1.1,

$$A_n(r, k) = \{\alpha \in \mathfrak{N}_n | \exists\, M^n \in \alpha \text{ with } \tau(M) \cong r\xi^k \oplus \eta\}.$$

It is easily seen that $A_n(r, k)$ is a subgroup of $\mathfrak{N}_n$, and if $A(r, k)$ is the weak direct sum $\Sigma_{n \geq rk} A_n(r, k)$, then it also follows that $A(r, k)$ is an algebra ideal of $\mathfrak{N}_*$. Most of this paper is devoted to the study of the ideals $A(r, 1)$. For convenience, we will write $A(r)$ for $A(r, 1)$.

3. The ideals $A(r)$. In this section some indecomposable classes of $A(r)$ are exhibited. Recall the manifolds $RP(n_1, \ldots, n_t)$ as defined in [7]. $RP(n_1, \ldots, n_t)$ is the real projective space bundle of $\lambda_1 \oplus \cdots \oplus \lambda_t \to RP(n_1) \times \cdots \times RP(n_t)$, where λ_i is the pullback of the canonical line bundle over the ith factor, and has dimension $n_1 + \cdots + n_t + t - 1$. In [7] Stong proves that for $t > 1$, $RP(n_1, \ldots, n_t)$ is indecomposable iff

$$\sum_{i=1}^{t} \binom{n_1 + \cdots + n_t + t - 2}{n_i}$$

is odd.

THEOREM 3.1. *There exist indecomposable classes* $\alpha_n \in A_n(r)$ *for*:
(a) $n = 4k, r = 4k - 2\ (k \geq 1)$,
(b) $n = 4k + 1, r = 4k - 1\ (k \geq 1)$,
(c) $n = 4k + 2, r = 4k - 1\ (k \geq 1)$, *and*
(d) $n = 2^p(2q + 1) - 1, r = 2^{p+1}q - 1\ (p \geq 2, q \geq 1)$.

Proof. The indecomposability of the classes to be defined follows from the above remark. For $RP(n_1,\ldots,n_t)$, let p be the projection $RP(n_1,\ldots,n_t) \to RP(n_1) \times \cdots \times RP(n_t)$, let γ be the canonical line bundle over $RP(n_1,\ldots,n_t)$ consisting of pairs

$$(\text{line in a fiber of } \lambda_1 \oplus \cdots \oplus \lambda_t, \text{ vector in that line}),$$

and let μ^{t-1} be the bundle along the fibers so that

$$\tau(RP(n_1,\ldots,n_t)) \cong p^*\tau(RP(n_1) \times \cdots \times RP(n_t)) \oplus \mu^{t-1}.$$

(a) Let α_{4k} be the class of $RP(4k - 3, 1, 0)$ which has τ isomorphic to $(4k - 2)p^*\lambda_1 \oplus \mu^2$.

(b) Let α_{4k+1} be the class of $RP(4k - 2, 1, 0)$ which has τ isomorphic to $(4k - 1)p^*\lambda_1 \oplus \mu^2$.

(c) Let α_{4k+2} be the class of $RP(1, 1, 0,\ldots,0)$ with $(4k - 1)$ zeros. This manifold has τ isomorphic to $1 \oplus 1 \oplus \mu^{4k}$, and it is well known that

$$\begin{aligned} 1 \oplus \mu^{4k} &\cong \gamma \otimes p^*(\lambda_1 \oplus \cdots \oplus \lambda_{4k+1}) \\ &\cong (\gamma \otimes p^*\lambda_1) \oplus \cdots \oplus (\gamma \otimes p^*\lambda_{4k+1}). \end{aligned}$$

For $i = 3,\ldots,4k + 1$, $p^*\lambda_i$ is trivial (pullback of canonical bundle over $RP(0) =$ point). Thus, it follows that

$$\begin{aligned} \tau(RP(1, 1, 0,\ldots,0)) &\cong 1 \oplus (\gamma \otimes p^*\lambda_1) \oplus (\gamma \otimes p^*\lambda_2) \oplus (4k - 1)\gamma \\ &\cong (4k - 1)\gamma \oplus \eta^3. \end{aligned}$$

(d) Let $\alpha_{2^p(2q+1)-1}$ be the class of $RP(2^{p+1}q - 2, 1, 0,\ldots,0)$ with $(2^p - 1)$ zeros. This manifold has τ isomorphic to $(2^{p+1}q - 1)p^*\lambda_1 \oplus \mu^{2^p}$. □

The question as to whether these are the largest values of r for which there is an indecomposable $\alpha_n \in A_n(r)$ is answered partially by

THEOREM 3.2. *Suppose* M^n *is a closed manifold.* (a) *If* $\tau(M^n) \cong (n - 1)\xi^1 \oplus \eta^1$, *then* M^n *bounds.* (b) *If* $n \equiv 2 \bmod 4$, $n \geq 6$, *and* $\tau(M^n) \cong (n - 2)\xi^1 \oplus \eta^2$, *then* M^n *bounds.*

REMARKS. (1) This theorem generalizes 2.2 in dimensions $\not\equiv 0 \bmod 4$.

(2) The result shows that the results of 3.1(a)–(c) are best possible. In fact, stronger results are obtained. Namely, the values of r in 3.1(a)–(c) are the largest for which there is a *nonzero* $\alpha_n \in A_n(r)$. (3) An example showing that 3.1(d) is not best possible will be presented later in this section.

To begin to establish 3.2(a), suppose $\tau(M^n) \cong (n-1)\xi^1 \oplus \eta^1$ and consider $[M^n, \xi, \eta]$ as an element of $\mathfrak{N}_n(RP(\infty) \times RP(\infty))$. The result is trivial for $n = 1$, so assume $n \geq 2$. Consider $X^n = RP(\lambda^1 \oplus (n-1)) \to RP(1)$ where λ^1 is the canonical line bundle over $RP(1)$. Let γ be the canonical line bundle over X^n, and also denote by λ^1 the pullback of λ^1 to X^n by the projection. Then one has $\tau(X) \cong (n-1)\gamma \oplus (\gamma \otimes \lambda)$, and there is the element $[X^n, \gamma, \gamma \otimes \lambda] \in \mathfrak{N}_n(RP(\infty) \times RP(\infty))$. If $c = w_1(\gamma)$ and $d = w_1(\gamma \otimes \lambda)$, then it is a routine calculation to show that for $0 \leq i \leq n$, $c^i d^{n-i}[X] = 1$ iff $i \equiv \bmod 2$.

LEMMA 3.3. *If* $[M^n, \xi, \eta] \neq 0$, *then* $[M^n, \xi, \eta] = [X^n, \gamma, \gamma \otimes \lambda]$.

Proof. Suppose $[M^n, \xi, \eta] \neq 0$ and let $w(\xi) = 1 + a$, $w(\eta) = 1 + b$. By the remarks above, it suffices to show that for $0 \leq i \leq n$, $a^i b^{n-i}[M^n] = 1$ iff $i \equiv n \bmod 2$.

For $n = 2$, one has $\tau(M^2) \cong \xi^1 \oplus \eta^1$, $w(M) = (1 + a)(1 + b) = 1 + (a + b) + ab$, and $ab[M] = w_2[M] = 0$ (by 2.1), so M^2 bounds. Therefore, $w_1^2[M] = 0$ which implies $a^2[M] = b^2[M] = 1$, so the result is true for $n = 2$.

Having assumed inductively the result true for $2 \leq n' < n$, suppose one has $[M^n, \xi, \eta] \neq 0$. Now observe that $[M^n, \xi, \eta]$ must have a nonzero characteristic number divisible by a. (To see this, note that if n is odd, the $w_1(\xi) = b$, so $b^n[M] = w_1 b^{n-1}[M] = \mathrm{Sq}^1 b^{n-1}[M] = 0$. If n is even, say $n = 2^p(2q + 1)$ $(p > 0, q \geq 0)$, one sees that

$$v_{2^p}(M) = a^{2^p} + b^{2^p} + a^{2^p - 1}b, \quad \text{so} \quad b^{2^p} = v_{2^p}(M) + a^{2^p} + a^{2^p - 1}b.$$

Hence,

$$\begin{aligned} b^n[M] &= \left\{v_{2^p}(M) + a^{2^p} + a^{2^p - 1}b\right\} b^{2^{p+1}q}[M] \\ &= \mathrm{Sq}^{2^p} b^{2^{p+1}q}[M] + ax[M] \end{aligned}$$

and

$$\mathrm{Sq}^{2^p} b^{2^{p+1}q}[M] = \binom{2^{p+1}q}{2^p} b^n[M] = 0.)$$

Dualizing ξ in M gives a submanifold $N^{n-1} \hookrightarrow M$ with

$$\tau(N) \cong (n-2)(\xi|N) \oplus (\eta|N), \quad \text{and} \quad ax[M] = (x|N)[N].$$

Since $[M, \xi, \eta]$ has a nonzero number divisible by a, $[N, (\xi|N), (\eta|N)] \neq 0$. The result follows by applying the induction hypothesis to $[N^{n-1}, (\xi|N), (\eta|N)]$. □

The result of 3.2(a) now follows, for if $[M^n, \xi, \eta] = 0$, then M^n bounds, and if $[M^n, \xi, \eta] = [X^n, \gamma, \gamma \otimes \lambda]$, then M is cobordant to X, and Conner and Floyd have shown [**3**] that X bounds.

The result of 3.2(b) is aided by the notion of a Yoshida manifold. As in [**10**] and [**11**], one defines a Yoshida manifold to be a closed manifold M^n which has Stiefel-Whitney numbers satisfying

$$w_\omega(M)[M] = 0 \quad \text{if } \omega \neq (2^{p_1}, 2^{p_2}, \ldots, 2^{p_t}).$$

Stong shows in [**10**] that if M^n is a Yoshida manifold, and n is not a power of 2, then M^n bounds.

Proof of 3.2(b). Let $n = 4k + 2$, $k \geq 1$, and suppose $\tau(M^{4k+2}) \cong 4k\xi^1 \oplus \eta^2$. It is claimed that the element

$$[M^{4k+2}, 4\xi] \in \mathfrak{N}_{4k+2}(HP(\infty))$$

is zero (it would then follow that M bounds). Let $w(\xi) = 1 + a$, $w(\eta) = 1 + b_1 + b_2$, and note that $[M, 4\xi]$ is determined by characteristic numbers $a^{4j} w_\omega(M)[M]$. For $k = 1$, one has $w(M^6) = (1 + a)^4(1 + b_1 + b_2)$ and $a^4 = w_4$, so to establish $[M^6, 4\xi] = 0$, it suffices to show M^6 bounds. Now $w_6[M] = 0$ (by 2.1), and $w_3^2[M] = 0$ since $w_3 = 0$. Also,

$$w_1^6[M] = b_1^6[M] = \mathrm{Sq}^3 b_1^3[M] = v_3 b_1^3[M] = b_1 b_2 b_1^3[M]$$
$$= b_1^2 \mathrm{Sq}^1 b_1 b_2[M] = b_1^2 \mathrm{Sq}^1 \mathrm{Sq}^1 b_2[M] = 0$$

since $\mathrm{Sq}^1 \mathrm{Sq}^1 = 0$. Since w_6, w_3^2 and w_1^6 determine on $\mathfrak{N}_6$, it follows that M^6 bounds.

Assume the result true for $1 \leq k' < k$ and that one has $[M^{4k+2}, 4\xi]$. Dualizing 4ξ in M gives $N^{4k-2} \hookrightarrow M$ with $\tau(N) \cong 4(k-1)(\xi|N) \oplus (\eta|N)$. Then $[N, 4(\xi|N)] = 0$ by induction, so $j > 0$ implies

$$a^{4j} w_\omega(M)[M] = \{a^{4j-4} w_\omega(M)|N\}[N] = 0.$$

Therefore, all numbers involving a^4 vanish, so $w(M)$ is equivalent in numbers to $1 + w_1 + w_2$, i.e., M is a Yoshida manifold, and for $k \geq 1$, $4k + 2$ cannot be a power of 2. Thus, M bounds, so the remaining numbers $w_\omega(M)[M]$ are zero. Therefore, $[M^{4k+2}, 4\xi] = 0$. □

It will now be shown that 3.1(d) is not best possible. Consider the indecomposable manifold $RP(7, 14, 0) \to RP(7) \times RP(14)$ and pull the fibration back over $RP(7) \times P^8 \times P^4 \times P^2$ by a degree 1 map f, where

$$P^8 = RP(\lambda \oplus 1) \to RP(7), \qquad P^4 = RP(\lambda \oplus 1) \to RP(3),$$

and

$$P^2 = RP(\lambda \oplus 1) \to RP(1).$$

(f is the identity on the first factor, and on $P^8 \times P^4 \times P^2$, f is the map which classifies the tensor product of the bundles along the fibers $P^8 \times P^4 \times P^2 \times \overset{\mu_1^1 \otimes \mu_2^1 \otimes \mu_3^1}{\to} RP(14)$). By Proposition 2.4 of [7], $M^{23} = f^*\{RP(7, 14, 0)\}$ is indecomposable. Moreover, M^{23} fibers over $RP(7) \times RP(7) \times RP(3) \times RP(1)$, so $\tau(M)$ has 18 linearly independent sections. Therefore, $\tau(M) \cong 18 \oplus \eta^5$.

Since this construction uses the fact that $\tau(RP(m))$ is trivial for $m = 1$, 3 and 7, it cannot be generalized to produce similar examples in other dimensions. Nonbounding examples (which are decomposable) are easily constructed in dimensions $4k + 3$, $k \geq 1$, by taking a product of a nonbounding P^5 with $RP(4k - 2)$. The product then has τ isomorphic to $(4k - 1)\lambda^1 \oplus \eta^4$ ($\tau(P^5)$ has a section). It will be shown momentarily that this result is best possible.

This discussion raises two questions: (1) Are there other examples of indecomposables in $A_n(r)$ with $n = 2^p(2_q + 1) - 1$ and $r > 2^{p+1}q - 1$? (2) What is the largest value of r for which there exist indecomposables in $A_n(r)$, $n = 2^p(2q + 1) - 1$ ($p > 2$, $q > 0$)? Theorem 3.1(d) is best possible for $p = 2$ by

THEOREM 3.4. *If the closed manifold M^{4k+3}, $k \geq 0$, has $\tau(M) \cong 4k\xi^1 \oplus \eta^3$, then M bounds.*

LEMMA 3.5 (a) *If $\tau(M^7) \cong 4\xi^1 \oplus \eta^3$, then M^7 bounds.* (b) *If $\tau(M^{11}) \cong 8\xi^1 \oplus \eta^3$, then M^{11} bounds.*

Proof. (a) A nonbounding P^7 must have $w_5(P)w_2(P)[P] \neq 0$. If $\tau(M^7) \cong 4\xi^1 \oplus \eta^3$, then $w_5(M) = w_1^4(\xi)w_1(\eta)$, $w_2(M) = w_2(\eta)$, so

$$w_5(M)w_2(M)[M] = w_1(\eta)w_2(\eta)w_1^4(\xi)[M] = w_1(M)w_2(M)w_1^4(\xi)[M]$$
$$= \mathrm{Sq}^3 w_1^4(\xi)[M] = 0.$$

(b) Assuming $\tau(M^{11}) \cong 8\xi^1 \oplus \eta^3$, let

$$w(\xi) = 1 + a,$$
$$w(\eta) = 1 + w_1 + w_2 + w_3 = 1 + w_1(M) + w_2(M) + w_3(M).$$

Then $a^8 w_3[M] = 0$ by 2.1, $a^8 w_1^3[M] = \mathrm{Sq}^1 a^8 w_1^2[M] = 0$, and $a^8 w_1 w_2[M] = \mathrm{Sq}^3 a^8[M] = 0$, so $w(M)$ is equivalent in numbers to $1 + w_1 + w_2 + w_3$. The proof that all $w_\omega[M]$ involving w_1, w_2 and w_3 are zero is routine, but tedious, and will be omitted. □

PROPOSITION 3.6. *If M^{4k+3}, $k \geq 0$, has $w(M)$ equivalent to $1 + w_1 + w_2 + w_3$ in numbers, then M bounds.*

Proof. Remarks in the proof of 3.5 give a basis for induction on k, so assume the result true for all $2 \leq k' < k$, and suppose $w(M^{4k+3}) \sim 1 + w_1 + w_2 + w_3$. Note that if M is nonbounding, then some Stiefel-Whitney number divisible by w_3 is nonzero (otherwise M is a Yoshida manifold with $4k + 3 = 2^s$ impossible).

Suppose $w_3^2 w_\omega[M] \neq 0$. Dualize w_3^2 in M to get $N^{4k-3} \hookrightarrow M$ with $\bar{w}(N) \sim 1 + w_1 + w_2 + w_3$ (use of restriction homomorphisms is omitted in this proof) and $w_3^2 w_\omega[M] = \bar{w}_\omega[N]$. By Proposition 5 of [**9**], N is cobordant to a Dold manifold $P(2^r - 1, 2^{s+1} - 2)$, $0 < r \leq s$. Then $4k - 3 = 2^r - 1 + 2^{s+2} - 4$, so $r = 1$ and $4k + 3 = 2^{s+2} + 3$ with $N^{2^{s+2}-3} \sim P(1, 2^{s+1} - 2)$ having $\bar{w} = 1 + d + cd$, $d \in H^2(P(1, 2^{s+1} - 2); Z_2)$, $c \in H^1(P(1, 2^{s+1} - 2); Z_2)$. The only nonzero number of N is then $\bar{w}_3 \bar{w}_2^{2^{s+1}-3}[N]$, so the only possible nonzero number of M involving w_3^2 is $w_3^3 w_2^{2^{s+1}-3}[M]$.

Consider $M \xrightarrow{2^s \tau} BO$ and pull back to BO_{3-2^s} so that $w \sim 1 + w_1^{2^s} + w_2^{2^s} + w_3^{2^s}$. Now $w_3^{2^s} w_\omega[M] = 0$ (note that $s > 1$ since $k > 2$), so one can dualize $w_2^{2^s}$, which gives $N^{2^{s+1}+3} \hookrightarrow M$ with

$$w(N) \sim (1 + w_1 + w_2 + w_3)/(1 + w_1^{2^s} + w_2^{2^s}),$$

and $w_2^{2^s} w_\omega[M] = w_\omega[N]$. Now $w_2^{2^s}$ contributes zero on N, for $w_2^{2^s} x_3[N] = w_2^{2^{s+1}} x_3[M]$, and

$$w_2^{2^{s+1}} w_1^3[M] = \mathrm{Sq}^1\big(w_2^{2^{s+1}} w_1^2\big)[M] = 0,$$

$$w_2^{2^{s+1}} w_1 w_2[M] = \mathrm{Sq}^3 w_2^{2^{s+1}}[M] = 0, \quad \text{and} \tag{3.7}$$

$$w_2^{2^{s+1}} w_3[M] = w_2^{2^{s+1}}\big(w_1 w_2 + \mathrm{Sq}^1 w_1 w_2\big)[M] = 2 w_1 w_2^{2^{s+1}+1}[M] = 0.$$

Therefore, $w(N) \sim (1 + w_1 + w_2 + w_3)/(1 + w_1^{2^s})$. Dualize $w_1^{2^s}$ in N to get $P^{2^{s+3}} \hookrightarrow N$ with $w(P) \sim (1 + w_1 + w_2 + w_3)/(1 + w_1^{2^{s+1}})$. By dimension, $w_1^{2^{s+1}} = 0$ in $H^*(P)$, so $w(P) \sim 1 + w_1 + w_2 + w_3$, and P bounds by induction. Therefore, $w_1^{2^s} w_\omega[N] = w_\omega[P] = 0$, i.e., $w_1^{2^s}$ contributes zero on N, so $w(N) \sim 1 + w_1 + w_2 + w_3$.

Hence, N bounds by induction, so $w_3^3 w_2^{2^{s+1}-3}[M] = w_3^3 w_2^{2^s-3}[N] = 0$. Therefore, all numbers of M involving w_3^2 are zero.

Now dualize w_3 in M to get $P^{4k} \hookrightarrow M$ with $P \to BO$ reducing to BO_2, so one can dualize w_2 in P to get $N^{4k-2} \hookrightarrow P \hookrightarrow M$ with $\bar{w}(N) \sim 1 + w_1 + w_2$ and $w_2 w_3 w_\omega[M] = \bar{w}_\omega[N]$. By Proposition 3′ of [**9**], N is cobordant to $RP(2^{p+1} - 2) \times RP(2^{q+1} - 2)$, $0 \le p \le q$. Then $4k - 2 = 2^{p+1} + 2^{q+1} - 4$ implies $p = 0$, so $N^{2^{q+1}-2} \sim RP(2^{q+1} - 2)$ having only $\bar{w}_1^{2^{q+1}-2}[N]$ nonzero. Therefore $w_2 w_3 w_1^{2^{q+1}-2}[M]$ is the only possible nonzero number involving $w_2 w_3$.

Dualize $w_1^{2^q}$ in M to get Q^{2^q+3} with

$$w(Q) \sim (1 + w_1 + w_2 + w_3)/\big(1 + w_1^{2^q}\big).$$

But $w_1^{2^q}$ contributes zero on Q, for $w_1^{2^q} x_3[Q] = w_1^{2^q} x_3[M]$ (see 3.7). Therefore $w(Q) \sim 1 + w_1 + w_2 + w_3$, so Q bounds by induction. Hence,

$$w_2 w_3 w_1^{2^{q+1}-1}[M] = w_2 w_3 w_1^{2^q-1}[Q] = 0.$$

Hence, the only possible nonzero number of M divisible by w_3 is

$$w_1^{4k} w_3[M] = w_1^{4k}\big(w_1 w_2 + \mathrm{Sq}^1 w_2\big)[M] = 2 w_1^{4k+1} w_2[M] = 0.$$

Therefore, M bounds. □

Proof of 3.4. Lemma 3.5 gives a basis for induction on k, so assume the result true for $1 \le k' < k$, and suppose $\tau(M^{4k+3}) \cong 4k\xi^1 \oplus \eta^3$. Let $a = w_1(\xi)$ and note that any w_ω divisible by a can be written $w_\omega = a^{4s} w_{\omega'}$, $0 < s \le k$, where $w_{\omega'}$ is a polynomial in $w(\eta)$. Dualizing $4s\xi$ in M gives $N^{4(k-s)+3} \hookrightarrow M$ with $\tau(N) \cong 4(k - s)(\xi | N) \oplus (\eta | N)$. If $s < k$, then N bounds by induction; if $s = k$, then $\dim N = 3$ and N bounds. Therefore, $a^{4s} w_{\omega'}[M] = w_{\omega'}(N)[M] = 0$. Hence, all numbers divisible by a are zero, so $w(M) \sim w(\eta)$ in numbers. The result follows by 3.6. □

4. The ideals $A(r, k)$, $k > 1$. Here the case $k > 1$ is treated briefly; results are not as extensive as those of §3. By 3.1, one has

COROLLARY 4.1. *Let $k \geq 1$. There exist indecomposables $\alpha_n \in A_n(r, k)$ for*:

(a) $n = 4m$, $r = [(4m - 2)/k]$ $(m \geq 1)$

(b) $n = 4m + 1$, $r = [(4m - 1)/k]$ $(m \geq 1)$

(c) $n = 4m + 2$, $r = [(4m - 1)/k]$ $(m \geq 1)$, *and*

(d) $n = 2^p(2q + 1) - 1$, $r = [(2^{p+1}q - 1)/k]$ $(p \geq 2, q \geq 1)$.

provided that $r \geq 2$ in each case ($[x]$ is the greatest integer $\leq x$).

There is a subalgebra $E = \Sigma_{k \geq 0} E_{2k}$ of $\mathfrak{N}_*$, where E_{2k} is the subset of classes of $\mathfrak{N}_{2k}$, represented by weakly almost complex manifolds for which all Chern numbers divisible by an odd-dimensional Chern class are zero. Floyd [**5**] showed there are generators x_i, $i > 1$, $i \neq 2^t - 1$, of $\mathfrak{N}_*$ such that

$$E \subset Z_2\left[x_i^{\varepsilon_i} | \varepsilon_{2k-1} = 8, \varepsilon_{2^k} = 8, \varepsilon_{2^k k(2j+1)} = 4\right]. \tag{4.2}$$

(It was later shown by Buhštaber [**1**] that equality holds in 4.2.)

If M^{rk+j} $(r \geq 2)$ has $\tau(M) \cong r\xi^k \oplus \eta^j$, then dualizing $(r - 2)\xi^k \oplus \eta^j$ in M gives $N^{2k} \hookrightarrow M$ with $\tau(N) \cong 2(\xi^k | N)$. Then $w_{rk+j}(M)[M] = w_{2k}(N)[N]$, and since $\tau(N) \cong 2(\xi | N)$, it follows that $[N] \in E_{2k}$. By 4.2 one sees that $E_{2k} = 0$ for $0 < 2k < 16$ and for $2k \not\equiv 0 \bmod 8$. Thus, one has the following generalization of Proposition 2.1:

PROPOSITION 4.3. *Suppose M^n has a strong almost tangent structure of type k. If $k = 4$ or if $k \not\equiv 0 \bmod 4$, then $w_n(M)[M] = 0$.*

Using this, it is easy to show that for $k = 2$, 4.1(a) is best possible. Also, for $k = 2$, 4.1(b) is best possible:

PROPOSITION 4.4. (a) *If M^{4m}, $m > 0$, is closed and $\tau(M) \cong 2m\xi^2$, then M bounds.* (b) *If M^{4m+1}, $m > 0$, is closed and $\tau(M) \cong 2m\xi^2 \oplus \eta^1$, then M bounds.*

Proof. (a) Let $w(\xi) = 1 + a_1 + a_2$ so $w(M) = (1 + a_1^2 + a_2^2)^m$. Then all Stiefel-Whitney numbers of M are sums of terms $a_1^{2J}a_2^{2j}[M]$, $2i + 4j = 4m$. For $i > 0$,

$$a_1^{2i}a_2^{2J}[M] = \mathrm{Sq}^1\left(a_1^{2i-1}a_2^{2j}\right)[M] = w_1 a_1^{2i-1}a_2^{2J}[M] = 0$$

since $w_1 = 0$, and for $i = 0$, one has $a_2^{2m}[M] = w_{4m}[M] = 0$ by 4.3. For (b), let $w(\xi) = 1 + a_1 + a_2$, $w(\eta) = 1 + b$ so

$$w(M) = (1 + a_1^2 + a_2^2)^m(1 + b).$$

Then all Stiefel-Whitney numbers of M are sums of terms $a_1^{2i}a_2^{2j}b^l[M]$ with $2i + 4j + l = 4m + 1$, so l must be odd. Then

$$a_1^{2i}a_2^{2j}b^l[M] = b(a_1^{2i}a_2^{2j}b^{l-1})[M] = w_1(a_1^{2i}a_2^{2j}b^{l-1})[M]$$
$$= \mathrm{Sq}^1(a_1^{2i}a_2^{2j}b^{l-1})[M] = 0$$

since $a_1^{2i}a_2^{2j}b^{l-1}$ is a square. □

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