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CAPILLARY SURFACES OVER OBSTACLES

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We consider the usual capillarity problem with the additional requirement that the capillary surface lies above some obstacle. This involves a variational inequality instead of a boundary value problem. We prove existence of a solution to the variational inequality and study the boundary regularity. In particular, global $C^{1,1}$-regularity is shown for a wider class of variational inequalities with conormal boundary condition.

Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$ and let

$$(0.1) \qquad A = -D_i(a^i(p)),^1 \qquad a^i(p) = p_i \cdot \left(1 + |p|^2\right)^{-1/2}$$

be the minimal surface operator. Then we study the variational inequality

$$(0.2) \qquad \langle Au + H(x,u), v - u\rangle \geq 0 \quad \forall v \in K,$$
$$K := \{ v \in H^{1,\infty} | v \geq \psi \}$$

where

$$(0.3) \qquad \langle Au, \eta\rangle = \int_\Omega a^i(Du) \cdot D_i\eta \, dx + \int_{\partial\Omega} \beta\eta \, dH_{n-1}.$$

Here H describes a gravitational field, ψ is the obstacle and β is the cosine of the contact angle at the boundary. We make the assumption that

$$(0.4) \qquad H = H(x,t) \in C^{0,1}(\mathbf{R}^n \times \mathbf{R}), \qquad \beta \in C^{0,1}(\partial\Omega)$$

satisfy the conditions

$$(0.5) \qquad \frac{\partial H}{\partial t} \geq \kappa > 0$$

and

$$(0.6) \qquad |\beta| \leq 1 - a, \qquad a > 0.$$

Under these assumptions Gerhardt **[2]** showed, that (0.2) admits a solution $u \in H^{2,p}(\Omega)$, if we impose on ψ the further condition

$$(0.7) \qquad -a^i(D\psi) \cdot \gamma_i \geq \beta \quad \text{on } \partial\Omega$$

[1] Here and in the following we sum over repeated indices.

where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is the exterior normal to $\partial\Omega$. The main theorem which we shall prove, is the following:

THEOREM 0.1. *Let $\partial\Omega$ be of class C^2, let $\psi \in H^{2,\infty}(\Omega)$ and assume that H and β satisfy* (0.4)–(0.6). *Then the variational inequality* (0.2) *admits a solution*

$$u \in H^{1,\infty}(\Omega) \cap H^{2,2}(\Omega) \cap H^{2,\infty}_{\mathrm{loc}}(\Omega)$$

with continuous tangential derivatives at the boundary. In the case $n = 2$ we have $u \in C^1(\bar{\Omega})$. Furthermore, if we assume that $\partial\Omega$ is of class $C^{3,\alpha}$, $\beta \in C^{1,1}(\partial\Omega)$ and that ψ satisfies (0.7) *then we have*

$$u \in H^{2,\infty}(\Omega).$$

REMARKS. (i) The physically interesting problem, where ψ is the bottom of a cylinder containing some liquid of prescribed volume, is also included in this setting: a solution of this problem fulfills (0.2), if we replace H by $(H + \lambda)$ with some Lagrange multiplier λ. (See Gerhardt [**2**, **3**]).

(ii) The boundary regularity results in Theorem 0.1 are valid for solutions of a much wider class of variational inequalities with conormal boundary condition, see §§3 and 4 below.

To prove the existence of a solution to (0.2) it is necessary to establish a priori estimates for the gradient of solutions to the corresponding boundary value problem:

$$Au + \tilde{H}(x, u) = 0 \quad \text{in } \Omega \tag{0.8}$$

$$-a^i(Du) \cdot \gamma_i = \beta \quad \text{on } \partial\Omega. \tag{0.9}$$

Using ideas of Ural'ceva [**12**] and Gerhardt [**2**] we can find a bound for $|Du|_\Omega$ which does not explicitly depend on $|\tilde{H}(\cdot, u)|_\Omega$.

At this place the author wishes to thank Claus Gerhardt for many helpful discussions.

NOTATION. We shall denote by $|\cdot|_\Omega$ the supremum norm on Ω and by $\|\cdot\|_p$ the norms of the L^p-spaces. By $c = c(\cdots)$ we shall denote various constants whereas indices will be used, if a constant recurs at another place.

1. Existence. To get a Lipschitz solution to (0.2), we consider the following related boundary value problems:

$$\begin{aligned} Au_\varepsilon + H(x, u_\varepsilon) + \mu\Theta_\varepsilon(u_\varepsilon - \psi) &= 0 \quad \text{in } \Omega \\ -a^i(Du_\varepsilon) \cdot \gamma_i &= \beta \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where $\mu > 0$ is a parameter tending to infinity and Θ_ε is a sequence of smooth monotone functions approximating the maximal monotone graph Θ:

$$(1.2) \qquad \Theta(t) = \begin{cases} 0, & t > 0, \\ [-1,0], & t = 0, \\ -1, & t < 0, \end{cases} \qquad \Theta_\varepsilon(t) = \begin{cases} 0, & t \geq 0, \\ -1, & t \leq -\varepsilon. \end{cases}$$

We want to use the following existence result from ([**2**], Theorem 2.1):

THEOREM 1.1. *Let $\partial\Omega$ be of class $C^{2,\alpha}$ and suppose that H and β are $C^{1,\alpha}$-functions in their arguments. Then the boundary value problem* (0.8), (0.9) *has a unique solution $u \in C^{2,\lambda}(\overline{\Omega})$, where λ, $0 < \lambda < 1$, is determined by the above quantities.*

Assuming for a moment these sharper differentiability condition on $\partial\Omega$, β and H, we get a unique regular solution u_ε of (1.1) for any ε, $0 < \varepsilon < 1$. In §2 we shall establish a priori estimates for u_ε:

THEOREM 1.2. *There is a large constant M, so that*

$$(1.3) \qquad |u_\varepsilon|_\Omega + |Du_\varepsilon|_\Omega \leq M$$

uniformly in ε and μ. Furthermore, for each ε, $0 < \varepsilon < 1$, we can choose μ as large that

$$(1.4) \qquad u_\varepsilon - \psi \geq -3\varepsilon.$$

Thus we conclude, that in the limit case a subsequence of the u_ε converges uniformly to some function $u \in H^{1,\infty}(\Omega)$, which satisfies (0.2).

Since the estimate (1.3) is independent of the sharper differentiability assumptions, an approximation argument shows, that the variational problem (0.2) has a solution $u \in H^{1,\infty}(\Omega)$ assuming only the weaker conditions.

2. A priori estimates for $|u|$ and $|Du|$. To derive an upper bound for u_ε, we multiply (1.1) with $\max(u_\varepsilon - k, 0)$ for an arbitrary $k \geq k_0 = \sup_\Omega \psi$. Observing that the critical term

$$(2.1) \qquad \int_{u_\varepsilon > k} \Theta_\varepsilon(u_\varepsilon - \psi)(u_\varepsilon - k)\, dx$$

vanishes because of $k \geq \sup \psi$, we get an uniform upper bound in view of the strict monotonicity of H.

For proving the estimate (1.4), we multiply (1.1) with

$$w = \max(\psi - u_\varepsilon - \delta, 0) \tag{2.2}$$

and denote by $A(\delta)$ the set $\{x \in \Omega | u_\varepsilon < \psi - \delta\}$. We get

$$\begin{aligned}\int_{A(\delta)} a^i(Du_\varepsilon)\cdot(D_i\psi - D_iu_\varepsilon)\,dx + \int_{\partial\Omega} \beta w\,dH_{n-1}\\ + \int_{A(\delta)} H(x,u_\varepsilon)(\psi - u_\varepsilon - \delta)\,dx\\ + \mu\cdot\int_{A(\delta)} \Theta_\varepsilon(u_\varepsilon - \psi)(\psi - u_\varepsilon - \delta)\,dx = 0.\end{aligned} \tag{2.3}$$

On $A(\delta)$ we have $\Theta_\varepsilon(u_\varepsilon - \psi) = -1$ and $H(x, u_\varepsilon) \le H(x, \psi)$ because of $\delta \ge \varepsilon$ and in view of the monotonicity of H. To estimate the boundary integral, we use (0.6) and the inequality

$$\int_{\partial\Omega} g\,dH_{n-1} \le \int_\Omega |Dg|dx + c(\Omega, n)\cdot\int_\Omega |g|dx, \qquad g \in H^{1,1} \tag{2.4}$$

which is proven in ([4], Lemma 1). We get

$$\begin{aligned} a\cdot\int_{A(\delta)} |Du_\varepsilon|dx + \mu\cdot\int_{A(\delta)} \psi - u_\varepsilon - \delta\,dx\\ \le (1 + 2|D\psi|_\Omega)|A(\delta)| + |H(\cdot,\psi)|_\Omega\cdot\int_{A(\delta)}\psi - u_\varepsilon - \delta\,dx\\ + c\cdot\int_{A(\delta)}\psi - u_\varepsilon - \delta\,dx\end{aligned} \tag{2.5}$$

or, better

$$\begin{aligned}\int_\Omega |Dw|dx + \mu\cdot\int_\Omega w\,dx \le c(a, |D\psi|_\Omega)|A(\delta)|\\ + (c_1 + |H(\cdot,\psi)|_\Omega)\cdot\int_\Omega w\,dx.\end{aligned} \tag{2.6}$$

Choosing now

$$\mu \ge \mu_1 + |H(\cdot,\psi)|_\Omega + c_1 \tag{2.7}$$

we get by the Sobolev imbedding theorem

$$\|w\|_{n/(n-1)} + \mu_1\cdot\int_\Omega w\,dx \le c|A(\delta)| \qquad \forall\,\delta \ge \varepsilon. \tag{2.8}$$

From this we derive the inequalities

$$\left.\begin{aligned}(\delta_1 - \delta_2)|A(\delta_1)| &\le c|A(\delta_2)|^{1+1/n}\\ (\delta_1 - \delta_2)|A(\delta_1)| &\le \mu_1^{-1}\cdot c|A(\delta_2)|\end{aligned}\right\} \quad \forall\,\delta_1 > \delta_2 \ge \varepsilon. \tag{2.9}$$

From a lemma due to Stampacchia ([**11**], Lemma 4.1) we now deduce from the first inequality

$$u_\varepsilon - \psi \geq -2\varepsilon - c\big(a, |D\psi|_\Omega\big)|A(2\varepsilon)|^{1/n} \tag{2.10}$$

and then from the second

$$|A(2\varepsilon)| \leq \mu_1^{-1} \cdot \varepsilon^{-1} \cdot c|A(\varepsilon)|. \tag{2.11}$$

Thus, inequality (1.4) follows by choosing μ_1 large enough, where μ_1 depends on ε, a, $|D\psi|_\Omega$, Ω.

The gradient bound will be established by a suitable modification of a proof in [**2**].

In view of the smoothness of $\partial\Omega$, we can extend β and γ into the whole domain Ω, so that $\beta \in C^{0,1}(\overline{\Omega})$ still satisfies (0.6) and so that the vectorfield γ is uniformly Lipschitz continuous in Ω and absolutely bounded by 1.We denote by S the graph of u_ε

$$S = \big\{ X = (x, x^{n+1})|x^{n+1} = u_\varepsilon(x)\big\} \tag{2.12}$$

and by $\delta = (\delta_1, \ldots, \delta_{n+1})$ the differential operators on S, i.e.

$$\delta_i g = D_i g - \nu_i \cdot \sum_{k=1}^{n+1} \nu^k \cdot D_k g, \qquad g \in C^1(\overline{\Omega}^{n+1}) \tag{2.13}$$

where $\nu = (\nu_1, \ldots, \nu_{n+1})$ is the exterior unit normal to S

$$\nu = \big(1 + |Du_\varepsilon|^2\big)^{-1/2} \cdot (-D_1 u_\varepsilon, \ldots, -D_n u_\varepsilon, 1). \tag{2.14}$$

As in [**2**] and [**12**] we want to prove that the function

$$v = \big(1 + |Du_\varepsilon|^2\big)^{1/2} + \beta \cdot D_k u_\varepsilon \cdot \gamma^k \equiv W + \beta \cdot D_k u_\varepsilon \cdot \gamma^k \tag{2.15}$$

is uniformly bounded in Ω. Notice, that

$$|Du_\varepsilon| \leq \big(1 + |Du_\varepsilon|^2\big)^{1/2} = W \leq \frac{1}{a} \cdot v. \tag{2.16}$$

During the proof we shall write u instead of u_ε and we set

$$\tilde{H}(x, u) := H(x, u) + \mu \cdot \Theta_\varepsilon(u - \psi). \tag{2.17}$$

We need the following lemmata:

LEMMA 2.1. *For any function $g \in C^1(\overline{\Omega})$ we have the inequality*

$$\begin{aligned} &\left(\int_S |g|^{n/(n-1)}\, dH_n\right)^{(n-1)/n} \\ &\quad \leq c_2(n) \cdot \left(\int_S |\delta g|\, dH_n + \int_S |\tilde{H}|\, |g|\, dH_n + \int_{\partial\Omega} |g| \cdot W\, dH_{n-1}\right). \end{aligned} \tag{2.18}$$

For functions vanishing on the boundary, this inequality was first established in **[9]**, whereas a proof of the general case can be found in **[2]**.

LEMMA 2.2. *On the boundary* $\partial\Omega$ *we have the estimate*

$$\left|\gamma^i \cdot a^{ij}\left(D_j v - D_j(\beta\gamma^k) \cdot D_k u\right)\right| \le c_3 \tag{2.19}$$

where $c_3 = c_3(\partial\Omega, |D\beta|_\Omega)$ *and* $a^{ij} = \partial a^i/\partial p_j$.

LEMMA 2.3. *For any positive function* $\eta \in H^{1,\infty}(\Omega)$ *we have the estimate*

$$\int_{\partial\Omega} v\eta \, dH_{n-1} \le \int_S |\delta\eta| dH_n + \int_S \left(|\tilde{H}| + |\delta\gamma|\right)\eta \, dH_n. \tag{2.20}$$

For a poof of these two lemmata see (**[2]**, Lemma 1.2 and Lemma 1.4).

Furthermore, from the proof of Lemma 1.3 in **[2]** we get the following inequalities:

LEMMA 2.4. *In the whole domain* Ω *we have*

$$a^{ij} D_j D_k u \cdot a^{k1} D_i D_1 u \ge \frac{1}{n} |\tilde{H}|^2 \tag{2.21}$$

$$\begin{aligned} &\left|a^{ij} D_j D_k u \cdot D_i(\beta\gamma^k)\right| \\ &\quad \le \eta \cdot a^{ij} D_j D_k u \cdot a^{k1} D_i D_1 u + c_\eta \cdot \left(1 + \frac{|\delta v|}{W}\right) \end{aligned} \tag{2.22}$$

where $0 < \eta < 1$ *is arbitrary and* $c_\eta = c_\eta(a, n, |D(\beta\gamma)|)$.

Now we are ready to bound the function v, or equivalently

$$w = \log v. \tag{2.23}$$

As in **[2]**, we start with the integral identity

$$\int_\Omega D_k a^i D_i \chi \, dx = -\int_\Omega D_k D_i a^i \chi \, dx + \int_{\partial\Omega} \gamma^i \cdot D_k a^i \chi \, dH_{n-1}. \tag{2.24}$$

Choosing now $\chi = (a^k + \beta\gamma^k)\eta$, $0 \le \eta \in H^{1,\infty}(\Omega)$ with $\operatorname{supp}\eta \subset \{w > h\}$, where h is large, we obtain in view of (1.1)

$$\begin{aligned} &\int_\Omega a^{ij}\left[D_j v - D_j(\beta\gamma^k) \cdot D_k u\right] D_i\eta + a^{ij} D_k D_j u \cdot a^{k1} D_1 D_i u \cdot \eta \, dx \\ &\qquad + \int_\Omega D_k \tilde{H} \cdot (a^k + \beta\gamma^k)\eta \, dx \\ &\quad = -\int_\Omega a^{ij} D_k D_j u \cdot D_i(\beta\gamma^k)\eta \, dx \\ &\qquad + \int_{\partial\Omega} \gamma^i \cdot a^{ij}\left[D_j v - D_j(\beta\gamma^k) \cdot D_k u\right]\eta \, dH_{n-1}. \end{aligned} \tag{2.25}$$

Remark that

$$D_j v = \left(a^k + \beta\gamma^k\right) \cdot D_k D_j u + D_j\left(\beta\gamma^k\right) \cdot D_k u. \tag{2.26}$$

In the following we shall use the relations

$$a^{ij} D_i g \cdot D_j g = W^{-1} |\delta g|^2 \qquad \forall\, g \in C^1(\bar{\Omega}) \tag{2.27}$$

$$\left|a^{ij} D_i g \cdot D_j \chi\right| \le W^{-1} \cdot |\delta g|\, |D\chi| \qquad \forall\, \chi \in C^1(\bar{\Omega}) \tag{2.28}$$

$$a \cdot W \le v \le 2 \cdot W \tag{2.29}$$

$$a^{ij} p_i q_j \le \frac{\varepsilon}{2} \cdot a^{ij} p_i p_j + \frac{1}{2\varepsilon} \cdot a^{ij} q_i q_j \qquad \forall\, \varepsilon > 0. \tag{2.30}$$

Now observe that

$$D_k \tilde{H} = \frac{\partial H}{\partial x_k} + \frac{\partial H}{\partial t} \cdot D_k u + \mu \Theta_\varepsilon' \cdot D_k(u - \psi). \tag{2.31}$$

Then in view of the assumptions (0.5) and (0.6) and in view of the Lemmata 2.2 and 2.4 we can deduce from (2.25)

$$\begin{aligned} &\int_\Omega a^{ij}\left[D_j v - D_j\left(\beta\gamma^k\right) \cdot D_k u\right] D_i \eta \, dx + \int_\Omega \frac{1}{2n} |\tilde{H}|^2 \eta \, dx \\ &\qquad \le c_3 \cdot \int_{\partial\Omega} \eta \, dH_{n-1} + c_4 \cdot \int_\Omega \left(\frac{|\delta v|}{W} + 1\right) \eta \, dx \end{aligned} \tag{2.32}$$

where $c_4 = c_4(|\delta(\beta\gamma)|_\Omega, |\partial/\partial x\, H(\cdot, u)|_\Omega)$. Here we used that $\operatorname{supp} \eta \subset \{w > h_0\}$, $h_0 = h_0(a, |D\psi|_\Omega)$ large. We choose

$$\eta = v \cdot \max(w - k, 0) \equiv v \cdot z \tag{2.33}$$

and set $A(k) = \{X \in S | w(x) > k\}$, $|A(k)| = H_n(A(k))$. Taking the relations (2.27)–(2.30) into account, we obtain in view of $dH_n = W\, dx$ and in view of Lemma 2.3

$$\int_{A(k)} |\delta z|^2 \, dH_n + \int_{A(k)} \frac{1}{n} \cdot |\tilde{H}|^2 z \, dH_n \le c \cdot |A(k)| + c \cdot \int_{A(k)} z \, dH_n \tag{2.34}$$

where $c = c(a, n, |D\gamma|_\Omega, |D\beta|_\Omega, |(\partial/\partial x) H(\cdot, u)|_\Omega)$. To proceed further, we need the following Lemma:

LEMMA 2.5. *For any $\varepsilon > 0$ the integral $\int_{A(k)} w - k\, dx$ can be estimated by*

$$\varepsilon \cdot \int_{A(k)} |\delta z|^2 \, dH_n + \varepsilon \cdot \int_{A(k)} |\tilde{H}|^2 z \, dH_n + c \cdot \varepsilon^{-1} |A(k)|. \tag{2.35}$$

Proof of Lemma 2.5. We shall use the identity

$$\int_\Omega a^i D_i\eta\, dx + \int_\Omega \tilde{H}\eta\, dx + \int_{\partial\Omega} \beta\eta\, dH_{n-1} = 0 \tag{2.36}$$

with $\eta = u \cdot \max(w - k, 0) = u \cdot z$. The boundary integral can be estimates with the help of (2.4) and we obtain in view of (0.6)

$$\begin{aligned} a \cdot \int_{\{w>k\}} W \cdot z\, dx &\le \int_{\{w>k\}} |\tilde{H}|\, |u| z\, dx + c \cdot \int_{\{w>k\}} |u|\, |Dw| dx \\ &\quad + c \cdot \int_{\{w>k\}} |u| z\, dx \\ &\le \varepsilon \cdot \int_{\{w>k\}} |\tilde{H}|^2 z\, dx + c \cdot \varepsilon^{-1} \cdot \int_{\{w>k\}} z\, dx \\ &\quad + \varepsilon \cdot \int_{\{w>k\}} |Dw|^2 W^{-1}\, dx + c \cdot \varepsilon^{-1} \cdot \int_{\{w>k\}} W\, dx \\ &\le \varepsilon \cdot \int_{\{w>k\}} W|\delta w|^2\, dx + \varepsilon \cdot \int_{\{w>k\}} |\tilde{H}|^2 z\, dx \\ &\quad + c \cdot \varepsilon^{-1} \cdot \int_{\{w>k\}} W\, dx. \end{aligned} \tag{2.37}$$

Here we used that $z \le W$ for $k \ge k_0$. The conclusion of the Lemma now immediately follows.

By Lemma 2.5 we deduce from (2.34) for $k \ge k_0$

$$\int_{A(k)} |\delta w|^2\, dH_n + \int_{A(k)} \frac{1}{n} |\tilde{H}|^2 z\, dH_n \le c \cdot |A(k)|. \tag{2.38}$$

Furthermore, from the Sobolev imbedding, Lemma 2.1 and from Lemma 2.3 we conclude

$$\begin{aligned} &\left(\int_S |z|^{n/(n-1)}\, dH_n \right)^{(n-1)/n} \\ &\quad \le c(n) \cdot \left(\int_S |\delta z| dH_n + \int_S |\tilde{H}| z\, dH_n + \int_\Omega W \cdot z\, dH_n \right) \\ &\quad \le c \cdot \left(\left(\int_S |\delta z|^2\, dH_n \right)^{1/2} |A(k)|^{1/2} \right. \\ &\qquad \left. + \varepsilon \cdot \int_S |\tilde{H}|^2 \cdot z\, dH_n + c_\varepsilon \cdot \int_S z\, dH_n \right). \end{aligned} \tag{2.39}$$

To estimate the first term on the righthand side we note that in view of (2.38) we have

$$\left(\int_S |\delta z|^2\, dH_n\right)^{1/2} \leq c|A(k)|^{1/2}. \tag{2.40}$$

Hence, we deduce from (2.38) and (2.39)

$$\begin{aligned}\left(\int_S |z|^{n/(n-1)}\, dH_n\right)^{(n-1)/n} &+ \int_S |\delta z|^2\, dH_n + \int_S \frac{1}{n}|\tilde{H}|^2 z\, dH_n \\ &\leq c|A(k)| + \varepsilon \cdot \int_{A(k)} |\tilde{H}|^2 z\, dH_n + c_\varepsilon \cdot \int_{A(k)} z\, dH_n.\end{aligned} \tag{2.41}$$

Applying again Lemma 2.5 we conclude finally

$$\left(\int_S |z|^{n/(n-1)}\, dH_n\right)^{(n-1)/n} \leq c \cdot |A(k)| \qquad \forall\, k \geq k_0. \tag{2.42}$$

The Hölder inequality yields

$$\int_S z\, dH_n \leq c|A(k)|^{1+1/n} \qquad \forall\, k \geq k_0 \tag{2.43}$$

and we are now in the same situation as in (2.8). It follows that

$$w = \log v \leq k_0 + c \cdot |A(k_0)|^{1/n} \tag{2.44}$$

where $k_0 = k_0(a, |D\psi|_\Omega, n)$ and $c = c(|(\partial/\partial x)H(\cdot, u)|_\Omega, a, n, |\delta\gamma|_\Omega, |D\beta|_\Omega, \Omega)$.

To complete the proof of the gradient bound, we have to establish an estimate for $|S| = \int_\Omega W\, dx$ independent of μ and ε. To accomplish this, we use (2.36) with $\eta = u - \psi$. We obtain

$$\begin{aligned}&\int_\Omega a^i(Du) \cdot D_i(u - \psi)\, dx + \int_\Omega H(x, u)(u - \psi)\, dx \\ &\quad + \mu \cdot \int_\Omega \Theta_\varepsilon(u - \psi)(u - \psi)\, dx + \int_{\partial\Omega} \beta \cdot (u - \psi)\, dH_{n-1} = 0.\end{aligned} \tag{2.45}$$

The critical term

$$\mu \cdot \int_\Omega \Theta_\varepsilon(u - \psi)(u - \psi)\, dx \tag{2.46}$$

is positive in view of the monotonicity of Θ_ε. Using again (0.5), (0.6) and (2.4) we conclude

$$a \cdot \int_\Omega W\, dx \leq c\big(|\Omega|, |u|_\Omega, |\psi|_\Omega, |H(\cdot, \psi)|_\Omega, |D\psi|_\Omega, a, n\big). \tag{2.47}$$

This completes the proof of Theorem 1.2.

REMARK. (i) As a consequence of (2.44) and (2.47) there is a gradient bound for solutions u of (0.8), (0.9), which does not depend on $|\tilde{H}(\cdot, u)|_\Omega$, but only on $|\tilde{H}(\cdot, 0)|_\Omega$.

(ii) After having finished the present article the author became acquainted with a paper of Lieberman [**8**] who obtained a gradient bound for solutions to conormal derivative problems.

3. C^1-Regularity. It is well known, that a solution of u of (0.2) satisfies

$$Au \in L^\infty(\Omega) \tag{3.1}$$

and therefore is in $H^{2,p}_{\mathrm{loc}}(\Omega)$ for any finite p.

To prove regularity results up to the boundary, we transform a neighbourhood $\Omega_\delta = \Omega \cap B_\delta(x_0)$ of a point $x_0 \in \partial\Omega$ with a C^2-diffeomorphism y into

$$B_1^+ = \{ x \in \mathbf{R}^n | \; |x| < 1, x^n > 0 \} \tag{3.2}$$

such that

$$\Gamma = y(\partial\Omega \cap B_\delta(x_0)) = \{ x \in \mathbf{R}^n | \; |x| < 1, x^n = 0 \}. \tag{3.3}$$

The transformed u satisfies in B_1^+ a local variational inequality of the same type as (0.2), where the transformed a^i depend now on x too. Furthermore, the relations

$$\begin{aligned} a^\rho(\hat{p}, p^n) &= a^\rho(\hat{p}, -p^n), \qquad 1 \le \rho \le n-1, \\ a^n(\hat{p}, p^n) &= -a^n(\hat{p}, -p^n) \end{aligned} \tag{3.4}$$

are not lost by the transformation.

In order to prove the continuity of the tangential derivatives of u, we shall use an approach due to Frehse [**1**]. We introduce the notations

$$[\xi]^p = |\xi|^{p-1} \cdot \xi, \qquad \forall \xi \in \mathbf{R}, \tag{3.5}$$

and

$$D_i^{\pm h} g(x) = \pm h^{-1} \cdot \{ g(x \pm h e_i) - g(x) \} \tag{3.6}$$

where e_i denotes the ith unit vector.

By the same arguments as in ([**1**], Lemma 2.1) we have

LEMMA 3.1. *Let u be a solution to* (0.2) *and let $0 \le \Phi \in H_0^{1,\infty}(B_1(0))$, $\operatorname{supp} \Phi \subset B_1$. Then for each $h \in]0, \operatorname{dist}(\operatorname{supp} \Phi, \partial B_1)[$ and each $p \ge 1$, $c \in \mathbf{R}$ there is an $\varepsilon > 0$ such that the functions*

$$u_\varepsilon := u + \varepsilon \cdot D_j^{-h}\big(\Phi \cdot D_j^h(u - \psi)\big), \qquad j = 1, \ldots, n-1, \tag{3.7}$$

and

$$(3.8)\quad u_\varepsilon^p := u + \varepsilon \cdot D_j^{-h}\left[\Phi \cdot D_j^h(u - \psi) - c\right]^p, \qquad j = 1, \ldots, n-1,$$

lie in K.

Now we can show the following Lemma

LEMMA 3.2. *The solution* u *of the local variational inequality obtained from* (0.2) *lies in* $H^{2,2}(B_{1/2}^+)$ *and satisfies*

$$(3.9)\qquad \int_{B_{1/2}^+} |D^2 u|^2 \cdot |x|^{2-n}\, dx < \infty.$$

Proof of Lemma 3.2. (i) We insert the function u_ε of Lemma 3.1 into the variational inequality and obtain

$$(3.10)\qquad \begin{aligned} &-\int_{B_1^+} D_j^h\left(a^i(x, Du)\right) D_i\left(\Phi D_j^h(u-\psi)\right) dx \\ &-\int_\Gamma D_j^h \beta \cdot \Phi D_j^h(u-\psi)\, d\hat{x} \\ &+\int_{B_1^+} H(x,u) \cdot D_j^{-h}\left(\Phi D_j^h(u-\psi)\right) dx \ge 0 \end{aligned}$$

in view of $1 \le j \le n-1$ and since $\Phi = \tau^2$ is a cut-off function in $C_0^\infty(B_1)$. The boundary integral can be estimated by

$$(3.11)\quad |D\beta| \left(\int_{B_1^+} \left| D\left(\tau^2 D_j^h(u-\psi)\right)\right| dx + c \cdot \int_{B_1^+} \tau^2 \left|D_j^h(u-\psi)\right| dx \right).$$

Since $u \in H^{1,\infty}(\Omega)$, the $a^{ij}(x, Du(x))$ are uniformly elliptic and we obtain by standard arguments that $D_j^h Du$ is uniformly bounded in $L^2(B_{1/2}^+)$ as $h \to 0$ and thus $D_j Du \in L^2(B_{1/2}^+)$. Now we deduce from this and from (3.1), that $D_n Du \in L^2(B_{1/2}^+)$.

(ii) Let $n \ge 3$. By Lemma 3.1 and by (i) we have the inequality

$$(3.12)\quad \left\langle Au + H(x,u), D_j\left(\Phi \cdot D_j(u-\psi)\right)\right\rangle \ge 0, \qquad 1 \le j \le n-1.$$

In order to find a suitable test function Φ, we define in $B_1(0)$

$$(3.13)\qquad b^{ij}(\hat{x}, x^n) = \begin{cases} a^{ij}(x; D\tilde{u}(x)), & x^n > 0, \\ a^{ij}(\hat{x}, -x^n; D\tilde{u}(x)), & x^n < 0, \end{cases}$$

where

$$\tilde{u}(\hat{x}, x^n) = \begin{cases} u(x), & x^n > 0, \\ u(\hat{x}, -x^n), & x^n < 0. \end{cases} \tag{3.14}$$

The function $\tilde{\psi}$ is defined similarly.

Now let $\delta_h \in L^\infty(B_1(0))$ satisfy $\delta_h \geq 0$, supp $\delta_h \subset B_1(0)$ and

$$\int_{B_1} \delta_h \, dx = 1, \quad \delta_h(\hat{x}, x^n) = \delta_h(\hat{x}, -x^n). \tag{3.15}$$

Since the b^{ij} are elliptic in B_1, there is a function $G_h \in H_0^{1,2}(B_1)$ so that

$$\int_{B_1} b^{ik} D_k v \cdot D_i G_h \, dx = \int_{B_1} \delta_h v \, dx \qquad \forall\, v \in H_0^{1,2}(B_1). \tag{3.16}$$

It is known (see [**1**, **6**]), that G_h is uniformly bounded in $H_0^{1,q}(B_1)$, $q < n/(n-1)$ and that $G_h \geq 0$. Furthermore, $G_h \to G$ in $H^{1,q}$, where G has the property

$$m|x|^{2-n} \leq G(x) \leq m^{-1}|x|^{2-n} \tag{3.17}$$

with some constant $m > 0$. The functions G_h satisfy

$$G_h(\hat{x}, x^n) = G_h(\hat{x}, -x^n). \tag{3.18}$$

To see this, we observe that $\hat{G}_h(\hat{x}, x^n) = G_h(\hat{x}, -x^n)$ is also a solution of (3.16) in view of the symmetry properties of δ_h and b^{ij}. Then, (3.18) follows from the uniqueness of G_h.

Now we can use (3.12) with $\Phi = \tau^2 G_h$, where $\tau \in C_0^\infty(B_1)$ satisfies $\tau \geq 0$, $\tau \equiv 1$ in $B_{1/2}$ and $\tau(\hat{x}, x^n) = \tau(\hat{x}, -x^n)$. We get

$$\begin{aligned} \int_{B_1^+} & a^{ik} D_k D_j u \cdot D_i D_j u \cdot \tau^2 G_h \, dx \\ & \leq |D\beta| \int_\Gamma |D_j(u - \psi)| G_h \tau^2 \, d\hat{x} \\ & \quad + \int_{B_1^+} a^{ik} D_k D_j u \cdot D_j(\psi - u) \cdot D_i G_h \tau^2 \, dx \\ & \quad + \int_{B_1^+} a^{ik} D_k D_j u \cdot D_i D_j \psi \cdot \tau^2 G_h \, dx \\ & \quad - \int_{B_1^+} a^{ik} D_k D_j u D_j(u - \psi) G_h \tau \cdot 2 D_i \tau \, dx \\ & \quad + \int_{B_1^+} \left(|H| + \left| \frac{\partial a^i}{\partial x_k} \right| \right) \left| D\left(G_h \tau^2 \cdot D_j(u - \psi) \right) \right| dx. \end{aligned} \tag{3.19}$$

The critical term

$$\begin{aligned}(3.20)\qquad &\int_{B_1^+} a^{ik} D_k D_j u \cdot D_j(\psi - u) \cdot D_l G_h \tau^2 \, dx \\ &= \frac{1}{2} \cdot \int_{B_1^+} a^{ik} D_k\Big(\tau^2\big(D_j(u-\psi)\big)^2\Big) \cdot D_i G_h \, dx + B\end{aligned}$$

where B stands for lower order terms, can be rewritten as

$$(3.21)\qquad \frac{1}{4} \cdot \int_{B_1} b^{ik} D_k\Big(\tau^2\big(D_j(\tilde{u} - \tilde{\psi})\big)^2\Big) \cdot D_l G_h \, dx + B.$$

This follows from the symmetry properties of $\tilde{u}$, $\tilde{\psi}$, τ, G_h and b^{ij}. But (3.21) equals

$$(3.22)\qquad \frac{1}{4} \cdot \int_{B_1} \delta_h \cdot \tau^2\big(D_j(\tilde{u} - \tilde{\psi})\big)^2 \, dx + B = B$$

since $\tau^2 \cdot (D_j(\tilde{u} - \tilde{\psi}))^2$ lies in $H_0^{1,2}(B_1)$, $j = 1,\ldots,n-1$. Thus we obtain from (3.19)—using ellipticity—that

$$(3.23)\qquad \int_{B_1^+} |D_k D_j u|^2 G_h \tau^2 \, dx \le \text{const.}$$

for $h \to 0$, $j = 1,\ldots,n-1$; $k = 1,\ldots,n$.

For $j = 1,\ldots,n-1$ the conclusion of the lemma now follows by a lower semicontinuity argument and by (3.17). For $j = n$ the conclusion follows from (3.1) and from the boundedness of

$$(3.24)\qquad \int_{B_{1/2}^+} |D_k D_j u|^2 G \, dx, \qquad k = 1,\ldots,n; j = 1,\ldots,n-1.$$

Now we are ready to establish the main inequality, from which we can start an iteration process. Therefore we insert the function u_ε^p (see Lemma 3.1) into the variational inequality, where $\Phi = \tau^2$ is a cut-off function. Passing to the limit $h \to 0$ we obtain

$$\begin{aligned}&- \int_{B_1^+} D_j a^l(x, Du) \cdot D_l[z - \hat{c}]^p \tau^2 \, dx \\ (3.25)\qquad &- \int_\Gamma D_j\beta \cdot \tau^2 [z - \hat{c}]^p \, d\hat{x} + \int_{B_1^+} H(x,u)\big(D_j\big(\tau^2[z-\hat{c}]^p\big)\big) \, dx \\ &- \int_{B_1^+} D_j a^i(x, Du) \cdot D_i\tau \cdot 2\tau[z - \hat{c}]^p \, dx \ge 0\end{aligned}$$

where we set $z = D_j u - D_j \psi$.

Due to (2.4) we can estimate the boundary integral by

$$(3.26)\quad |D\beta|\cdot\left(\int_{B_1^+}|D\tau|\cdot 2\tau[z-\hat{c}]^p\right)dx + \int_{B_1^+}\tau^2\cdot p|z-\hat{c}|^{p-1}|Dz|dx + c\cdot\int_{B_1^+}\tau^2|z-\hat{c}|^p\,dx.$$

Using ellipticity and Hölder's inequality we deduce from (3.25) after some calculation the main inequality

$$(3.27)\quad \int_{B_1^+}\left|D\left(\tau[z-\hat{c}]^{(p+1)/2}\right)\right|^2 dx \le p^2\cdot c\cdot\int_{B_1^+}|z-\hat{c}|^{p-1}\left(|D\tau|^2+\chi_\tau\right)dx$$

where χ_τ is the characteristic function of supp τ and $c = c(|z|_\Omega, |H(\cdot, u)|_\Omega, |\partial a^i/\partial x_k|, |D\beta|, |D\gamma|)$. Here, we used that (3.27) will be only applied with $|\hat{c}| \le |z|_\Omega$.

From inequality (3.27) we can start an iteration as in ([**1**], Lemma 1.3 and 1.4). We obtain for $R \le \frac{1}{2}$

$$(3.28)\quad \operatorname{osc}\left\{ z(x)|x\in B_R^+(0)\right\} \le c\cdot\left(R^{2-n}\int_{**}|Dz|^2\,dx\right)^{1/n} + c\cdot R^\alpha$$

for $n \ge 3$ and $\alpha = 2\cdot(n-2)\cdot n^2$,

and for $n = 2$

$$(3.29)\quad \operatorname{osc}\left\{ z(x)|x\in B_R^+(0)\right\} \le c\cdot\left(\int_{**}|Dz|^2\,dx\right)^{1/2-2/(t+4)} + c\cdot R^{2/(t+4)}\cdot\left(\int_{*}|Dz|^2\,dx\right)^{1/2-2/(t+4)} \qquad \forall\, t>0.$$

We used the notation $(**) = B_{2R}^+ - B_R^+$ and $(*) = B_{2R}^+$.

Since $R^{2-n} \le c\cdot |x|^{2-n}$ on $(**)$, we obtain by Lemma 3.2 that

$$(3.30)\quad R^{2-n}\cdot\int_{**}|Dz|^2\,dx \le c\cdot\int_{**}|Dz|^2|x|^{2-n}\,dx$$

is small if R is small. Together with (3.28) and (3.29) this means the continuity of $z = D_j u - D_j\psi$.

Again following Frehse's proof in ([**1**], Chap. 3) we conclude that in the case $n = 2$ $D_n(u-\psi)$ too is uniformly continuous.

Remark. Obviously this regularity result applies to any elliptic operator

$$A = -D_i\left(a^i(x, Du)\right)$$

if the a^i's satisfy the symmetry condition (3.4). It is not clear, whether Lemma 3.2 can be established without this assumption.

4. Estimates in $H^{2,\infty}(\Omega)$. In the following we shall consider a slightly more general problem than considered in the introduction. Let u_0 be a solution of the variational inequality

$$(4.1)\qquad \langle Au_0 + Hu_0, v - u_0\rangle \geq 0 \qquad \forall v \in K,$$
$$K := \{ v \in H^{1,\infty}(\Omega) | v \geq \psi \}$$

where A is an elliptic operator and

$$(4.2)\qquad \langle Au, \eta\rangle = \int_\Omega a^i D_i\eta \, dx + \int_{\partial\Omega} \beta\eta \, dH_{n-1},$$
$$Au = -D_i(a^i(x, u, Du)), \quad Hu = H(x, u, Du).$$

It is well known, that u_0 satisfies

$$(4.3)\qquad Au_0 \in L^\infty(\Omega)$$

and therefore is of class $H^{2,p}_{\text{loc}}(\Omega)$ for any finite p, if the coefficients are smooth enough. Furthermore, if we assume that

$$(4.4)\qquad -a^i(x, \psi, D\psi) \cdot \gamma_i \geq \beta \quad \text{on } \partial\Omega$$

holds we have (see [2]) $u_0 \in H^{2,p}(\Omega)$ and u_0 satisfies

$$(4.5)\qquad -a^i(x, u_0, Du_0) \cdot \gamma_i = \beta \quad \text{on } \partial\Omega.$$

Recently, Gerhardt [5] showed that a solution of the corresponding Dirichlet problem lies in $H^{2,\infty}(\Omega)$, if the boundary data are of class C^3.

We shall prove the following

THEOREM 4.1. *Let $\partial\Omega$ be of class $C^{3,\alpha}$, $\beta \in C^{1,1}(\partial\Omega)$ and assume that $\psi \in H^{2,\infty}(\Omega)$ satisfies* (4.4). *Let the a^i's be of class C^2 in x and u and of class C^3 in the p-variable. Moreover, assume that H is of class $C^{0,1}$ in all its arguments. Then any solution of the variational inequality* (4.1) *is in* $H^{2,\infty}(\Omega)$.

As in [5], we want to show uniform a priori estimates for the solutions of approximating problems. Since a solution u_0 of (4.1) is of class $H^{2,p}$ in view of (4.4), there is a constant M with

$$(4.6)\qquad 1 + |u_0|_\Omega + |Du_0|_\Omega \leq M.$$

Thus, we can replace A and H by operators $\hat{A}$ and $\hat{H}$ so that

$$(4.7)\qquad \hat{A}u_0 + \hat{H}u_0 = Au_0 + Hu_0$$

and so that the corresponding boundary value problems are always solvable (see [5] for details).

Furthermore, we can choose a constant γ so large that the operator

$$\hat{A}u + \hat{H}u + \gamma u \tag{4.8}$$

is uniformly monotone, i.e.

$$\begin{aligned}\langle \hat{A}u_1 + \hat{H}u_1 + \gamma u_1 - \hat{A}u_2 - \hat{H}u_2 - \gamma u_2, u_1 - u_2\rangle \\ \geq c \cdot \|u_1 - u_2\|_{1,2}^2, \qquad c > 0.\end{aligned} \tag{4.9}$$

We shall write A and H instead of $\hat{A}$ and $\hat{H}$ in the following. Let us assume for the moment, that the a^i's and H are of class C^4 in their arguments. Then we consider the boundary value problems

$$\begin{aligned}Au + Hu + \gamma u + \mu\Theta(u - \psi) &= \gamma u_0 \quad &&\text{in } \Omega,\\ -a^i(x, u, Du) \cdot \gamma_i = \beta - \delta &= \beta_1 &&\text{on } \partial\Omega\end{aligned} \tag{4.10}$$

where $\delta > 0$ is small and where now

$$\Theta(t) = \begin{cases} 0, & t > 0, \\ -t^2, & t \leq 0. \end{cases} \tag{4.11}$$

Again μ is a parameter tending to infinity. In view of our assumptions on A and H, the boundary value problem (4.10) has always a solution $u \in C^{3,\alpha}(\bar{\Omega})$. We want to show, that the second derivatives of u are bounded independent of μ and δ. In the limit case $\mu \to \infty$, u tends to a solution $\tilde{u}_0$ of (4.1), where β is replaced by β_1. On $\partial\Omega$, $\tilde{u}_0$ satisfies

$$- a^i(x, \tilde{u}_0, D\tilde{u}_0) \cdot \gamma_i = \beta_1. \tag{4.12}$$

Removing then the sharper differentiability assumptions and letting δ tend to zero we shall conclude, that $\tilde{u}_0$ tends to u_0 which therefore lies in $H^{2,\infty}(\Omega)$.

As a first step we need the following Lemma.

LEMMA 4.1. *Let u be a solution of* (4.10). *Then* $u - \psi \geq -c \cdot \mu^{-1/2}$ *and*

$$\mu \cdot |\Theta(u - \psi)| \leq c^2 \tag{4.13}$$

where

$$c^2 = \sup_{\Omega} |A\psi + H\psi|, \qquad c > 0. \tag{4.14}$$

Proof of Lemma 4.1. We multiply the inequality

$$Au - A\psi + Hu - H\psi + \gamma(u - \psi) + \mu\Theta(u - \psi) + c^2 \geq 0 \tag{4.15}$$

by $v = \min(u - \psi + c \cdot \mu^{-1/2}, 0)$ and obtain

$$\int_\Omega \left(a^i(x,u,Du) - a^i(x,\psi,D\psi)\right) \cdot D_i v\,dx \tag{4.16}$$
$$+\mu \int_\Omega \left(\Theta(u-\psi) + c^2\mu^{-1}\right) v\,dx$$
$$+\int_\Omega (Hu - H\psi + \gamma(u-\psi)) v\,dx$$
$$+\int_{\partial\Omega} \left(a^i(x,\psi,D\psi)\cdot\gamma_i + \beta\right) v\,dH_{n-1} \le 0.$$

The conclusion now essentially follows from the boundary condition on ψ (4.4).

We deduce from this Lemma that

$$Au \in L^\infty(\Omega) \tag{4.17}$$

with an uniform bound and

$$\|u\|_{2,p} \le c, \qquad \forall\, 1 \le p < \infty, \tag{4.18}$$

where the constant depends on p, $\|\psi\|_{2,\infty}$, $\partial\Omega$ and other known quantities.

We shall denote by f^i any vectorfield such that

$$\|f^i\|_p \le c\left(1 + \|u\|_{2,p}\right)^m \tag{4.19}$$

for any $1 \le p \le \infty$, where c and m are arbitrary constants depending on p. Furthermore, f denotes any function which can be estimated as in (4.19).

As in §3 we assume the equation (4.10) to hold in $B_1^+ = \{x \in B_1(0) \mid x^n > 0\}$. Then the boundary condition takes the form

$$-a^n = \beta_2(x) \quad \text{on } \Gamma = \left\{ x \in B_1 \mid x^n = 0 \right\} \tag{4.20}$$

where β_2 is related to β_1 by some positive factor depending on the transformation.

LEMMA 4.2. *The solution* $\tilde{u}_0$ *of*

$$\langle A\tilde{u}_0 + H\tilde{u}_0 + \gamma(\tilde{u}_0 - u_0), v - \tilde{u}_0 \rangle \ge 0, \qquad \forall v \in K, \tag{4.21}$$

where

$$\langle A\tilde{u}_0, \eta\rangle = \int_\Omega a^i D_i\eta\,dx + \int_{\partial\Omega} \beta_1 \eta\,dH_{n-1} \tag{4.22}$$

satisfies the strict inequality

$$\tilde{u}_0 > \psi \quad \text{on } \partial\Omega. \tag{4.23}$$

Proof of Lemma 4.2. In view of (4.12) and (4.4) we have

$$-a^i(x, \tilde{u}_0, D\tilde{u}_0)\cdot\gamma_i < -a^i(x, \psi, D\psi)\cdot\gamma_i \quad \text{on } \partial\Omega \tag{4.24}$$

or equivalently

$$-a^n(x, \tilde{u}_0, D\tilde{u}_0) < -a^n(x, \psi, D\psi) \quad \text{on } \Gamma. \tag{4.25}$$

Now assume that there is $x_0 \in \partial\Omega$ such that

$$\tilde{u}_0(x_0) = \psi(x_0). \tag{4.26}$$

It follows that $D_j(\tilde{u}_0 - \psi)(x_0) = 0$, $\forall 1 \le j \le n-1$. Thus, we obtain from (4.25)

$$\begin{aligned} 0 < &\int_0^1 a^{nj}(x_0, t\tilde{u}_0 + (1-t)\psi, tD\tilde{u}_0 + (1-t)D\psi) \\ &\qquad\qquad \times\big(D_j(\tilde{u}_0 - \psi)(x_0)\big)\,dt \\ &+ \int_0^1 \frac{\partial a^n}{\partial u}(x_0, t\tilde{u}_0 + (1-t)\psi, tD\tilde{u}_0 + (1-t)D\psi) \\ &\qquad\qquad \times\big((\tilde{u}_0 - \psi)(x_0)\big)\,dt \\ = &\int_0^1 a^{nn}(\cdots)\cdot D_n(\tilde{u}_0 - \psi)(x_0)\,dt. \end{aligned} \tag{4.27}$$

But in view of $\tilde{u}_0 \ge \psi$ we have

$$D_n(\tilde{u}_0 - \psi) \le 0 \quad \text{at } x_0. \tag{4.28}$$

Thus, the contradiction is a consequence of ellipticity.

Since we already know that in the case $\mu \to \infty$ the solutions u of the approximating problems (4.10) tend to $\tilde{u}_0$ uniformly, we can assume in the following that μ is so large that

$$u > \psi \quad \text{on } \partial\Omega. \tag{4.29}$$

In particular we have

$$\Theta(u-\psi) = \Theta'(u-\psi) = 0 \quad \text{on } \partial\Omega. \tag{4.30}$$

Now we are ready to estimate the second tangential derivatives of u.

LEMMA 4.3. *The second tangential derivatives of u can be estimated by*

$$\sup_{B^+_{1/2}} |D_\rho D_\sigma u| \le c\cdot(1 + \|u\|_{2,\infty})^{\varepsilon} \tag{4.31}$$

for any ε, $0 < \varepsilon < 1$, where c depends on ε, $\|u\|_{2,p}$ and known quantities.

Proof of Lemma 4.3. Following ideas in [5] and [7] we shall estimate the quantity

$$\lambda\cdot a^{kl}D_kD_lu \pm D_\sigma D_\rho u, \qquad 1 \le \rho, \sigma \le n-1, \tag{4.32}$$

from below. As in [5] we derive the differential inequality

$$(4.33)\qquad -D_i\big(a^{ij}D_jw\big)+\gamma w+\mu\Theta'(w-\bar{w})\geq f+D_if^i$$

where

$$(4.34)\qquad \begin{aligned} w&=\lambda\cdot a^{kl}D_kD_lu\pm D_rD_su,\\ \bar{w}&=\lambda\cdot a^{kl}D_kD_l\psi\pm D_rD_s\psi, \end{aligned}\qquad 1\leq r,s\leq n,$$

and λ is large.

We set $r=\rho$, $s=\sigma$ and multiply (4.33) with

$$(4.35)\qquad w_k\cdot\eta^2=\min(w\cdot\eta^2+k,0)\cdot\eta^2$$

where $\eta\equiv 1$ in $B_{1/2}$ and supp $\eta\subset B_1$ and

$$(4.36)\qquad k\geq k_0=\sup_\Omega|\bar{w}|.$$

Using ellipticity and (4.19) we obtain

$$(4.37)\qquad \int_{B_1^+}|Dw|^2\eta^4\,dx+\gamma\cdot\int_{B_1^+}w_k^2\,dx$$
$$\leq c\cdot(1+\|u\|_{2,\infty})^m|A(k)|$$
$$+\int_\Gamma|f^n\cdot w_k|d\hat{x}+\int_\Gamma\big|a^{nj}D_jw\cdot\eta^2\cdot w_k\big|d\hat{x}$$

where $A(k)$ is the set $\{x\in B_1^+|w\cdot\eta^2<-k\}$. The first boundary integral can be estimated by

$$(4.38)\qquad \|f\|_\infty\cdot\left(\int_{B_1^+}|Dw_k|dx+c\cdot\int_{B_1^+}w_k\,dx\right)$$
$$\leq\varepsilon\cdot\int_{B_1^+}|Dw|^2\eta^4\,dx+c\cdot(1+\|u\|_{2,\infty})^m|A(k)|.$$

To estimate the second boundary integral, we conclude from the equation in view of (4.30) that

$$(4.39)\qquad D_jw=D_jF+D_jD_\rho D_\sigma u$$

where $D_jF=f$. In order to estimate the critical term

$$(4.40)\qquad a^{nj}D_kD_\rho D_\sigma u$$

we differentiate the boundary condition (4.20) and obtain

$$(4.41)\qquad -a^{nj}D_jD_\sigma u=D_\sigma\beta_2+\frac{\partial a^n}{\partial u}\cdot D_\sigma u+\frac{\partial a^n}{\partial x_\sigma}$$

and

$$(4.42)\qquad -a^{nj}D_jD_\sigma D_\rho u = D_\sigma D_\rho \beta_2 + D_\rho\left(\frac{\partial a^n}{\partial u}\cdot D_\sigma u + \frac{\partial a^n}{\partial x_\sigma}\right) + D_\rho(a^{nj})\cdot D_j D_\sigma u.$$

But this equals f and so we have

$$(4.43)\qquad \int_\Gamma |a^{nj}D_j w\cdot\eta^2\cdot w_k|d\hat{x} \le \int_\Gamma |f\cdot w_k|d\hat{x}$$

which can be estimated as in (4.38). Finally, we conclude

$$(4.44)\qquad \int_{B_1^+}|Dw_k|^2\,dx + \gamma\cdot\int_{B_1^+} w_k^2\,dx \le c\cdot(1+\|u\|_{2,\infty})^m\cdot|A(k)|$$

for any $k \ge k_0$. Now the conclusion of the Lemma follows from the same arguments as in ([**5**], Theorem 2.2).

To get a similar bound for the mixed derivatives $D_nD_\sigma u$, we remark that due to (4.41)

$$(4.45)\qquad -a^{nn}D_nD_\sigma u = g + a^{n\rho}D_\rho D_\sigma u \quad \text{on } \Gamma$$

with some bounded function g and so—again using $a^{nn} > 0$—we deduce that

$$(4.46)\qquad |D_nD_\sigma u| \le c\left(1+|D_\sigma D_\rho u|\right) \le \hat{c}_\varepsilon\cdot(1+\|u\|_{2,\infty})^\varepsilon$$

holds on Γ. Repeating now the proof of Lemma 4.3 with $w = \lambda\cdot a^{kl}D_kD_lu \pm D_nD_\sigma u$ and $k \ge \hat{k}_0 = k_0 + \hat{c}_\varepsilon(1+\|u\|_{2,\infty})^\varepsilon$, we conclude that (4.46) holds in $B_{1/2}^+$ since no boundary integrals occur.

Finally, using the equation we can estimate D_nD_nu in terms of $D_\sigma D_\rho u$ and $D_nD_\sigma u$. Thus, we obtain

$$(4.47)\qquad \|u\|_{2,\infty,B_{1/2}^+} \le c_\varepsilon\cdot(1+\|u\|_{2,\infty})^\varepsilon$$

for any ε, $0<\varepsilon<1$.

As $\partial\Omega$ is compact, this estimate holds in a boundary neighbourhood. In the interior of Ω the estimate can be derived by a version of the proof of Lemma 4.3. Thus, we have an a priori estimate for $\|u\|_{2,\infty,\Omega}$ depending only on known quantities, but not on μ and δ.

Letting now μ tend to infinity, u tends to the (unique) solution $\tilde{u}_0$ of (4.21). Then, letting δ tend to zero, we arrive at a function $\hat{u}\in H^{2,\infty}(\Omega)$ solving the variational inequality

$$(4.48)\qquad \begin{aligned} &\langle A\hat{u} + H\hat{u} + \gamma(\hat{u}-u_0), v-\hat{u}\rangle \ge 0, \qquad \forall v\in K,\\ &\langle A\hat{u},\eta\rangle = \int_\Omega a^iD_i\eta\,dx + \int_{\partial\Omega}\beta\eta\,dH_{n-1}\end{aligned}$$

where A and H satisfy the sharper differentiability assumptions. By an approximation argument we conclude, that (4.48) admits a solution $\hat{u} \in H^{2,\infty}(\Omega)$ assuming only the weaker conditions, since the estimates are independent of the sharper assumptions. The conclusion

$$\hat{u} = u_0 \tag{4.49}$$

now follows from the uniqueness of a solution of (4.48).

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