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AN APPROXIMATION THEOREM FOR EQUIVARIANT LOOP SPACES IN THE COMPACT LIE CASE

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Let V be a real orthogonal countable-dimensional representation of the Lie group G and denote by $\Omega^V\Sigma^V X$ the space of maps $S^V \to \Sigma^V X = X \wedge S^V$, where S^V denotes the one-point compactification of V and where X is an arbitrary G-space with stationary basepoint (if V is infinite-dimensional, $\Omega^V\Sigma^V X$ is taken as the natural colimit over spaces indexed on the finite-dimensional submodules of V). Since G acts on $\Omega^V\Sigma^V X$ by conjugation, the fixed-set $(\Omega^V\Sigma^V X)^G$ is the subspace of G-equivariant maps. We present here an approximation to $(\Omega^V\Sigma^V X)^G$ in the stable case (V large). This approximation will take the form of a space of "configurations" of G-orbits in V.

In the Geometry of Iterated Loop Spaces **[M1]**, J. P. May carries out this program using an approximation

$$\alpha_n\colon C_n X \to \Omega^n\Sigma^n X$$

for $1 \leq n$. The map α_n is a homotopy-equivalence when X is connected, and in general is a group-completion. This means that α_n is an H-map between an H-space and a group-like H-space, and $(\alpha_n)_*\colon H_*(C_n X) \to H_*(\Omega^n\Sigma^n X)$ is a localization of the ring $H_*(C_n X)$ at its multiplicative submonoid $\pi_0 X$ for field coefficients. (See **[M2**, Ch. 15].)

We wish to carry out such a program in the equivariant case, where all spaces are acted upon by a group G. The right space to approximate in place of $\Omega^n\Sigma^n X$ is the space $\Omega^V\Sigma^V X$. Such an approximation exists in the case where G is finite (**[H1]**, **[S2]**). But the case where G is a compact Lie group is much deeper, and our approximation to $(\Omega^V\Sigma^V X)^G$ is therefore not the greatest generality one could hope for. Indeed, a suitable approximation to $\Omega^V\Sigma^V X$ (even in the stable case) would suffice to prove an equivariant "regcognition principle"—a simple test enabling one to determine whether a given space admits deloopings of all orders. (When G is finite, May, Hauschild and Waner have developed such a principle.) In the last section we indicate what one could hope for in this regard, and plan to address the actual development of a recognition principle in a future paper.

1. Definitions and notations. Let G be a compact Lie group, and V a finite-dimensional real representation of G. Throughout the text, $\mathcal{S}_j$ will stand for the symmetric group on j letters.

In this section, we will define a space $C_G(V, X)$ of "configurations" of G-orbits in V which will be the finite-dimensional version of an approximation to $(\Omega^V\Sigma^V X)^G$.

If H is a closed subgroup of G and Y is any G-space we will denote by Y^H and $Y_{(H)}$ the subspaces

$$(1.1)\qquad \begin{aligned} Y^H &= \{ y \in Y | G_y \geq H \} \quad \text{(the subspace of } Y \text{ fixed by } H), \\ Y_{(H)} &= \{ y \in Y | G_y \sim H \}, \end{aligned}$$

where (H) stands for the class of subgroups conjugate to H, and

$$G_y = \{ g \in G | yg = y \}$$

is the isotropy group of y. In particular, if W is a representation, W^H is a vector space acted on by the normalizer NH of H, $W_{(H)}$ is a G-manifold, and $W_{(H)} = W_{(H')}$ if $H \sim H'$.

Let $L(H)$ denote the tangent space of G/H at the coset $H \cdot e$. Note that if $H' = k^{-1}Hk$ is conjugate to H, there is a G-map

$$(1.1a)\qquad G/H \to G/H' \quad \text{taking} \quad Hg \mapsto Hk^{-1}gk$$

and this induces a linear isomorphism $k_*\colon L(H) \to L(H')$. Note that if $k \in NH$, k_* is an automorphism of $L(H)$, so that $L(H)$ is a representation of NH. Let $K(H)$ denote $L(H)^H$.

If Y is any G-space, define the space of configurations of G-orbits

$$(1.2)\qquad F^G(Y, j) = \{ (y_1, \ldots, y_j) \in Y^j | y_1 \cdot G, \ldots, y_j \cdot G \text{ are pairwise disjoint orbits} \}.$$

This can be thought of as a generalization of a "configuration space" of n-tuples of distinct points, which is so important in the nonequivariant case ([**M1**], [**B2**]).

Let X be a based G-space (with G-fixed basepoint), and then define

$$F^G(V_{(H)}, j)\, \overline{\times}_{\mathcal{S}_j} (\Sigma^{L(H)^H} X^H)^j$$

as the subspace of

$$F^G(V_{(H)}, j) \times_{\mathcal{S}_j} \Big(\coprod_{H' \sim H} \Sigma^{(H')^{H'}} X^{H'} \Big)^j$$

consisting of points $((v_1, \ldots, v_j), (x_1, \ldots, x_j))$ such that $v_i \in V^{H'}$ if $x_i \in \Sigma^{K(H')} X^{H'}$, for $i = 1, \ldots, j$.

Then let

$$(1.3)\qquad C_G(V,X)_{(H)} = \coprod_{j\geq 0} F^G(V_{(H)}, j)\,\overline{\times}_{\mathscr{S}_j}(\Sigma^{K(H)}X^H)^j/\approx$$

where $\approx$ is the equivalence relation generated by

$$(1.4a)\qquad \big((v_1,\dots,v_j),(x_1,\dots,x_j)\big) \approx \big((v_1\cdot g, v_2,\dots,v_j);(x_1\cdot g, x_2,\dots,x_j)\big)$$

for all $v_\iota \in V_{(H)} \cap V^{H_\iota}$, $x_i \in \Sigma^{L(H_\iota)^{H_\iota}}X^{H_\iota}$, and

$$(1.4b)\qquad \big((v_1,\dots,v_j),(x_1,\dots,x_j)\big) \approx \big((v_2,\dots,v_j),(x_2,\dots,x_j)\big)$$

if $x_1 = *$. If $K \leq G$, and U is a sub-K-space of V, let

$$C_K(U,X)_{(H)} = \coprod_{j\geq 0} F^K(U\cap V_{(H)}, j)\,\overline{\times}_{\Sigma_j}(\Sigma^{L(H)^H}X^H)^j/\approx .$$

Finally, let

$$(1.5)\qquad C_G(V,X) = \prod_{(H)} C_G(V,X)_{(H)}$$

where (H) ranges over the set of conjugacy classes of closed subgroups of G, and the product is the "weak product" (the direct limit of the finite subproducts via basepoint inclusions). Thus a point of $C_G(V,X)$ is represented by a tuple $(v_1,\dots,v_j; x_1,\dots,x_j)$ where $v_i \in V$ and

$$x_i \in (\Sigma^{L(G_{v_i})}X)^{G_{v_i}}.$$

We denote such points with square brackets in place of parentheses: $[v_1,\dots,v_j; x_1,\dots,x_j]$.

Now that we have defined the approximating space, we need an approximating map. Ideally this would be a map $\alpha\colon C_G(V,X) \to (\Omega^V\Sigma^V X)^G$, which would then be shown to be a group-completion under certain hypotheses. However, the orbits in $C_G(V,X)$ are not "thick enough" to make it convenient to define such a map directly, and (analogously to **[C3]**, **[M2]**) we will turn to an intermediate space $\overline{C}_G(V,X)$ which will map to both $C_G(V,X)$ and $(\Omega^V\Sigma^V X)^G$.

First we will define a space of "thick orbits" in V. Let $\overline{F}^G(V)$ denote the spaces of discs D in V such that

(i) D is a normal slice at its center v to the orbit $v\cdot G$ in V, and

(ii) the map $D\times_H G \to V$ taking (d,g) to dg is an embedding, where $H = G_v$.

There is an injection $\iota\colon \overline{F}^G(V) \to V\times \mathbf{R}$ given by sending D to $(v$, radius $D)$, and we topologize $\overline{F}^G(V)$ as a subspace of $V\times\mathbf{R}$ via ι.

Let $\bar{F}^G(V)_{(H)}$ denote the subspace of $\bar{F}^G(V)$ of discs whose centers lie in $V_{(H)}$.

Finally, define

(1.6) $$\bar{F}^G(V, j)_{(H)} = \left\{(D_1, \ldots, D_j) \in \left(\bar{F}^G(V)_{(H)}\right)^j \mid D_1G, \ldots, D_jG \text{ are pairwise disjoint sets}\right\}.$$

Now, in parallel with the previous definition, let

$$\bar{C}^G(V, X)_{(H)} = \coprod_{j \geq 0} \bar{F}^G(V, j)_{(H)} \times_{\Sigma_j} [\Sigma^{K(H)} X^H]^j / \approx, \quad \text{and}$$

(1.7) $$\bar{C}_G(V, X) = \left\{(D_1, \ldots, D_j; [x_1, t_1], \ldots, [x_j, t_j]) \in \prod_{(H)} \bar{C}_G(V, X)_{(H)} \middle| D_1G, \ldots, D_jG \text{ are disjoint}\right\}.$$

Note that $C_G(V, X)$ and $\bar{C}_G(V, X)$ depend only on G, V and the G-homotopy type of X.

Now we are ready to define our approximating maps.

(1.8) LEMMA. *Let $\gamma_{(H)}$: $\bar{F}^G(V)_{(H)} \to V_{(H)}$ take a disc D to its centerpoint. This induces a $G \wr \Sigma_j$-equivariant homotopy equivalence*

$$\gamma'_{(H)}: \bar{F}^G(V, j)_{(H)} \to \bar{F}^G(V_{(H)}, j)$$

and so a homotopy equivalence

$$\gamma: \bar{C}_G(V, X) \to C_G(V, X).$$

(*See* [**C5**] *Lemma* 2.9.)

The other half of the approximation is a map

$$\alpha_V: \bar{C}_G(V, X) \to (\Omega^V \Sigma^V X)^G$$

which we define as follows.

Suppose D is a disc in $\bar{F}^G(V)_{(H)}$ with center $v \in V^H$ and $[x, t] \in (\Sigma^{L(H)} X)^H$. D spans the normal space N to $v \cdot G$ at v, and there is a homeomorphism

$$\psi = \psi_D: D \to N$$

given by

(1.9) $$\psi_D(v + v') = \frac{v'}{\frac{1}{2}\operatorname{diam}(D) - \|v'\|}.$$

The inclusion $G/H \to V$ taking Hg to $v \cdot g$ induces an identification ϕ of $L(H)$ with the tangent space T to $v \cdot G$ at v.

Define a map

$$\alpha[D, x, t]\colon S^V \to X \wedge S^V \approx X \wedge S^T \wedge S^N$$

by letting

$$(1.10)\quad \alpha[D, x, t](u) = \begin{cases} * & \text{if } u = \infty \text{ or } u \in V-(D \cdot G), \\ [xg, \phi(t)g, \psi(n)g] & \text{if } u = n \cdot g \text{ with } n \in D, g \in G. \end{cases}$$

This is a G-map, since

$$\alpha[D, x, t](n \cdot g\hat{g}) = [xg\hat{g}, \phi(t)g\hat{g}, \psi(n)g\hat{g}] = [xg, \phi(t)g, \psi(n)g] \cdot \hat{g}.$$

Since $D \times_H G \to D \cdot G$ is a homeomorphism, $\alpha[D, x, t] = \alpha[Dg, xg, tg]$. In fact, if $n \in D$,

$$\begin{aligned}(1.11)\quad \alpha[Dg, xg, tg](ng \cdot \hat{g}) &= [xg \cdot \hat{g}, \phi(tg)\hat{g}, \psi(ng)\hat{g}] \\ &= [x \cdot g\hat{g}, \phi(t)g\hat{g}, \psi(n)g\hat{g}] = \alpha[D, x, t](n \cdot g\hat{g}).\end{aligned}$$

Now we can define $\alpha = \alpha_v$.

Let $z = [D_1,\ldots,D_j; [x_1, t_1],\ldots,[x_j, t_j]] \in \overline{C}_G(V, X)$. Then we define

$$S^V \xrightarrow{\alpha(z)} X \wedge S^V$$

by

$$(1.12)\quad \alpha(z)(u) = \begin{cases} * & \text{if } \alpha[D_i, x_i, t_i](u) = * \text{ for all } i, \\ u_i & \text{if } \alpha[D_i, x_i, t_i](u) = u_i \neq * \text{ for some } i. \end{cases}$$

This is well-defined since the sets $D_i \cdot G$ are pairwise disjoint, and (1.11) holds. There is a general lemma which guarantees its continuity:

(1.13) Lemma. *Let X be a filtered space, η_j: $F_jX \to (F_1X)^j$, and let M be a partial monoid with jth multiplication μ_j: $M_j \to M$. Let α: $X \to M$ be a function and $\bar{\alpha}_j$ a function $(F_1X)^j \to M_j$ such that for all $j \geq 1$,*

$$\begin{tikzcd} F_jX \arrow[r, "\alpha|F_jM"] \arrow[d, "\eta_j"'] & M \\ (F_1X)^j \arrow[r, "\bar{\alpha}_j"] \arrow[dr, "(\alpha|F_1X)^j"'] & M_j \arrow[u, "\mu_j"'] \\ & M^j \arrow[u, hook] \end{tikzcd}$$

commutes. Then α is continuous if $\alpha|F_1X$ is.

Proof. α is continuous if $\alpha|F_jX$ is continuous for all j. □

In the present case, $(\Omega^V\Sigma^VX)^G$ is a partial monoid with $(\Omega^V\Sigma^VX)^G_j$ as the set of j-tuples $\langle f_1,\ldots,f_j\rangle$ of maps from S^V to Σ^VX such that no two of $f_1(v),\ldots,f_j(v)$ are different from $*$ for any $v \in S^V$, and composition is done by combining $f_1,\ldots,f_j$ as in (1.12).

The space

$$Z = \prod_{(H)}\left(\coprod_{j\geq 0} \bar{F}^G(V_{(H)}, j)\bar{x}\left((\Sigma^{L(H)}X)^H\right)^j\right)$$

is filtered by

$$(1.14)\quad F_jZ = \Big\{ z|z_{(H)} \neq * \text{ for only finitely many } (H)\text{'s}, \\ (H_1),\ldots,(H_k), z_{(H_i)} \in \bar{F}^G\big(V_{(H_i)}, l_i\big) \times \left((\Sigma^{L(H_i)}x)^{H_i}\right)^{l_i} \\ \text{and } l_1 + \cdots + l_k \leq j\Big\}.$$

The space Z projects onto $\bar{C}_G(V, X)$ via a quotient map ρ, and the composite $\alpha\rho$ satisfies the hypotheses of the lemma and hence is continuous; thus α is continuous.

We can now state the main theorems.

(1.15) THEOREM. *Let W be a G-vector space containing an infinite-dimensional trivial representation $\mathbf{R}^\infty$, such that W is the direct limit of its finite-dimensional subspaces. Define*

$$\bar{C}_G(W, X) = \varinjlim \bar{C}_G(V, X),$$

and

$$\Omega^W\Sigma^WX = \varinjlim \Omega^V\Sigma^VX,$$

where V ranges over finite-dimensional G-subspaces of W, and let $\alpha_W = \varinjlim \alpha_V$. Then if X is a countable G-CW complex,

$$\alpha_W\colon \bar{C}_G(W, X) \to (\Omega^W\Sigma^WX)^G \text{ is a group-completion.}$$

Let $A(G)$ denote the Burnside ring of G. In the finite case, this is the universal enveloping ring for the semi-ring of isomorphism types of finite G-sets, under the operations of disjoint union and Cartesian product. Tom Dieck defines it analogously for Lie groups. (See **D2**.) We will compute the structure of $\pi_0(C_G(W, X))$, and (1.8) and (1.15) will allow us to deduce the additive part of tom Dieck's result in [**D1**]:

(1.16) COROLLARY. *Define $W = G\mathbf{R}^\infty$ to be the direct sum $\oplus_i V_i^\infty$, where $V_1, V_2, \ldots$ are the irreducible real representations of G, and V_i^∞ denotes the direct sum of infinitely many copies of V_i. Then*

$$A(G) \cong \varinjlim_{V < W} [S^V, S^V]$$

where V runs over all finite-dimensional G-subspaces of W and the direct limit is taken over suspension homomorphisms.

We also have a splitting theorem:

(1.17) COROLLARY. *$(\Omega^W \Sigma^W X)$ is equivalent to a product*

$$\prod_{(H)} \varinjlim_{V < W} \operatorname{Map}_0\left(V^H/\left(V^H - V_{(H)}\right), \Sigma^{V^H} X^H\right)^{NH}.$$

Finally it is worth noting how nicely this result restrict to the finite case:

(1.18) THEOREM. *Let*

$$C(V, X) = \left(\coprod_{j<0} F(V, j) \times_{\Sigma_j} X^j\right) / \sim$$

be the usual configuration space. If $W \geq \mathbf{R}^\infty$, and X is a countable G-CW complex, then $C(W, X)$ is a based G-space, and there is a map

$$\alpha \colon C(W, X) \to \Omega^W \Sigma^W X$$

such that $\alpha^H \colon C(W, X)^H \to (\Omega^W \Sigma^W X)^H$ is a group-completion for all $H \leq G$.

Proof. Since G is finite, $L(H) = 0$ for all H, and G/H is a finite set. Define a homeomorphism

$$h \colon C_G(W, X) \xrightarrow{\approx} C(W, X)^G$$

by

$$h\left[\langle v_1, \ldots, v_j \rangle ; x_1, \ldots, x_j\right] = \left[\langle v_i \cdot g | i = 1, \ldots, j, g \in G \rangle, \langle x_i \cdot g \rangle\right].$$

The approximation

$$C_G(W, X) \underset{\simeq}{\overset{\gamma}{\leftarrow}} K(W, X) \xrightarrow{\alpha} \Omega^W \Sigma^W X$$

using little convex bodies K restricts to

$$\alpha^G\colon C(W,X)^G \underset{\approx}{\overset{h}{\leftarrow}} C_G(W,X) \underset{\approx}{\overset{\gamma}{\leftarrow}} \overline{C}_G(W,X) \overset{\alpha_W}{\to} (\Omega^W\Sigma^W X)^G;$$

restricting to a subgroup H lets us conclude that α^H is a group-completion for all $H < G$. □

The nice point here is that we needn't approximate the various fixed-point sets of $\Omega^W\Sigma^W X$ separately when G is finite.

2. Fiberings of equivariant function spaces. Let V be a real representation of G.

Here we will produce fiberings involving various subspaces of $(\Omega^V\Sigma^V X)^G$, and we will use these later to reduce the proof of the main theorem to consideration of spaces of functions which are nontrivial only on maximal orbits.

(2.1) an *orbit-type family* $\mathscr{F}$ for G is a collection of closed subgroups such that if $H \in \mathscr{F}$ and K is subconjugate to H, then $K \in \mathscr{F}$.

An *orbit of class* $\mathscr{F}$ is any G-orbit isomorphic to G/H for some $H \in \mathscr{F}$. If 1 denotes the trivial subgroup, the *maximal orbits* are those of class $\{1\}$.

We will examine subspaces of $(\Omega^V\Sigma^V X)^G$. Let K be a subgroup of G, let U be a sub-K-space of V, and let Y be any based K-space. Then define

(2.2) $(\Omega^U Y)^K_{\mathscr{F}} = \left\{ f \in (\Omega^U Y)^K \mid f(U \cap V^H) = * \text{ for all } H \notin \mathscr{F} \right\}.$

That is, $(\Omega^U Y)^K_{\mathscr{F}}$ consists of K-maps f which are non-trivial only on orbits which are of class $\mathscr{F}$ in V.

We will construct fibrations in the situation where $\mathscr{F}_1$ and $\mathscr{F}_2$ are *successive* orbit families, that is, $\mathscr{F}_1 \subseteq \mathscr{F}_2$ and $\mathscr{F}_2 - \mathscr{F}_1$ consists of just one conjugacy class (H), so that $\mathscr{F}_1$ contains all proper subgroups of H. In this case we may form the following sequence

$$(2.3) \qquad (\Omega^V Y)^G_{\mathscr{F}_1} \xrightarrow{i} (\Omega^V Y)^G_{\mathscr{F}_2} \xrightarrow{p} (\Omega^{V^H} Y^H)^{NH}_{\mathscr{F}_2}$$

where i is inclusion of subspaces and p is obtained by restricting each element of its domain to S^{V^H}.

(2.4) THEOREM. *The sequence* (2.3) *is a fiber sequence onto the image of* p, *which is a union of components of* $(\Omega^{V^H} Y^H)^{NH}_{\mathscr{F}_2}$.

Proof. Suppose we have an NH-map

$$h\colon Z \wedge I^+ \wedge S^{V^H} \to Y^H$$

such that $h(Z \wedge I^+ \wedge S^{V^K}) = *$ for all $K \leq H$ such that $K \notin \mathcal{F}_2$ that is, for all K such that $H \lneqq K \leq NH$. Suppose also that we have a G-map

$$H_0\colon Z \wedge \{0\}^+ \wedge S^V \to Y$$

such that $pH_0 = h_0$.

Since h is NH-equivariant and nontrivial only on $(Z \wedge I^+ \wedge (S^{V^H})_{(H)})$, it extends uniquely to a G-map

$$\bar{h}\colon Z \wedge I^+ \wedge (S^{V^H} \cdot G) \to Y^H \cdot G \to Y$$

where $S^{V^H} \cdot G \subseteq S^V$. In fact, this inclusion is a cofibration, and hence the homotopy extension property for $(S^V, S^{V^H} \cdot G)$ implies that H_0 extends to a lift H of h. Also it is clear that $p^{-1}(*) = (\Omega^V Y)^G_{\mathcal{F}_1}$.

This establishes the result, including the fact that the image of p is a union of components. $\square$

3. Decomposition of configuration spaces. It is convenient to develop a decomposition-theory for the spaces $\bar{C}_G(V, X)$ parallel to that developed in §2 for mapping spaces. It turns out that the theory and the arguments are quite a bit simpler.

Recall the notations of the previous section. Define

$$\bar{C}_K(U, Y)_{\mathcal{F}}$$

to be subspace of $\bar{C}_K(U, Y)$ consisting of configurations $[D_1, \ldots, D_j; [x_1, t_1], \ldots, [x_j, t_j]]$, where for each i the center point v_i of D_i gives rise to an orbit $v_i \cdot G$ in V of type $\mathcal{F}$; that is, $v_i \in V_{(H)}$ for some $H \in \mathcal{F}$. Then

$$\bar{C}_K(U, Y) = \coprod_{(H) \subset \mathcal{F}} \bar{C}_K(U, Y)_{(H)}. \tag{3.1}$$

If H is a subgroup of G, maximal in $\mathcal{F}$, we can define a homeomorphism

$$k\colon \bar{C}_G(V, X)_{(H)} \to \bar{C}_{NH}(V^H, X^H)_{(H)} = \bar{C}_{NH}(V^H, X^H)_{\mathcal{F}} \tag{3.2}$$

as follows. Let

$$k\big[D_1, \ldots, D_j; [x_1, t_1], \ldots, [x_j, t_j]\big] = \big[D_1', \ldots, D_j'; [x_1', t_1'], \ldots, [x_j', t_j']\big],$$

where D_i' is related to D_i as follows: if $v_i \in V^{gHg^{-1}}$, let $v_i' = v_i g$, and let D_i' be the disc normal to the orbit $v_i' \cdot NH$ at v_i' in V^H, with the same radius as D_i. Let $t_i' = g_*(t_i)$, where

$$g_*\colon L(gHg^{-1})^{gHg^{-1}} \to L(H)^H \cong \left(\tau(NH/H)_{eH}\right)^H$$

is the map in (1.12), and let $x_i' = x_i \cdot g$.

The equivalence relation $\approx$ in (1.4) and (1.6) shows that k is well-defined, and a well-defined inverse can be given by sending $[D_i', [x_i', t_i']]$ to $[D_i'', [x_i', t_i']]$, where D_i'' is the disc with the same diameter as D_i', normal in V to $v_i' \cdot G$.

Thus we may define a product bundle

$$\text{(3.3)} \qquad \overline{C}_G(V, X)_{\mathscr{F}_1} \xrightarrow{i} \overline{C}_G(V, X)_{\mathscr{F}_2} \xrightarrow{p} \overline{C}_{NH}(V^H, X^H)_{\mathscr{F}_2}$$

where i is inclusion and p is the projection

$$\prod_{(K) \subset \mathscr{F}_2} \overline{C}_G(V, X)_{(K)} \to \overline{C}_G(V, X)_{(H)}$$

composed with k.

4. Duality in equivariant function spaces. The previous two sections reduce the main work of proving (1.15) to showing that α restricts to an equivalence of $\overline{C}_{NH}(W^H, X^H)_{\mathscr{F}}$ and $(\Omega^{W^H}\Sigma^{W^H}X^H)^{NH}_{\mathscr{F}}$ when H is maximal in $\mathscr{F}$. This "maximal orbit type" case can be attacked by something similar to methods used by Becker and Schultz in **[B1]**.

In this section, we will obtain an equivalence

$$\text{(4.1)} \qquad \varepsilon\colon (\Omega^{W^H}\Sigma^{W^H}X^H)^{NH}_{\mathscr{F}} \to \Omega^\infty\Sigma^\infty\left(EJ^+ \wedge_J \Sigma^L X^H\right)$$

where $\mathbf{R}^\infty \leq W$, H is maximal in $\mathscr{F}$, $J = NH/H$ and $L = L(J/1)$ is the Lie algebra of J. The space $\overline{C}_{NH}(W^H, X^H)_{\mathscr{F}}$ is analyzed in the following section.

Recall the following definitions and notation from **[B1]**. A *sectioned bundle* is a bundle $\xi\colon E \to B$ equipped with a section $\Delta_\xi\colon B \to E$ (so that $\xi \circ \Delta_\xi = \text{id}$). If B is fixed, sectioned bundles over B form a topological category, and if ξ, η are sectioned bundles over B, define

$$\text{Bund}_0(\xi, \eta)$$

to be the space of morphisms from ξ to η. These are bundle maps f over B such that $f \circ \Delta_\xi = \Delta_\eta$.

The fiber $\xi^{-1}(b)$ of ξ over b may be thought of as a based space with basepoint $\Delta_\xi(b)$, and we may construct several functors on the category of sectioned bundles over B from standard functors on the category of based spaces. For example, if ξ and η are sectioned bundles over B, let $\xi \wedge \eta$

denote the "fiberwise smash product" of ξ and η; the fiber of $\xi \wedge \eta$ over b is

$$(\xi \wedge \eta)^{-1}(b) = \xi^{-1}(b) \wedge \eta^{-1}(b),$$

with basepoint $\Delta_\xi(b) \wedge \Delta_\eta(b)$.

If $A \subseteq B$, $\xi|A$ is defined to be the bundle $\xi^{-1}(A) \to A$ whose projection and section are the restrictions of ξ and Δ_ξ.

If X is a based space, let $\dot{X}$ denote the product bundle $p\colon X \times B \to B$ with $p(x, b) = b$ and $\Delta(b) = (*, b)$.

If $\alpha\colon E \to B$ is a vector bundle, let $\bar{\alpha}$ denote the based sphere bundle obtained by taking the fiberwise one-point compactification of E and letting Δ be the cross-section at infinity. Note that $\overline{\alpha \oplus \beta}$ is canonically isomorphic to $\bar{\alpha} \wedge \bar{\beta}$.

There is a functor T from sectioned bundles to based spaces defined by

$$T(\xi) = E/\Delta_\xi(B).$$

If α is a vector bundle, $T(\bar{\alpha})$ is just the usual Thom space of α, also denoted $T\alpha$ or B^α. If $A \subseteq B$, define

$$(B, A)^\xi = E/\big(\Delta(B) \cup \xi^{-1}(A)\big).$$

We may also define a category of pairs (ξ, ξ') where ξ is a sectioned bundle of B and ξ' is a subbundle of $\xi|A$; if (η, η') is another such pair let the morphism space

$$\mathrm{Bund}_0(\xi, \xi'; \eta, \eta')$$

be the subspace of $\mathrm{Bund}_0(\xi, \eta)$ of maps sending ξ' into η'.

(4.2) LEMMA. (i) *There is a natural isomorphism* $T(\dot{X} \wedge \xi) \approx X \wedge T(\xi)$.

(ii) *T is a continuous functor, and induces a homeomorphisms*

$$\mathrm{Bund}_0(\xi, \xi|A; \dot{X}, *) \xrightarrow{\approx} \mathrm{Map}_0\big((B, A)^\xi, X\big).$$

This is a simple check.

If H is maximal in $\mathscr{F}$, Y is an NH/H-space, and V is a sub-NH-space of W^H, then $(\Omega^V \Sigma^V Y)^{NH}_{\mathscr{F}}$ is homeomorphic to the mapping space

$$\mathrm{Map}\big(S^V, S^V - (V \cap W_{(H)}); \Sigma^V Y, *\big)^J$$

where $J = NH/H$. We will thus need to make some remarks about mapping spaces of G-manifolds.

(4.3) REMARKS. Let $(M, \partial M)$ be a compact smooth manifold with boundary, on which NH acts such that $M = M_{(H)}$. Then J is a finite group acting on M, so M/J is a manifold, and

$$\mu\colon M \to M/J$$

is a principal G-bundle. Suppose that M is contained in a J-vector space V such that $\dim M = \dim V$. Then the natural projection
$$M \times_J V \to M/J$$
is equivalent to the Whitney sum of the tangent bundle τ of M/J and the bundle
$$\pi\colon M \times_J L \to M/J$$
where L is the Lie algebra of J.

Consider the mapping space $\mathrm{Map}_J(M, \partial M; \Sigma^V Y, *)$ of J-maps from M to $\Sigma^V X$ taking ∂M to $*$, where Y is a based J-space. This can be identified with the space of sections of the bundle
$$\eta\colon M \times_J \Sigma^V Y \to M/J$$
taking the value $*$ on $\partial M/J$. This is the space of bundle maps
$$\mathrm{Bund}_0(\dot{S}^0, \dot{S}^0|(\partial M/J); \eta, *).$$
From the note on $M \times_J V$, we see that η may be decomposed as

(4.4) $$\eta = \bar{\tau} \wedge \bar{\pi} \wedge \xi$$

where ξ: $M \times_J Y \to M/J$ is the natural projection with section $\Delta[m] = [m, *]$.

Choose some embedding i: $M/J \to \mathbf{R}^s$ and let v be its normal bundle. Then we have a Pontryagin-Thom map
$$C\colon S^s \to (M/J, \partial M/J)^v,$$
and a natural isomorphism of $v \oplus \tau$ to the trivial vector bundle with fiber $\mathbf{R}^s$. Define a map $\varepsilon(M)$ as the composite

(4.5)
$$\begin{array}{c}
\mathrm{Map}(M, \partial M; \Sigma^V Y, *)^J \\
\downarrow \wr \\
\mathrm{Bund}_0(\dot{S}^0, \dot{S}^0|(\partial M/J); \bar{\tau} \wedge \bar{\pi} \wedge \xi, *) \\
\downarrow \sigma \\
\mathrm{Bund}_0(\bar{v}, \bar{v}|(\partial M/J)\colon \overline{v \oplus \tau} \wedge \bar{\pi} \wedge \xi, *) \\
\| \\
\mathrm{Bund}_0(\bar{v}, \bar{v}|(\partial M/J); \dot{S}^s \wedge \bar{\pi} \wedge \xi, *) \\
\downarrow T \\
\mathrm{Map}_0\left((M/J, \partial M/J)^v, \Sigma^s (M/J)^{\bar{\pi} \wedge \xi}\right) \\
\downarrow C^* \\
\mathrm{Map}_0\left(S^s, \Sigma^s (M/J)^{\bar{\pi} \wedge \xi}\right) \\
\| \\
\Omega^s \Sigma^s (M^+ \wedge_J \Sigma^L Y),
\end{array}$$
where σ is suspension by $\bar{v}$.

The manifold we wish to study is a manifold $M = M(V)$ constructed by letting $\delta > 0$ be small enough that

$$M = \left[V - B_\delta\left(V - V \cap W_{(H)}\right)\right] \cap \overline{B_{1/\delta}(0)}$$

is a deformation retract of $V \cap W_{(H)}$, where

$$B_\delta(A) = \text{the } \delta\text{-neighborhood of } A.$$

Then

$$(\Omega^V \Sigma^V Y)^{NH}_{\mathscr{F}} \approx \mathrm{Map}_0(M/\partial M, \Sigma^V Y)^J,$$

because $M/\partial M$ is homeomorphic to $S^V/(S^V - V \cap W_{(H)})$.

We cannot prove that $\varepsilon(M(V))$ is an equivalence if V is finite-dimensional. However, let W be a representation of G containing an orbit isomorphic to G/H and a copy of $\mathbf{R}^\infty$. Then there is a sequence of finite-dimensional sub-NH-spaces

$$(4.6) \qquad V_1 < V_2 < \cdots < W^H$$

such that $W^H = \bigcup_n V_n$. Define $M_n = M(V_n)$ for $n = 1, 2, 3, \ldots$, and choose δ;s so that

$$(M_1, \partial M_1) \subset (M_2, \partial M_2) \subset \cdots.$$

The union $\bigcup_n M_n$ is a free J-space which can be shown to be contractible and hence may be thought of as the total space EJ of the universal bundle of J.

We may also choose the embeddings i_n: $M_n/J \to \mathbf{R}^{s_n}$ so that $s_n < s_{n+1}$, and so that

$$(4.7) \qquad \begin{array}{ccc} M_n/J & \xrightarrow{i_n} & \mathbf{R}^{s_n} \\ \downarrow & & \downarrow i' \\ M_{n+1}/J & \xrightarrow{i_{n+1}} & \mathbf{R}^{s_{n+1}} \end{array}$$

commutes, where $i'(\underline{t}) = (\underline{t}, \underline{0})$ is the standard inclusion. It follows that

$$(4.8) \qquad \begin{array}{ccc} \mathrm{Map}\left(M_n, \partial M_n; \Sigma^{V_n} Y, *\right)^J & \xrightarrow{\varepsilon(M_n)} & \Omega^{s_n}\Sigma^{s_n}\left(M_n^+ \wedge_J \Sigma^L Y\right) \\ j\downarrow & & \downarrow \sigma \\ \mathrm{Map}\left(M_{n+1}, \partial M_{n+1}; \Sigma^{V_{n+1}} Y, *\right)^J & \xrightarrow{\varepsilon(M_{n+1})} & \Omega^{s_{n+1}}\Sigma^{s_{n+1}}\left(M_{n+1}^+ \wedge_J \Sigma^L Y\right) \end{array}$$

commutes, where σ is suspension composed with the inclusion induced from $M_n \hookrightarrow M_{n+1}$, and j is defined as making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Map}(M_n, \partial M_n; \Sigma^{V_n}Y, *)^J & \approx & (\Omega^{V_n}\Sigma^{V_n}Y)^J_{\mathscr{F}} \\ j\downarrow & & \downarrow\sigma \\ \mathrm{Map}(M_{n+1}, \partial M_{n+1}; \Sigma^{V_{n+1}}Y, *)^J & \approx & (\Omega^{V_{n+1}}\Sigma^{V_{n+1}}Y)^J_{\mathscr{F}} \end{array}$$

(4.10) PROPOSITION. *Let $U = W^H$ and $Y = X^H$. Taking the direct limit over the sequence* (4.6) *via diagram* (4.8), *define ε to be the composite*

$$(\Omega^U\Sigma^U Y)^{NH}_{\mathscr{F}} \overset{h}{\approx} \varinjlim \mathrm{Map}(M_n, \partial M_n; \Sigma^{V_n}Y, *)$$

$$\xrightarrow{\varinjlim \varepsilon(M_n)} \varinjlim \Omega^{s_n}\Sigma^{s_n}(M_n^+ \wedge_J \Sigma^L Y) = \Omega^\infty\Sigma^\infty(EJ^+ \wedge_J \Sigma^L Y).$$

Then ε is an equivalence of H-spaces.

Proof. The H-structure on $(\Omega^U\Sigma^U Y)^{NH}_{\mathscr{F}}$ comes from the loop multiplication, since $\mathbf{R}^\infty \leq W^H$, and this is carried over to the mapping spaces since if $V_n \oplus \mathbf{R} \subseteq V_{n+1}$, then $M_n \times \mathbf{R} \subseteq M_{n+1}$. The map ε is an H-map since we may choose the V_n's to have the form $V_n' \oplus \mathbf{R}$, where $U = (\bigcup_n V_n') \oplus \mathbf{R}$, and we can choose $i_n\colon M_n/J \to \mathbf{R}^{s_n}$ to have the form

$$(4.11) \qquad M_n/J \approx M(V_n')/J \times I \xrightarrow{i_n' \times 1} \mathbf{R}^{s_n - 1} \times I \to \mathbf{R}^{s_n}$$

where $I = [0,1] \subseteq \mathbf{R}$.

The proof that ε is an equivalence occupies the remainder of this section.

Some of the maps we will use are duality maps from a fiberwise duality involving $\bar{\pi} \wedge \xi$. We recall some facts of equivariant topology.

(4.12) LEMMA. *Let (X, A) be a pair of finite G-CW complexes. Then X embeds in the unit ball of a representation V such that*

(i) *X is a G-Euclidean neighborhood retract (G-ENR) of some invariant neighborhood U,*

(ii) *there is a G-deformation $\overline{U} - X \to \partial\overline{U}$, and*

(iii) *A embeds in a hemisphere E^+ of the unit sphere with* (i) *and* (ii) *restricting appropriately.*

In this case, an argument similar to that of Atiyah in [**A1**] *shows that $\overline{U}/\partial\overline{U}$ is S-dual to X/A, and $\overline{U}/(\partial\overline{U} \cup \overline{U} \cap E^+)$ is S-dual to X.*

A fiberwise duality is a map $\gamma \wedge \hat{\gamma} \to \dot{S}^t$ of bundles which restricts to a duality on fibers. The above lemma proves the existence of a fiberwise dual for a bundle γ whose fibers are finite G-CW complexes. Hence assume from now on that Y is finite, and let $\hat{\gamma}$ be the dual to the bundle $\gamma = \bar{\pi} \wedge \xi$, and i': $M \times_J \Sigma^L Y \to M/J \times \mathbf{R}^t$ the inclusion as a fiberwise ENR.

In this case, we may define an embedding

$$(4.13) \qquad j\colon M \times_J \Sigma^L Y \xrightarrow{i'} M/J \times \mathbf{R}^t \xrightarrow{\iota \times 1} \mathbf{R}^s \times \mathbf{R}^t = \mathbf{R}^{s+t}.$$

Then we can prove

(4.14) PROPOSITION. *The Thom space* $(M/J, \partial M/J)^{\hat{\gamma} \wedge \bar{v}}$ *is S-dual to* $M^+ \wedge {}_J\Sigma^L Y$, *where v is normal bundle of i.*

Proof. Let $B = M/J$, $E = M \times_J \Sigma^L Y$; thus $\partial B = (\partial M)/G$. Let γ: $E \to B$ be the bundle, i: $B \to \mathbf{R}^s$, and i': $E \to B \times \mathbf{R}^t$. Finally let γ': $U \to B$ be the projection to B of the neighborhood U of $\mathrm{im}(i')$ in $B \times \mathbf{R}^t$, and define $\hat{\gamma}$: $\overline{U}/\partial\overline{U} \to B$ to be the *fiberwise* collapse of $\partial\overline{U}$. Now $B^\gamma = E/\Delta_\gamma(B)$. Define i and i' so that the embedding $(i \times 1) \circ i'$ sends E into $I^{s+t} \subset \mathbf{R}^{s+t}$ with $\Delta(B)$ embedded in $I^s \times 0$. By Atiyah's argument, $E/\Delta(B)$ is dual to $I^{s+t} - E$, which is equivalent to $I^{s+t}/\overline{I^{s+t} - U'}$, U' being a regular neighborhood of E in I^{s+t}. This is the same as collapsing out $\gamma^{-1}(\partial B)$, and collapsing the boundary of $(\gamma')^{-1}(b) \times v^{-1}(b)$ to a point for all $b \in B$.

But this is just the Thom space of $(\gamma'/\partial\gamma') \wedge (v/\partial v) = \hat{\gamma} \wedge \bar{v}$ over $(B, \partial B)$. Hence

$$B^\gamma \text{ is } S\text{-dual to } (B, \partial B)^{\hat{\gamma} \wedge \bar{v}}. \qquad \square$$

In the above proposition, the duality map

$$\left(M^+ \wedge {}_J\Sigma^L Y\right) \wedge \left(M/J, \partial M/J\right)^{\bar{v} \wedge \hat{\gamma}} \to S^{s+t}$$

comes from the embedding j in (4.13). This and the fiberwise duality $\gamma \wedge \hat{\gamma} \to \dot{S}^t$ induce maps

$$(4.15) \qquad \mathrm{Bund}_0(\mu, \mu' \wedge \gamma) \to \mathrm{Bund}_0(\mu \wedge \hat{\gamma}, \mu' \wedge \dot{S}^t)$$

and

$$\mathrm{Map}_0\left(Y, M^+ \wedge {}_J\Sigma^L Y\right) \to \mathrm{Map}_0\left(Y \wedge \left(M/J, \partial M/J\right)^{\bar{v} \wedge \hat{\gamma}}, S^{s+t}\right)$$

which will be generically denoted by D. We now use these maps to complete the proof that ε is an equivalence.

By construction $\varepsilon = \varinjlim C_n^* \circ T_n \circ \sigma(\bar{\upsilon}_n)$. The connectivity of the fiber of $\bar{\upsilon}_n$ tends to infinity with n, and a suspension theorem **[J1]** applies to prove that $\varinjlim \sigma(\bar{\upsilon}_n)$ is an equivalence

To show that $\varinjlim C_n^* \circ T_n$ is a equivalence, we note that the following diagram commutes for $M = M_n$:

(4.16)

$$\begin{tikzcd}
\mathrm{Bund}_0\left(\bar{\upsilon}, \bar{\upsilon}|(\partial M/J); \overline{\tau \oplus \upsilon} \wedge \gamma, *\right) P \arrow[rr, "D"] \arrow[d, "T"'] & & \mathrm{Bund}_0\left(\bar{\upsilon} \wedge \hat{\gamma}, \bar{\upsilon} \wedge \hat{\gamma}|(\partial M/J); \dot{S}^{s+t}, *\right) \arrow[d, "T"] \\
\mathrm{Map}_0\left((M/J, \partial M/J)^{\upsilon}, \Sigma^s(M^+ \wedge_J \Sigma^L Y)\right) \arrow[d, "C^*"'] & & \mathrm{Map}_0\left((M/J, \partial M/J)^{\bar{\upsilon} \wedge \hat{\gamma}}, \Sigma^{s+t}(M/J^+)\right) \arrow[d, "p"] \\
\mathrm{Map}_0\left(S^s, \Sigma^s(M^+ \wedge_J \Sigma^L Y)\right) \arrow[dd, "\sigma"'] & & \mathrm{Map}_0\left((M/J, \partial M/J)^{\bar{\upsilon} \wedge \hat{\gamma}}, S^{s+t}\right) \arrow[dd, "\sigma"] \\
& M^+ \wedge_J \Sigma^L Y \arrow[ul, "\sigma"'] \arrow[ur, "D"] & \\
\mathrm{Map}_0\left(S^N, \Sigma^N(M^+ \wedge_J \Sigma^L Y)\right) \arrow[rr, "D"] & & \mathrm{Map}_0\left(\Sigma^N(M/J, \partial M/J)^{\bar{\upsilon} \wedge \hat{\gamma}}, S^{s+t+N}\right)
\end{tikzcd}$$

where the σ's are suspensions, N is any number $\geq s$, and p: $\Sigma^{s+t}(M/J^+) \to S^{s+t}$ collapses M/J to a point. By (4.2) (ii), the composite pT is a homeomorphism since $\overline{\tau \oplus \upsilon} = \dot{S}^{s+t}$ is trivial.

Passing to the limit over M_n, the suspensions and duality maps become equivalences, and hence so does $\varinjlim C_n^* T_n$.

This was all done assuming X finite, but now a simple colimit argument allows us to deduce the same result when X is a countable G-CW complex as in the hypotheses to (1.15). □

5. Duality and configuration spaces. This is parallel to §4; we will exhibit homotopy-equivalences

$$\mu\colon C_{NH}(U, Y)_{\mathscr{F}} \to C\left(\mathbf{R}^\infty, EJ^+ \wedge_J \Delta^L Y\right) \tag{5.1}$$

where U, Y, and L are as in (4.10).

Let $V_1 < V_2 < \cdots < U$ be as in (4.6), and $M_n = M(V_n)$ for $n = 1, 2, \ldots$. Now define $C_{NH}(M|\partial M, Y)$ as the space of configurations in $C_{NH}(V, Y)_{\mathscr{F}}$ whose V-coordinates all lie in $M - \partial M$. This suggestive notation is to indicate that they may be thought of as approximations to $\mathrm{Map}_{NH}(M/\partial M, \Sigma^V Y)$, though this will not be developed here. Note that the natural homeomorphism $M - \partial M \approx V \cap W_{(H)}$ induces a homeomorphism

$$C_{NH}(V, Y)_{\mathscr{F}} \approx C_{NH}(M|\partial M, Y). \tag{5.2}$$

Recall the embedding $i\colon M/G \to \mathbf{R}^s$ and define

$$(5.3)\qquad \phi\colon C_{NH}(M|\partial M, Y) \to C(\mathbf{R}^s, M^+ \wedge {}_J\Sigma^L Y)$$

by sending

$$(m_1,\ldots,m_k; [x_1, l_1],\ldots,[x_k, l_k]) \quad \text{to}$$
$$(i[m_1],\ldots,i[m_k]; [m_1, x_1, l_1],\ldots,[m_k, x_k, l_k])$$

and

$$\begin{aligned}\psi\colon C(\mathbf{R}^s, M^+ \wedge {}_J\Sigma^L Y) &\to C_{NH}(\mathbf{R}^s \times (M - \partial M), Y)_{\mathscr{F}}\\ &= C_{NH}(M(\mathbf{R}^s \times V)|\partial M(\mathbf{R}^s \times V), Y)\end{aligned}$$

by

$$(p, [m, x, l]) \mapsto (p, j(m); [x, l])$$

where $j\colon M \to M$ is a self-embedding with $j(M) \subseteq M - \partial M$.

We may choose $V_1, V_2,\ldots$ so that

$$V_n \oplus \mathbf{R}^{s_n} < V_{n+1},$$

and so that

$$(5.4)\qquad \begin{tikzcd} (M_n \times \mathbf{R}^{s_n})/J \arrow[r, hook, "i"] \arrow[d, phantom, "\wr"] & M_{n+1}/J \arrow[dd, "i_{n+1}"] \\ M_n/J \times \mathbf{R}^{s_n} \arrow[d, "i_n \times 1"'] & \\ \mathbf{R}^{s_n} \times \mathbf{R}^{s_n} \arrow[r, hook, "i'"] & \mathbf{R}^{s_{n+1}} \end{tikzcd}$$

commutes.

Then the following diagram commutes:

(5.5)

$$\begin{tikzcd} C_{NH}(V_n, Y)_{\mathscr{F}} \arrow[r, phantom, "\cong"] \arrow[dddd, hook] & C_{NH}(M_n|\partial M_n, Y) \arrow[dr, "\phi_n"] \arrow[dd, hook, "C(i_0)"'] & \\ & & C(\mathbf{R}^{s_n}, M_n^+ \wedge {}_J\Sigma^L Y) \arrow[dl, "\psi_n"] \arrow[dd, hook] \\ & C_{NH}(M_n \times \mathbf{R}^{s_n}|\partial M_n \times \mathbf{R}^{s_n}, Y) \arrow[dd, hook] & \\ & & C(\mathbf{R}^{s_{n+1}}, M_n^+ \wedge {}_J\Sigma^L Y) \arrow[d, "(C1, i_n)"] \\ C_{NH}(V_{n+1}, Y)_{\mathscr{F}} \arrow[r, phantom, "\cong"] & C_{NH}(M_{n+1}|\partial M_{n+1}, Y) \arrow[r, "\phi_{n+1}"] & C(\mathbf{R}^{s_{n+1}}, M_{n+1}^+ \wedge {}_J\Sigma^L Y). \end{tikzcd}$$

We pass to limits and obtain the composite

$$(5.6)\qquad \mu \left[\begin{array}{c} C_{NH}(U,Y)_{\mathscr{F}} \\ \| \\ \varinjlim C_{NH}(V_n,Y)_{\mathscr{F}} \\ \|\wr \\ \varinjlim C_{NH}(M_n|\partial M_n,Y) \\ \varinjlim \phi_n \downarrow \quad \uparrow \varinjlim \psi_n \\ \varinjlim C(\mathbf{R}^{s_n}, M_n^+ \wedge {}_J\Sigma^L Y) \\ \| \\ C(\mathbf{R}^\infty, EJ^+ \wedge {}_J \Sigma^L Y) \end{array} \right.$$

which is an equivalence of H-spaces.

A similar theorem is provable for $\overline{C}_{NH}(U,Y)$; in fact there is an equivalence $\bar{\mu}$ such that

$$\begin{array}{ccc} C_{NH}(U,Y) & \overset{\gamma}{\leftarrow} & \overline{C}_{NH}(U,Y) \\ \mu\downarrow & & \downarrow\bar{\mu} \\ C(\mathbf{R}^\infty, EJ^+\wedge {}_J\Sigma^L Y) & \overset{\gamma'}{\leftarrow} & C_\infty(EJ^+\wedge {}_J\Sigma^L Y) \end{array}$$

commutes, where γ' is the map replacing each little cube with its center-point.

6. Proof of the main theorem. This section contains the proof of (1.15); in fact, we will prove the following general statement:

(6.1) If $\mathbf{R}^\infty \leq W$, then the restriction of α,

$$\alpha_{\mathscr{F}}\colon C_G(W,X)_{\mathscr{F}} \to (\Omega^W\Sigma^W X)^G_{\mathscr{F}},$$

is a group completion for any orbit-type family $\mathscr{F}$ of closed subgroups of G.

The proof of (6.1) consists of a series of lemmas.

(6.2) LEMMA. *If H is maximal in $\mathscr{F}$, then the induced map*

$$\alpha^{NH}\colon C_{NH}(W^H, X^H)_{\mathscr{F}} \to (\Omega^{W^H}\Sigma^{W^H}X^H)^{NH}_{\mathscr{F}}$$

is a group-completion.

Proof. Assembling the results of §§4–5, we see that there is a diagram of H-spaces and H-maps:

$$\begin{array}{ccccc} C_{NH}(W^H,X^H)_{\mathscr{F}} & \overset{\gamma}{\leftarrow} & \overline{C}_{NH}(W^H,X^H)_{\mathscr{F}} & \overset{\alpha}{\rightarrow} & (\Omega^{W^H}\Sigma^{W^H}X^H)^{NH}_{\mathscr{F}} \\ \downarrow\mu & & \bar{\mu}\downarrow & & \downarrow\varepsilon \\ C(\mathbf{R}^\infty, EJ^+\wedge {}_J\Sigma^L X^H) & \overset{\gamma'}{\leftarrow} & C_\infty(EJ^+\wedge {}_J\Sigma^L X^H) & \overset{\alpha'}{\rightarrow} & \Omega^\infty\Sigma^\infty(EJ^+\wedge {}_J\Sigma^L X^H) \end{array}$$

which commutes up to homotopy, where α' is the nonequivariant approximation, and $\gamma, \gamma', \mu, \bar{\mu}$, and ε are equivalences. Thus (6.2) follows from the fact that α' is a group-completion ([C1], [C4], [S1]). □

(6.3) LEMMA. *Let A, C be H-spaces. Let F, E, B be grouplike H-spaces (i.e., application of the functor $\pi_0(-)$ yields a group), and $F \xrightarrow{i} E \xrightarrow{p} B$ a fiber sequence such that the following diagram is homotopy-commutative*:

$$\begin{array}{ccc} E & \xrightarrow{\simeq} & (\pi_0 E) \times E_0 \\ p\downarrow & & \downarrow \pi_0(p) \times (p|E_0) \\ B & \xrightarrow{\simeq} & (\pi_0 B) \times B_0 \end{array}$$

Then if

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & F \\ i_1 \downarrow & & \downarrow i \\ A \times C & \xrightarrow{\gamma} & E \\ pr_2 \downarrow & & \downarrow p \\ C & \xrightarrow{\beta} & B \end{array}$$

is a commutative diagram of H-spaces and H-maps, and α, β are group-completions, then γ is a group-completion and $E \simeq F \times B$.

Proof. Clearly it suffices to show this for the case where B is connected ($B = B_0$) or discrete ($B \cong \pi_0 B$). The discrete case is easy algebra, and we consider only the connected case.

The product of group-completions is a group-completion [C2], [M4], so that there is an H-map $\bar{\gamma}$: $F \times B \to E$ such that

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\alpha"] \arrow[dd] & \\ F \arrow[rr, "="] \arrow[dd] & & F \arrow[dd] \\ & A \times C \arrow[dl, "\alpha \times \beta"'] \arrow[dr, "\gamma"] \arrow[dd] & \\ F \times B \arrow[rr, dashed, "\bar{\gamma}"] \arrow[dd] & & E \arrow[dd] \\ & C \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & \\ B \arrow[rr, "="] & & B \end{tikzcd}$$

homotopy-commutes. Hence $\bar{\gamma}$ is an equivalence, and γ is a group-completion. □

Now partially order the conjugacy classes of subgroups of G by defining

$$(H) < (K)$$

if H is subconjugate to K. Then an orbit-type family is precisely the union of the classes in an initial segment of this partial order.

By letting $\mathscr{F}$ run through some cofinal sequence of initial segments, one obtains the following by induction (see, for example, McClure [**M5**]):

(6.4) LEMMA. *The statement* (6.1) *is true for all* $\mathscr{F}$ *if*

(i) (6.1) *is true for* $\mathscr{F} = \{1\}$.

(ii) *whenever* $\mathscr{F}_1$ *and* $\mathscr{F}_2 = \mathscr{F}_1 \cup (H)$ *are a successive pair of families, and* (6.1) *is true for* $\mathscr{F}_1$, *then it is true for* $\mathscr{F}_2$, *and*

(iii) *whenever* $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_n \subset \cdots$ *is a chain of families and* $\mathscr{F} = \bigcup_n \mathscr{F}_n$, *then if* (6.1) *is true for all the* $\mathscr{F}_n$, *it is true for* $\mathscr{F}$.

We use this last lemma to prove (6.1). Hypothesis (i) is just the special case $\mathscr{F} = \{1\}$, $H = 1$, $NH = G$ of Lemma (6.2). Hypothesis (iii) is easily verified by looking at the homology of

$$C_G(W, X)_{\mathscr{F}} = \varinjlim C_G(W, X)_{\mathscr{F}_n} \quad \text{and} \quad (\Omega^W \Sigma^W X)^G_{\mathscr{F}} = \varinjlim (\Omega^W \Sigma^W X)^G_{\mathscr{F}_n}.$$

Finally, to verify hypothesis (ii), we take together the results of §§2–3 and note that the following diagram commutes:

$$\begin{array}{ccc}
C_G(W, X)_{\mathscr{F}_1} & \xrightarrow{\alpha^G_{\mathscr{F}_1}} & (\Omega^W \Sigma^W X)^G_{\mathscr{F}_1} \\
i \downarrow & & \downarrow i' \\
C_G(W, X)_{\mathscr{F}_2} & \xrightarrow{\alpha^G_{\mathscr{F}_2}} & (\Omega^W \Sigma^W X)^G_{\mathscr{F}_2} \\
p \downarrow & & \downarrow p' \\
C_{NH}(W^H, X^H)_{\mathscr{F}_2} & \xrightarrow{\alpha^{NH}_{\mathscr{F}_2}} & (\Omega^{W^H} \Sigma^{W^H} X^H)^{NH}_{\mathscr{F}_2}
\end{array}$$

By hypothesis $\alpha^G_{\mathscr{F}_1}$ is a group-completion, and by (6.2), $\alpha^{NH}_{\mathscr{F}_2}$ is a group-completion. Hence by (6.3), $\alpha^G_{\mathscr{F}_2}$ is a group completion. □

Hence (6.1) is true for all $\mathscr{F}$ and (1.15) follows.

7. Other results. We offer a proof of Corollary (1.16). Recall that $A(G)$ is additively the free abelian group generated by elements $[G/H]$ for which $|NH: H| < \infty$. If $A^+(G)$ is the submonoid of elements with non-negative coefficients [**D1**], then $A(G)$ is the universal enveloping group of $A^+(G)$.

A consequence of (1.15) is that

$$\pi_0(\alpha\gamma^{-1})\colon \pi_0(C_G(W, S^0)) \to \pi_0((\Omega^W S^W)^G) = [S^W, S^W]_G$$

is the inclusion of $\pi_0(C_G(W, S^0))$ into its universal enveloping group. Hence (1.16) can be shown by constructing an isomorphism

$$\Phi\colon A^+(G) \to \pi_0(C_G(W, S^0))$$

of monoids.

To keep our notation clear, let $S^0 = \{*, a\}$. Then any point in $C_G(W, S^0)$ may be written in the form

$$z = [v_1,\dots,v_j;\, a, a,\dots,a;\, t_1,\dots,t_n]$$

where $v_i \in W$ and $t_i \in L(G_{v_i})^{G_{v_i}}$. If some v_i has isotropy group G_{v_i} with $|NG_{v_i}\colon G_{v_i}| = \infty$, then $L(G_{v_i})^{G_{v_i}} \neq 0$, and so there is a path from z to the point

$$\begin{aligned} z' &= [v_1,\dots,v_j;\, a,\dots,a;\, t_1,\dots,t_i' = \infty,\dots,t_j] \\ &= [v_1,\dots,\hat{v}_i,\dots,v_j;\, a,\dots,a;\, t_1,\dots,\hat{t}_i,\dots,t_j]. \end{aligned}$$

It follows that any element of $\pi_0(C_G(W, S^0))$ may be represented by a point of the form

$$[v_1,\dots,v_j;\, a,\dots,a;\, 0,\dots,0]$$

where G_{v_i} has finite index in its normalizer.

Now let $(H_1),\dots,(H_n),\dots$ be the conjugacy classes of subgroups of G, and choose $w_1,\dots,w_n,\dots$ such that

$$w_n \in W_{(H_n)}.$$

Then define Φ as above by letting

$$\Phi([G/H_i]) = \{[w_i;\, a;\, 0]\}$$

and extending to $A^+(G)$ by additivity.

An inverse $\Psi\colon \pi_0(C_G(W, S^0)) \to A^+(G)$ to Φ may be defined by letting

$$\Psi\{[v_1,\dots,v_j;\, a,\dots,a;\, 0,\dots,0]\} = \sum_{n=1}^{j} [G/G_{v_i}],$$

where $G_{v_1},\dots,G_{v_j}$ are of finite index in their normalizers. Then it is easily verified that $\psi\Phi = \mathrm{id}$ and that

$$\Phi\psi\{z\} = \{z'\}$$

where z' is in the same path-component as z.

8. Other questions. The ambition highlighted in the introduction, of finding a model for $\Omega^V\Sigma^V X$ which would serve as a basis for a recognition principle, is still unsatisfied. Three basic and natural questions spring up and need to be answered:

(i) To what do the natural inclusion maps

$$(\Omega^W\Sigma^W X)^G \hookrightarrow (\Omega^W\Sigma^W X)^H$$

correspond on the configuration-space level, for $H \leq G$?

(ii) Can we construct a manageable global model $C(W, X)$ so that

$$(C(W, X))^H = C_H(W, X) \text{ for all } H \leq G$$

(as for the case where G is finite)?

(iii) What can be said for the "unstable" case where $\mathbf{R}^n \leq W$ but $\mathbf{R}^\infty \nleq W$?

Related to these questions is that of the multiplicative structure in the Burnside ring:

(iv) Is there a natural ring space structure $C_G(W, X) \times C_G(W, X) \to C_G(W, X)$ corresponding to multiplication in $A(G)$ via (1.16)?

Finally, we are examining the following question along with (i)–(iv), which are all work in progress.

(v) Recall the homotopical model $\tilde{C}_n X$ for $\Omega^n\Sigma^n X$ [**C3**]. Is there a similar model for $(\Omega^W\Sigma^W X)^G$, and how does it relate to $C_G(W, X)$?

This last question may need to be answered before we can approach any of the others.

References

[A1] M. F. Atiyah, *Thom complexes*, Prof. London Math. Soc. (3), **11** (1961), 291–310.

[B1] J. C. Becker and R. E. Schultz, *Equivariant function spaces and stable homotopy theory* I, Comm. Math. Helv., **49** (1974), 1–34.

[B2] J. M. Boardman and R. M. Vogt, *Homotopy-everything H-spaces*, Bull. Amer. Math. Soc., **74** (1968), 1117–1122.

[C1] J. L. Caruso, Ph.D. Thesis, Univ. of Chicago (1979).

[C2] J. L. Caruso, F. R. Cohen, J. P. May, and L. R. Taylor, *James maps, Segal maps, and the Kahn-Priddy theorem*, to appear.

[C3] J. L. Caruso and S. Waner, *An approximation to* $\Omega^n\Sigma^n X$, Trans. Amer. Math. Soc., **265** (1981), 147–162.

[C4] F. R. Cohen, T. Lada, and J. P. May, *The Homology of Iterated Loop Spaces*, Lecture Notes in Mathematics, vol. **533** (1976). Springer-Verlag.

[C5] F. R. Cohen, J. P. May, and L. R. Taylor, *Splittings of certain spaces CX*, Math. Proc. Cam. Phil. Soc., **84** (1978), 465–496.

[D1] T. tom Dieck, *The Burnside ring and equivariant stable homotopy*, Chicago Lecture Notes, 1975.

[D2] ______, *Transformation Groups and Representation Theory*, Springer volume 766 (1979).

[H1] H. Hauschild, *Zerspaltung äquivariante homotopiemengen*, Math. Ann., **230** (1977), 279–292.
[J1] I. M. James, *Bundles with special structure*, J. Ann. Math., **89** (1969), 359–390.
[M1] J. P. May, *The Geometry of Iterated LLoop Spaces*, Springer vol. 271, 1972.
[M2] ______, *Classifying spaces and fibrations*, Mem. Amer. Math. Soc., no. 155, 1972.
[M3] ______, *The Homotopical Foundations of Algebraic Topology*, Acad. Press, to appear.
[M4] J. P. May and R. Thomason, *The uniqueness of infinite loop space machines*, Topology, **17** (1978), 205–224.
[M5] J. E. McClure, *Localization and Splitting of Equivariant Homology and Cohomology Theories*, §5, preprint, 1980.
[S1] G. B. Segal, *Configuration spaces and iterated loop spaces*, Invent. Math., **21** (1973), 213–221.
[S2] ______, *Some results in equivariant homotopy theory*, preprint, 1979.

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