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CLOPEN REALCOMPACTIFICATION OF A MAPPING

TAKESI ISIWATA

In this note, we give a necessary and sufficient condition on φ: $X \to Y$ for $\upsilon\varphi$ to be an open perfect mapping of υX onto υY and other related results.

Throughout this paper, by a space we mean a completely regular Hausdorff space and mappings are continuous and we assume familiarity with [1] whose notation and terminology will be used throughout. We denote by φ: $X \to Y$ a map of X onto Y, by βX (υX) the Stone-Čech compactification (Hewitt realcompactification) of X and by $\beta\varphi$ ($\upsilon\varphi = (\beta\varphi)|\upsilon X$) the Stone extension (realcompactification) over βX (υX) of φ.

Concerning clopenness of $\upsilon\varphi$ of a clopen map φ: $X \to Y$ the following results are known.

THEOREM A (*Ishii* [4]). *If* φ: $X \to Y$ *is an open quasi-perfect map, then* $\upsilon\varphi$ *is an open perfect map of* υX *onto* υY.

THEOREM B (*Morita* [8]). *If* φ: $X \to Y$ *is a clopen map such that the boundary of each fiber is relatively pseudocompact, then* $\upsilon\varphi$ *is also a clopen map of* υX *onto* υY.

In §2, concerning Theorem A we give a necessary and sufficient condition on φ for $\upsilon\varphi$ to be an open perfect map of υX onto υY without using the theory of hyper-spaces (Theorem 2.3 below) and a necessary and sufficient condition on φ for $\upsilon\varphi$ to be an open RC-preserving map of υX onto υY under some condition (Theorem 2.6 below).

We use the following notation and abbreviation: $C(X)$ is the set of real-valued continuous functions defined on X, $C(X; \varphi) = \{f \in C(X); f$ is φ-bounded$\}$, Bd A = the boundary of A, usc = upper semicontinuous, lsc = lower semicontinuous and ω (ω_1) = the first infinite (uncountabel) ordinal, clopen = closed and open.

1. Definitions and Lemmas.

1.1. DEFINITION. Let φ: $X \to Y$. $f \in C(X)$ is said to be *φ-bounded* if $\sup\{|f(x)|; x \in \varphi^{-1}(y)\} < \infty$ for every $y \in Y$. Whenever f is φ-bounded, we put

$$f^s(y) = \sup\{f(x); x \in \varphi^{-1}(y)\} \quad \text{and}$$

$$f^i(y) = \inf\{f(x); x \in \varphi^{-1}(y)\} \quad \text{for each } y \in Y.$$

A subset A of X is *relatively pseudocompact* if $f|A$ is bounded for each $f \in C(X)$. $\varphi\colon X \to Y$ is said to be

(1) *WZ* if $\mathrm{cl}_{\beta X}\varphi^{-1}y = (\beta\varphi)^{-1}y$ for each $y \in Y$ [5].

(2) W_rN if $\mathrm{cl}_{\beta X}\varphi^{-1}R = (\beta\varphi)^{-1}(\mathrm{cl}_{\beta Y}R)$ for every regular closed set R of Y [3].

(3) *$*$-open* (*W^*-open*) if $\mathrm{int}(\mathrm{cl}\,\varphi U) \supset \varphi U$ ($\mathrm{int}(\mathrm{cl}\,\varphi U) \neq \varnothing$) for every open set U of X [2, 7].

(4) *β-open* if φ is $*$-open and W_rN.

(5) a *d^*-map* if $\bigcap \mathrm{cl}\,\varphi Z_n = \varnothing$ for any decreasing sequence $\{Z_n\}$ of zero sets of X with empty intersection [6].

(6) *RC-preserving* (*an RC-map*) if φR is regular closed(closed) for every regular closed set R of X [2].

We note that (1) a closed map is a Z-map and a Z-map is *WZ* [5], (2) an open map is $*$-open and a $*$-open map is W^*-open [7], (3) a space Y is cb$*$ iff any d^*-map onto Y is hyper-real, i.e., $\upsilon\varphi$ is a perfect map onto υY [6], (4) an *RC*-preserving map is *RC* and (5) an open *WZ*-map is β-open by 1.2 (1, 5) below. Thus it is easy to see that if φ is β-open, then $(\beta\varphi)|Z\colon Z \to (\beta\varphi)Z$ is β-open for each Z with $X \subset Z \subset \beta X$. $Y \supset B$ is said to be *φ-d^** if $(\beta\varphi)^{-1}B \subset \upsilon X$. By 1.2(4) below, φ is a d^*-map iff Y is φ-d^*.

LEMMA 1.2. *Let $\varphi\colon X \to Y$.*

(1) *If φ is WZ, then φ is open iff $\beta\varphi$ is open* [5].

(2) *if φ is open* (*WZ*), *then f^i is usc* (*lsc*) *and f^s is lsc* (*usc*) *for every $f \in C(X\colon \varphi)$* (*for example, see* [5]).

(3) *If φ is open WZ, then f^i and $f^s \in C(Y)$ for every $f \in C(X; \varphi)$* [5].

(4) *φ is a d^*-map iff $(\beta\varphi)^{-1}Y \subset \upsilon X$* [6].

(5) *φ is β-open iff $\beta\varphi$ is open* [7].

(6) *If φ is an RC-map, then φ is WZ* [3].

(7) *φ is RC-preserving iff φ is a W^*-open RC-map* [2].

2. Main Theorems.

LEMMA 2.1. *Let $\varphi\colon X \to Y$. Then the following are equivalent*:

(1) *φ is WZ* (*open*).

(2) *f^i is lsc* (*usc*) *for every $f \in C(X; \varphi)$*

(3) *f^s is usc* (*lsc*) *for every $f \in C(X; \varphi)$*.

Proof. (2) $\Leftrightarrow$ (3) is evident. (1) $\Rightarrow$ (2). From 1.2(2).

We will prove (2) $\Rightarrow$ (1). Suppose that φ is not WZ. Then there are $y \in Y$ and $p \in \beta X$ with $p \in (\beta\varphi)^{-1}y - \mathrm{cl}_{\beta X}\varphi^{-1}y$. Since $p \notin \mathrm{cl}_{\beta X}\varphi^{-1}y$, there is $g \in C(\beta X)$ such that $p \in \mathrm{int}_{\beta X}Z(g)$ and $g = 1$ on $\mathrm{cl}_{\beta X}\varphi^{-1}y$. Let us put $f = g|X$. Then $f \in C(X)$, $f^i(y) = 1$, $A = Z(f) \neq \varnothing$ and $p \in \mathrm{cl}_{\beta X}A$. On the other hand, $\mathrm{cl}_{\beta Y}\varphi A = \mathrm{cl}_{\beta Y}(\beta\varphi)A = (\beta\varphi)\mathrm{cl}_{\beta X}A \ni (\beta\varphi)p = y$. This shows $y \in \mathrm{cl}\,\varphi A$ and hence for each neighborhood V of y, there is $z \in V$ with $f^i(z) = 0$, i.e., f^i is not lsc.

Now suppose that φ is not open. Then there are a point x and an open set $U \ni x$ such that $V - \varphi U \neq \varnothing$ for every open set $V \ni y = \varphi(x)$. Let $f \in C(X;\ \varphi)$ such that $x \in \mathrm{int}\,Z(f) \subset U$ and $f = 1$ on $X - U$. Obviously $f^i(y) = 0$ and $f^i = 1$ on $V - \varphi U$. This shows that f^i is not usc.

Using 2.1, it is easy to see the following:

THEOREM 2.2. *φ: $X \to Y$ is open WZ iff f^i and $f^s \in C(Y)$ for every $f \in C(X;\ \varphi)$ equivalently,*

$$C(Y) = \{f^i; f \in C(X;\ \varphi)\} = \{f^s; f \in C(X;\ \varphi)\}.$$

THEOREM 2.3. *φ: $X \to Y$ is a β-open d^*-map iff $v\varphi$ is an open perfect map of vX onto vY.*

Proof. $\Leftarrow$) From 1.2(1, 4, 5) and $(\beta\varphi)^{-1}Y \subset (\beta\varphi)^{-1}vY = vX$. $\Rightarrow$) By 1.2(5), $\beta\varphi$ is open. We will prove that $v\varphi$ is a perfect map onto vY. To do this, it suffices to show that $(\beta\varphi)p = q \in \beta Y - vY$ for every $p \in \beta X - vX$. Let $p \in \beta X - vX$. Then there is $f \in C(\beta X)$ with $p \in Z(f) \subset \beta X - vX$. $\beta\varphi$ being open WZ by 1.2(5), it follows from 2.2 that $f^i \in C(\beta Y)$, $f^i(q) = 0$ and $f^i > 0$ on Y. This shows $q \in \beta Y - vY$, so $v\varphi$ is a perfect map onto vY. Since $\beta(v\varphi) = \beta\varphi$ and $\beta\varphi$ is open, $v\varphi$ is open by 1.2(1). Thus $v\varphi$ is an open perfect map of vX onto vY.

2.4. EXAMPLE. Let $X = [0, \omega_1]^2 - \{(\omega_1, \alpha);\ \omega \leq \alpha \leq \omega_1\}$, $Y = [0, \omega_1]$ and φ the projection of X onto Y. It is obvious that φ is not WZ and hence not closed and $\varphi^{-1}(\omega_1)$ is not compact. On the other hand $\beta\varphi$: $\beta X = vX = [0, \omega_1]^2 \to Y = vY = \beta Y$ is open perfect (compare with the assumption of Theorem A).

2.5. LEMMA. *If φ: $X \to Y$ is a $*$-open RC-map, then φ is open.*

Proof. Let U be open in X and $x \in U$. Take a regular closed set R with $x \in \mathrm{int}\,R \subset R \subset U$. Since φ is a $*$-open RC-map, we have $y = \varphi(x) \in \mathrm{int}(\mathrm{cl}\,\varphi(\mathrm{int}\,R)) \subset \varphi R \subset \varphi U$, so $y \in \mathrm{int}\,\varphi U$. Thus φ is open.

In the following we put

$Y_d = \{ y \in Y; \varphi^{-1}y$ is open but not relatively pseudocompact$\}$,

$Y_e = X - Y_d$.

THEOREM 2.6. *φ: $X \to Y$ is a β-open map such that Y_e is φ-d^* iff $\upsilon\varphi$ is an open RC-preserving map of υX onto υY such that $\mathrm{cl}_{\upsilon Y} Y_e$ is $(\upsilon\varphi)$-d^*.*

Proof. $\Leftarrow$) Since $\upsilon\varphi$ is open WZ by 1.2(6, 7), $\beta\varphi$ is open by 1.2(1) and φ is a β-open map by 1.2(5). The fact that $\mathrm{cl}_{\upsilon Y} Y_e$ is $(\upsilon\varphi)$-d^* implies that Y_e is φ-d^*.

$\Rightarrow$) (1) We will first prove that if $p \in \beta X - \upsilon X$ and $(\beta\varphi)p = q \in \upsilon Y$, then there is a clopen subset D of Y such that $q \in \mathrm{cl}_{\upsilon Y} D$, $D \subset Y_d$ and $\mathrm{cl}_{\upsilon Y} D \cap \mathrm{cl}_{\upsilon Y} Y_e = \varnothing$. There is $f \in C(\beta X)$ with $p \in Z(f) \subset \beta X - \upsilon X$. By 1.2(5), $\beta\varphi$ is open. Thus $f^i \in C(\beta Y)$. Since Y_e is $\varphi - d^*$, $f^i > 0$ on Y_e and hence $Z(f^i) \cap Y_e = \varnothing$. Since $f^i(q) = 0$, $q \in \upsilon Y$ and $Z(f^i)$ is closed. $D = Z(f^i) \cap Y_d = Z(f^i) \cap Y$ is a non-empty clopen discrete subset of Y contained in Y_d. $\mathrm{Cl}_{\upsilon Y} D = Z(f^i) \cap \upsilon Y$ implies $q \in \mathrm{cl}_{\upsilon Y} D$ and $\mathrm{cl}_{\upsilon Y} D \cap \mathrm{cl}_{\upsilon Y} Y_e = \varnothing$.

(2) Let us put $\mathscr{D} = \{ D \subset Y_d;\ D$ is a clopen subset of $Y\}$ and $\mathrm{cl}_{\upsilon Y}\mathscr{D} = \bigcup\{\mathrm{cl}_{\upsilon Y} D;\ D \in \mathscr{D}\}$. Then it is easy to see the following

$$\upsilon Y = \mathrm{cl}_{\upsilon Y}\mathscr{D} \cup \mathrm{cl}_{\upsilon Y} Y_e, \qquad \mathrm{cl}_{\upsilon Y}\mathscr{D} \cap \mathrm{cl}_{\upsilon Y} Y_e = \varnothing$$

and

$$(\beta\varphi)^{-1}\mathrm{cl}_{\upsilon Y} Y_e \subset \upsilon X.$$

(3) *$\upsilon\varphi$ is onto υY.* Let $q \in \mathrm{cl}_{\upsilon Y} D$, $D \in \mathscr{D}$. For each $y \in D$, let us pick a point $p(y)$ from $\varphi^{-1}y$ and put $A = \{ p(y); y \in D\}$. Then A is a discrete closed C-embedded subset of X. Thus $\upsilon A = \mathrm{cl}_{\upsilon X} A$ is homeomorphic to $\mathrm{cl}_{\upsilon Y} D$ under the map $\upsilon\varphi$. Thus we have $\upsilon\varphi(\upsilon X) = \upsilon Y$.

(4) *$\upsilon\varphi$ is an RC-map.* Let F be regular closed in υX and $E = (\upsilon\varphi)F$. Suppose that there is $q \in \mathrm{cl}_{\upsilon Y} E - E$. By (2) and the clopenness of $\varphi^{-1}y$, $y \in Y_d$, we have $q \notin Y_d \cup \mathrm{cl}_{\upsilon Y} Y_e$. Thus there is $D \in \mathscr{D}$ with $q \in \mathrm{cl}_{\upsilon Y} D$ and $\mathrm{cl}_{\upsilon Y} D \cap \mathrm{cl}_{\upsilon Y} Y_e = \varnothing$ by (2). Since $\beta\varphi$ is open by 1.2(5), $\upsilon\varphi$ is also $*$-open and we have that $E \supset (\upsilon\varphi)\mathrm{int}_{\upsilon X} F$ is dense in $\mathrm{cl}_{\upsilon Y} E$ because F is regular closed. Let $M = E \cap D \cap Y_d$. Then $q \in \mathrm{cl}_{\upsilon Y} M$. Let us pick a point $p(y)$ from $\varphi^{-1}(y) \cap F$, $y \in M$. $A = \{ p(y); y \in M\}$ is a discrete closed C-embedded subset of X and hence $\upsilon A = \mathrm{cl}_{\upsilon X} A \subset F$ and υA is homeomorphic to $\upsilon M = \mathrm{cl}_{\upsilon Y} M$, so $q \in E$ a contradiction.

(5) *$\upsilon\varphi$ is open RC-preserving.* Since $\upsilon\varphi$ is an RC-map, $\upsilon\varphi$ is WZ by 1.2(6). Thus the openness of $\beta\varphi$ implies that $\upsilon\varphi$ is open by 1.2(1) and RC-preserving by 1.2(7).

As a direct consequence of the above theorem, we have the following corollary which is a generalization of the result obtained in [5] if X is realcompact and $\varphi\colon X \to Y$ is an open WZ map with $\mathrm{Bd}\,\varphi^{-1}y$ = compact for each $y \in Y$, then Y is also realcompact.

COROLLARY 2.7. *If X is realcompact and $\varphi\colon X \to Y$ is a β-open map such that Y_e is φ-d^*, then Y is also realcompact.*

THEOREM 2.8. *Let $\varphi\colon X \to Y$ and $Z = (\beta\varphi)^{-1}Y_d \cup vX$. Then the following are equivalent:*

(1) *Z is a realcompact and φ is a β-open map such that Y_e is φ-d^*.*

(2) *$\varphi' = (\beta\varphi)|Z$ is an open perfect map of Z onto vY.*

(3) *$v\varphi$ is a clopen map of vX onto vY such that* $\mathrm{Bd}(v\varphi)^{-1}q$ *is compact for every $q \in vY$.*

(4) *$v\varphi$ is a clopen map of vX onto vY such that $(vY)_e$ is $(v\varphi)$-d^*.*

Proof. (1) $\Rightarrow$ (2) If $Z = \beta X$, then $\varphi' = \beta\varphi$ and φ' is an open perfect map onto vY. Let $p \in \beta X - Z$ and $q = (\beta\varphi)p$. Then $Z = vZ$, $\beta Z = \beta X$ and there is $f \in C(\beta X)$ such that $p \in Z(f) \subset \beta X - Z$ and $0 \leq f \leq 1$. Since $\beta\varphi$ is open WZ and Y_e is φ-d^*, it is easy to see that $f^i \in C(\beta Y)$, $f^i(q) = 0$ and $f^i > 0$ on Y. Thus $q \in \beta Y - vY$, so φ' is a perfect map onto vY. The openness of φ' follows from 1.2(1, 5).

(2) $\Rightarrow$ (3) We shall show that $v\varphi$ is closed. Let F be closed in vX and $q \in \mathrm{cl}_{vY}(v\varphi)F - (v\varphi)F$. Since φ' is perfect and every point of Y_d is isolated, we have $q \notin Y_d$, so $(\beta\varphi)^{-1}q = (v\varphi)^{-1}q$ is disjoint from $\mathrm{cl}_Z F$, and hence $q \notin \varphi'(\mathrm{cl}_Z F)$, a contradiction. Thus $v\varphi$ is closed. The verifications of other parts are easy. (3) $\Rightarrow$ (4) Evident.

(4) $\Rightarrow$ (1) Since $v\varphi$ is clopen, $\beta(v\varphi) = \beta\varphi$ is open by 1.2(1) and hence φ is β-open by 1.2(5). Since $vY = (vY)_e \cup Y_d$, the $(v\varphi)$-d^*-ness of $(vY)_e = vY - Y_d$ implies the φ-d^*-ness of Y_e. Since $Y_d = (vY)_d$ and $(vY)_e$ is $(v\varphi)$-d^*, we have $Z = (\beta\varphi)^{-1}vY$, and hence $\varphi'\colon Z \to vY$ is an open perfect map which shows that Z is realcompact.

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TOKYO GAKUGEI UNIVERSITY
(184) 4-1-1 NUKUIKITA-MACHI
KOGANEI-SHI, TOKYO, JAPAN