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ON THE RADIAL MAXIMAL FUNCTION OF DISTRIBUTIONS

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We show that if the radial maximal function of a distribution $f \in \mathscr{D}(R^n)'$ belongs to $L^p(R^n)$, then f belongs to $H^p(R^n)$. This gives an affirmative answer to the question posed by Aleksandrov and Havin.

1. Introduction. Functions and distributions considered are real-valued. For $x = (x_1, \dots, x_n) \in R^n$, $t > 0$, $a > 0$ and for a measurable function $h(x)$ defined on R^n let

$$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2},$$

$$B(x,t) = \{ y \in R^n : |x - y| < t \},$$

$$Q(x,t) = \{ (y,s) : y \in B(x,t),\ s \in (0,t) \},$$

$$(h)_t(x) = t^{-n} h(x/t)$$

and

$$\|h\|_{\Lambda_a} = \sup_{x \in R^n,\ t>0} \ \inf_{P:\ \deg P \le a} t^{-n-a} \int_{B(x,t)} |h(y) - P(y)|\, dy,$$

where the infimum is taken over all polynomials $P(y)$ of degree $\le a$. Let

$$\Lambda_a(R^n) = \left\{ h \in L^1_{\mathrm{loc}}(R^n) : \|h\|_{\Lambda_a} < +\infty \right\}.$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, where α_i's are nonnegative integers, let

$$|\alpha| = \sum_{i=1}^{n} \alpha_i,$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

A linear functional f defined on $\mathscr{D}(R^n)$, which is the set of all C^∞-functions with compact support, is said to belong to $\mathscr{D}(R^n)'$ if to every compact set $K \subset R^n$ there correspond positive constants c and a such that

$$|f(\theta)| \leq c \sum_{\alpha:\, |\alpha| \leq a} \|D^\alpha \theta\|_{L^\infty},$$

whenever $\theta \in \mathscr{D}(R^n)$ and $\operatorname{supp} \theta \subset K$.

A linear functional f defined on $\mathscr{D}(R^n)$ is said to belong to $\mathscr{S}(R^n)'$ if there exist positive constants c and a such that

$$|f(\theta)| \leq c \sum_{\alpha:\, |\alpha| \leq a} \|(1 + |x|)^a D^\alpha \theta(x)\|_{L^\infty}$$

whenever $\theta \in \mathscr{D}(R^n)$.

Let

$$(1.1) \qquad \varphi \in \mathscr{D}(R^n), \quad \int_{R^n} \varphi(x)\, dx = 1 \quad \text{and} \quad \operatorname{supp} \varphi \subset B(0,1).$$

For $f \in \mathscr{D}(R^n)'$ and $x \in R^n$, let

$$f_\varphi^+(x) = \sup_{t>0} |f*(\varphi)_t(x)|,$$

where $f*\theta(x) = f(\theta(x - \cdot\,))$. In their celebrated paper [**2**], C. Fefferman and E. M. Stein characterized $H^p(R^n)$ as a subspace of $\mathscr{S}(R^n)'$. Among several other characterizations of $H^p(R^n)$, they showed that $H^p(R^n)$, $0 < p < +\infty$, can be identified with

$$(1.2) \qquad \left\{ f \in \mathscr{S}(R^n)' \colon f_\varphi^+ \in L^p(R^n) \right\}.$$

[They showed this fact more generally for rapidly decreasing functions φ satisfying $\int_{R^n} \varphi(x)\, dx = 1$. See [**2**] Theorem 11.] Consequently, the space (1.2) turns out to be independent of the choice of φ. In the proof of "(1.2) $\subset H^p(R^n)$", the condition $f \in \mathscr{S}(R^n)'$ is used to take Fourier transform and to give certain estimates on the growth of the function $f*(\varphi)_t(x)$ defined on $R_+^{n+1} = \{(x,t)\colon x \in R^n, t > 0\}$.

Recently, in "Linear and complex analysis problem book" [**4**] p. 346, Aleksandrov and Havin asked whether the condition $f \in \mathscr{S}(R^n)'$ in (1.2) can be replaced by the condition $f \in \mathscr{D}(R^n)'$, namely whether (1.2) is equal to

$$(1.3) \qquad \left\{ f \in \mathscr{D}(R^n)' \colon f_\varphi^+ \in L^p(R^n) \right\}.$$

[As they pointed out, if $p \in [1, +\infty)$, then the answer is YES.] In this paper we answer their question affirmatively. It is enough to show that

$$(1.4) \qquad (1.3) \subset \mathscr{S}(R^n)'$$

for all $p \in (0,1)$. The argument in this paper is a refinement of the author's paper [7].

For $\varepsilon > 0$ let

$$f^+_{\varphi,\varepsilon}(x) = \sup_{t \in (0,\varepsilon]} |f * (\varphi)_t(x)|.$$

Our result is the following.

THEOREM. *Assume* (1.1). *Let* $\varepsilon > 0$, $p \in (0,1)$, $f \in \mathscr{D}(R^n)'$, $x_0 \in R^n$, $t_0 > 0$, $\psi \in \mathscr{D}(R^n)$ *and* $\operatorname{supp}\psi \subset B(x_0, t_0)$. *Then*

$$|f(\psi)| \le C\left\{\int_{B(x_0,(1+\varepsilon)t_0)} f^+_{\varphi,\varepsilon t_0}(x)^p\,dx\right\}^{1/p} \|\psi\|_{\Lambda_{n(1/p-1)}},$$

where C is a constant depending only on ε, p, φ and n.

In particular we have

$$(1.5) \qquad |f(\theta)| \le C_{p,\varphi,n}\|f^+_\varphi\|_{L^p}\|\theta\|_{\Lambda_{n(1/p-1)}}$$

for any $p \in (0,1)$, any $f \in \mathscr{D}(R^n)'$ and any $\theta \in \mathscr{D}(R^n)$. Now, we explain how (1.4) follows from (1.5). Let $\rho \in \mathscr{D}(R^n)$ and $\rho(x) \equiv 1$ near the origin. Put $\sigma(x) = \rho(x/2) - \rho(x)$. Then for any $\theta \in \mathscr{D}(R^n)$ we have $\theta(x) \equiv \theta(x)\rho(x/2^k)$ with k sufficiently large depending on the support of θ. Thus

$$\begin{aligned} f(\theta) &= f\left(\theta(\cdot)\left(\rho(\cdot) + \sum_{k=0}^{\infty}\sigma\left(\frac{\cdot}{2^k}\right)\right)\right) \\ &= f(\theta(\cdot)\rho(\cdot)) + \sum_{k=0}^{\infty} f\left(\theta(\cdot)\sigma\left(\frac{\cdot}{2^k}\right)\right) \end{aligned}$$

because $\sum_{k=0}^{\infty}$ is actually a finite sum. This equality combined with (1.5) tells us

$$\begin{aligned} |f(\theta)| &\le C_{p,\varphi,n}\|f^+_\varphi\|_{L^p}\left\{\|\theta(\cdot)\rho(\cdot)\|_{\Lambda_{n(1/p-1)}} + \sum_{k=0}^{\infty}\left\|\theta(\cdot)\sigma\left(\frac{\cdot}{2^k}\right)\right\|_{\Lambda_{n(1/p-1)}}\right\} \\ &\le C_{p,\varphi,n,\rho}\|f^+_\varphi\|_{L^p} \sum_{\alpha:\,|\alpha|<n(1/p-1)+1} \|(1+|x|)D^\alpha\theta(x)\|_{L^\infty}. \end{aligned}$$

Therefore, $f^+_\varphi \in L^p$ implies $f \in \mathscr{S}(R^n)'$ and we get (1.4).

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2. Proof of the Theorem.

MAIN LEMMA. *Assume* (1.1). *Let* $0 < a < b$ *and* $\varepsilon > 0$. *Let* $\mathscr{E} \subset R_+^{n+1}$ *be a measurable set that satisfies*

$$\int\int_{\mathscr{E}\cap Q(x,t)} dy\,ds \le C_{2.1} t^{n+1} \tag{2.1}$$

for any $(x,t) \in R_+^{n+1}$. *Let* $\psi \in \Lambda_b(R^n)$ *and*

$$\operatorname{supp}\psi \subset B(0,1). \tag{2.2}$$

Then there exists a measurable function $k(x,t)$ *defined on* R_+^{n+1} *such that*

$$k(x,t) = 0 \text{ on } \mathscr{E}, \tag{2.3}$$

$$k(x,t) = 0 \text{ on } R_+^{n+1}\backslash(B(0,1+\varepsilon)\times(0,\varepsilon)), \tag{2.4}$$

$$|k(x,t)| \le C_{2.2}\min\left(t^a\|\psi\|_{\Lambda_a}, t^b\|\psi\|_{\Lambda_b}\right) \tag{2.5}$$

and

$$\psi(x) = \int\int_{R_+^{n+1}} (\varphi)_s(y-x)k(y,s)\,dy\frac{ds}{s}, \tag{2.6}$$

where $C_{2.1}$ *and* $C_{2.2}$ *are positive constants depending only on* a, b, ε, φ *and* n.

We prove this in §3.

LEMMA 2.1. *Let* $0 < a < a'$. *Let* $h \in \Lambda_{a'}(R^n)$ *and* $\operatorname{supp} h \subset B(0,1)$. *Let* ν *be a signed measure on* R_+^{n+1} *such that* $\operatorname{supp}\nu$ *is bounded and that*

$$|\nu|(Q(x,t)) \le t^{n+a} \tag{2.7}$$

for any $(x,t) \in R_+^{n+1}$, *where* $|\nu|$ *is the total variation of* ν. *Let*

$$g(x) = \int\int_{R_+^{n+1}} (h)_s(x-y)\,d\nu(y,s).$$

Then

$$\|g\|_{\Lambda_a} \le C_{2.3}\|h\|_{\Lambda_{a'}},$$

where $C_{2.3}$ *is a constant depending only on* a, a' *and* n.

Proof. We may assume $\|h\|_{\Lambda_{a'}} = 1$. Take any ball $B = B(z, t)$, where $z \in R^n$ and $t > 0$. Put

$$D_1 = \{(y, s) \in R_+^{n+1}: s \in (0, t), |y - z| < t + s\}$$

and

$$D_2 = \{(y, s) \in R_+^{n+1}: s \geq t, |y - z| < t + s\}.$$

If $x \in B$, then

$$\begin{aligned} g(x) &= \int\int_{D_1} (h)_s(x - y)\, d\nu(y, s) + \int\int_{D_2} (h)_s(x - y)\, d\nu(y, s) \\ &= g_1(x) + g_2(x). \end{aligned}$$

By (2.7)

$$\int_B |g_1(x)|\, dx \leq \|h\|_{L^1} \int\int_{D_1} d|\nu|(y, s) \leq Ct^{n+a}. \tag{2.8}$$

On the other hand, since

$$\begin{aligned} \|g_2\|_{\Lambda_{a'}} &\leq \int\int_{D_2} \|(h)_s\|_{\Lambda_{a'}} d|\nu| \leq \int\int_{D_2} s^{-n-a'} d|\nu| \\ &\leq \sum_{j=0}^{\infty} C(2^j t)^{-n-a'} \int\int_{D_2 \cap (Q(z, 2^{j+1}t) \setminus Q(z, 2^j t))} d|\nu| \\ &\leq \sum C(2^j t)^{-n-a'} (2^j t)^{n+a} \quad \text{by (2.7)} \\ &= Ct^{a-a'} \quad \text{by } a' > a, \end{aligned}$$

we get

$$\begin{aligned} \inf_{P:\, \deg P \leq a'} \int_B |g_2(x) - P(x)|\, dx &\leq \|g_2\|_{\Lambda_{a'}} t^{n+a'} \\ &\leq Ct^{a-a'} t^{n+a'} = Ct^{n+a}. \end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9) gives

$$\sup_{z \in R^n,\, t > 0}\ \inf_{P:\, \deg P \leq a'} t^{-n-a} \int_{B(z, t)} |g(x) - P(x)|\, dx \leq C. \tag{2.10}$$

It is known that if $a' \geq a$ and if supp g is compact, then the left-hand side of (2.10) is equivalent with $\|g\|_{\Lambda_a}$. [For example see Taibleson and Weiss [**6**].] This concludes the proof of Lemma 2.1. □

LEMMA 2.2. *Let $a \geq 0$, $q > 1$ and $h \in L^1_{\mathrm{loc}}(R^n)$. Let ν be a positive measure on R^{n+1}_+ that satisfies* (2.7) *for any $(x, t) \in R^{n+1}_+$. Then*

$$\left\{\int \int_{R^{n+1}_+} \left|h * s^{-n}\chi_{B(0,s)}(y)\right|^{q(1+a/n)} d\nu(y, s)\right\}^{1/(q(1+a/n))} \leq C_{2.4}\|h\|_{L^q}$$

where χ_B denotes the characteristic function of B and where $C_{2.4}$ is a constant depending only on q, a and n.

This is due to L. Carleson and P. Duren. [See **[1]**. This is proved in **[7]**, p. 583, too.]

Now, we begin the proof of our Theorem. In the following part of this section the letter C denotes various constants that depend only on ε, p, φ and n. We will show that if $\psi \in \mathscr{D}(R^n)$ satisfies (2.2), then

$$\text{(2.11)} \qquad |f(\psi)| \leq C\left\{\int_{B(0,1+2\varepsilon)} f^+_{\varphi,\varepsilon}(x)^p\, dx\right\}^{1/p} \|\psi\|_{\Lambda_{n(1/p-1)}}.$$

Our Theorem follows from (2.11) combined with translation and dilation. In order to show (2.11) we may assume

$$\text{(2.12)} \qquad \int_{B(0,1+2\varepsilon)} f^+_{\varphi,\varepsilon}(x)^p\, dx < +\infty.$$

Let $A > 1$ and

$$\text{(2.13)} \qquad \mathscr{E} = \{(y, s) \in R^n \times (0, \varepsilon): \\ |f * (\varphi)_s(y)|^{p/2} > A f^{+p/2}_{\varphi,\varepsilon} * s^{-n}\chi_{B(0,s)}(y)\}.$$

LEMMA 2.3. *Let p, ε, φ and f be as in the Theorem. Let $\mathscr{E}$ be defined by* (2.13). *Let $(y, s) \in (B(0, 1 + \varepsilon) \times (0, \varepsilon)) \setminus \mathscr{E}$. Then*

$$|f * (\varphi)_s(y)|^{p/2} \leq A\left(f^+_{\varphi,\varepsilon}\chi_{B(0,1+2\varepsilon)}\right)^{p/2} * s^{-n}\chi_{B(0,s)}(y).$$

This is clear from the definition of $\mathscr{E}$.

LEMMA 2.4. *Let p, ε, φ, f and $\mathscr{E}$ be as in Lemma* 2.3. *Then*

$$\text{(2.14)} \qquad \int \int_{\mathscr{E} \cap Q(x,t)} dy\, ds \leq C_{2.5}A^{-1}t^{n+1}$$

for any $(x, t) \in R^{n+1}_+$, where $C_{2.5}$ is a constant depending only on n.

Proof. Since $\mathscr{E} \subset R^n \times (0, \varepsilon)$, it is enough to show (2.14) only for the case $t \in (0, \varepsilon)$. First note that

$$\int_{\mathscr{E}\cap(B(x,t/2)\times\{t\})} dy \le \int_{B(x,t/2)} |f*(\varphi)_t(y)|^{p/2} \cdot \left\{At^{-n}\int_{B(y,t)} f^+_{\varphi,\varepsilon}(z)^{p/2}\,dz\right\}^{-1} dy$$

by the definition of $\mathscr{E}$

$$\le \int_{B(x,t/2)} |f*(\varphi)_t(y)|^{p/2}\,dy \cdot \left\{At^{-n}\int_{B(x,t/2)} f^+_{\varphi,\varepsilon}(z)^{p/2}\,dz\right\}^{-1}$$

$$\le A^{-1}t^n.$$

If $0 < s < t$, then dividing $B(x, t/2)$ into the union of the balls of radius $s/2$ with bounded overlap gives

$$\int_{\mathscr{E}\cap(B(x,t/2)\times\{s\})} dy \le CA^{-1}t^n.$$

Thus

$$\int\int_{\mathscr{E}\cap Q(x,t)} dy\,ds = \int_0^t ds \int_{\mathscr{E}\cap(B(x,t)\times\{s\})} dy \le CA^{-1}t^{n+1}. \qquad \square$$

We return to the proof of (2.11). Since $f \in \mathscr{D}(R^n)'$, there exist constants a_0 and c_0, depending on f and ε, such that

$$|f(\theta)| \le c_0\|\theta\|_{\Lambda a_0} \tag{2.15}$$

whenever

$$\theta \in \mathscr{D}(R^n) \text{ and } \operatorname{supp}\theta \subset B(0, 1 + 2\varepsilon), \tag{2.16}$$

[where Λa_0 means Λ_{a_0}. It is easy to show that if θ satisfies (2.16), then $C\|\theta\|_{\Lambda a_0}$ dominates $\sum_{\alpha:|\alpha|<a_0}\|D^\alpha\theta\|_{L^\infty}$.] Let

$$a = n(1/p - 1) \tag{2.17}$$

and

$$b = \max(a, a_0) + 1. \tag{2.18}$$

Let $C_{2.1}(a, a_0, \varepsilon, \varphi)$ and $C_{2.2}(a, a_0, \varepsilon, \varphi)$ be the constants $C_{2.1}$ and $C_{2.2}$ determined by the Main Lemma from the above data a, b, ε, and φ. Let

$$A(a, a_0, \varepsilon, \varphi) = C_{2.5}/C_{2.1}(a, a_0, \varepsilon, \varphi). \tag{2.19}$$

Then by Lemma 2.4 we can apply the Main Lemma to the above a [in (2.17)], b [in (2.18)], ε, φ, $\psi[\in \mathscr{D}(R^n)$ with (2.2)] and $\mathscr{E}$ [in (2.13) with $A = A(a, a_0, \varepsilon, \varphi)$] and get $k(x, t)$ that satisfies (2.3)–(2.6).

For $r > 0$ let

$$\psi_r(x) = \int\int_{y\in R^n,\, s>r} (\varphi)_s(y-x)k(y,s)\,dy\frac{ds}{s}.$$

Since $\operatorname{supp}\psi_r \subset B(0, 1+2\varepsilon)$ by (2.4) and (1.1) and since

$$\psi_r \to \psi \quad \text{in } \Lambda_{a_0} \quad \text{as } r \to +0$$

by (2.5) [with (2.18)], (2.6) and Lemma 2.1, we get

$$\begin{aligned} f(\psi) &= \lim_{r\to +0} f(\psi_r) \\ &= \lim_{r\to +0}\int\int_{y\in R^n,\, s>r} f*(\varphi)_s(y)k(y,s)\,dy\,\frac{ds}{s}. \end{aligned}$$

Thus

$$\begin{aligned} |f(\psi)| &\le \int\int_{R_+^{n+1}} |f*(\varphi)_s(y)||k(y,s)|\,dy\,\frac{ds}{s} \\ &\le \int\int A(a, a_0, \varepsilon, \varphi)^{2/p} \\ &\qquad \times \left(\left(f_{\varphi,\varepsilon}^+\chi_{B(0,1+2\varepsilon)}\right)^{p/2} * s^{-n}\chi_{B(0,s)}(y)\right)^{2/p}|k(y,s)|\,dy\,\frac{ds}{s} \\ &\qquad\qquad \text{by Lemma 2.3, (2.3) and (2.4)} \\ &\le \int\int A(\cdots)^{2/p}(\cdots * \cdots)^{2/p}C_{2.2}(a, a_0, \varepsilon, \varphi)s^a\|\psi\|_{\Lambda_a}\,dy\,\frac{ds}{s} \\ &\qquad\qquad \text{by (2.5)} \\ &= A(\cdots)^{2/p}C_{2.2}(\cdots)\int\int(\cdots * \cdots)^{2/p}s^a\,dy\,\frac{ds}{s}\|\psi\|_{\Lambda_a} \\ &\le A(\cdots)^{2/p}C_{2.2}(\cdots)C\left(\int_{B(0,1+2\varepsilon)} f_{\varphi,\varepsilon}^+(x)^p\,dx\right)^{1/p}\|\psi\|_{\Lambda_a} \\ &\qquad\qquad \text{by Lemma 2.2 and (2.17).} \end{aligned}$$

So far we have shown that if φ, ε and p are as in the Theorem, if $\psi \in \mathscr{D}(R^n)$ satisfies (2.2) and if $f \in \mathscr{D}(R^n)'$ satisfies (2.15) for any θ satisfying (2.16), then

$$\begin{aligned} |f(\psi)| &\le A(a, a_0, \varepsilon, \varphi)^{2/p}C_{2.2}(a, a_0, \varepsilon, \varphi) \\ &\qquad \cdot C\left(\int_{B(0,1+2\varepsilon)} f_{\varphi,\varepsilon}^+(x)^p\,dx\right)^{1/p}\|\psi\|_{\Lambda_a}. \end{aligned} \tag{2.20}$$

The constants $C_{2.2}$ and A in (2.20) are independent of c_0 in (2.15). But unfortunately they depend on a_0. So, in order to get (2.11) we have to remove their dependency on a_0.

Let

$$\varepsilon' = ((1 + 2\varepsilon)/(1 + \varepsilon) - 1)/2.$$

For $\theta \in \mathscr{D}(R^n)$ let

$$\tilde{f}(\theta) = f\big(\theta((1 + \varepsilon)^{-1})\big).$$

If $\theta \in \mathscr{D}(R^n)$ and if $\operatorname{supp}\theta \subset B(0, 1 + 2\varepsilon')$, then

$$|\tilde{f}(\theta)| \leq c_0 \big\| \theta\big((1 + \varepsilon)^{-1} \cdot\big) \big\|_{\Lambda a_0} = c_0 (1 + \varepsilon)^{-a_0} \|\theta\|_{\Lambda a_0},$$

by (2.15) and by $\operatorname{supp}\theta((1 + \varepsilon)^{-1} \cdot) \subset B(0, 1 + 2\varepsilon)$. Thus replacing f and ε in (2.20) by $\tilde{f}$ and ε' respectively gives that if $\psi \in \mathscr{D}(R^n)$ satisfies (2.2), then

$$\begin{aligned}
(2.21) \quad |\tilde{f}(\psi)| &\leq A\big(a, a_0, \varepsilon', \varphi\big)^{2/p} C_{2.2}\big(a, a_0, \varepsilon', \varphi\big) \\
&\quad \cdot C\left(\int_{B(0, 1 + 2\varepsilon')} \tilde{f}^{+}_{\varphi, \varepsilon'}(x)^p \, dx \right)^{1/p} \|\psi\|_{\Lambda_a} \\
&= \left\{ A(\cdots)^{2/p} C_{2.2}(\cdots) C\left(\int_{B(0, 1 + 2\varepsilon)} f^{+}_{\varphi, \varepsilon/2}(x)^p \, dx \right)^{1/p} \right\} \\
&\quad \cdot (1 + \varepsilon)^{n(1 - 1/p)} \|\psi\|_{\Lambda_a} \\
&= \{c_1\} (1 + \varepsilon)^{n(1 - 1/p)} \|\psi\|_{\Lambda_a}.
\end{aligned}$$

So if

$$(2.22) \qquad \theta \in \mathscr{D}(R^n) \text{ and } \operatorname{supp}\theta \subset B(0, 1 + \varepsilon),$$

then applying (2.21) to $\psi(x) = \theta((1 + \varepsilon)x)$ gives

$$|f(\theta)| = |\tilde{f}(\psi)| \leq c_1 (1 + \varepsilon)^{n(1 - 1/p)} \|\psi\|_{\Lambda_a} = c_1 \|\theta\|_{\Lambda_a}.$$

The constant c_1 depends on f but $c_1 < +\infty$ by (2.12). Thus if we replace the condition (2.16) by the condition (2.22) [i.e. if we replace ε in (2.16) by $\varepsilon/2$], then we can take a_0 in (2.15) to be a $(= n(1/p - 1))$ with c_0 replaced by c_1.

So, replacing a_0 and ε in (2.20) by a and $\varepsilon/2$ respectively gives

$$(2.23)\quad |f(\psi)| \le A(a, a, \varepsilon/2, \varphi)^{2/p} C_{2.2}(a, a, \varepsilon/2, \varphi) \cdot C\left(\int_{B(0,1+\varepsilon)} f^{+}_{\varphi,\varepsilon/2}(x)^p\,dx\right)^{1/p} \|\psi\|_{\Lambda_a}$$
$$\le A(\cdots)^{2/p} C_{2.2}(\cdots) C\left(\int_{B(0,1+2\varepsilon)} f^{+}_{\varphi,\varepsilon}(x)^p\,dx\right)^{1/p} \|\psi\|_{\Lambda_a}.$$

The constants A and $C_{2.2}$ in (2.23) depend only on a, ε and φ [i.e. on p, ε and φ]. Thus we get (2.11) and conclude the proof of the Theorem.

3. The proof of the Main Lemma. Take $\eta \in \mathscr{D}(R^n)$, depending only on ε and b, so that

$$(3.1)\qquad \operatorname{supp}\eta \subset B(0, \varepsilon/(4e)),$$

$$(3.2)\qquad \int_{R^n} \eta(x)\,dx = 1$$

and that

$$(3.3)\qquad \int_{R^n} \eta(x) x^\alpha\,dx = 0 \quad \text{if } 1 \le |\alpha| \le b,$$

where α is a multi-index defined in §1. Put

$$\eta_+(x) = \eta(x) + (\eta)_e(x) \quad \text{and} \quad \eta_-(x) = \eta(x) - (\eta)_e(x).$$

Note that

$$(3.4)\qquad \operatorname{supp}\eta_+,\ \operatorname{supp}\eta_- \subset B(0, \varepsilon/4)$$

and

$$(3.5)\qquad \int \eta_-(x) x^\alpha\,dx = 0 \quad \text{if } |\alpha| \le b$$

by (3.1) and (3.3). In this section the letter C denotes various positive constants that depend only on a, b, ε, η, φ and n.

LEMMA 3.1. *Let a, b, ε, φ and ψ be as in the Main Lemma. Let $\varepsilon' \in (0,1)$. Then there exist a measurable function $\ell(x,t)$ defined on R^{n+1}_+ and $\psi_1, \ldots, \psi_L \in \Lambda_b(R^n)$ such that*

$$(3.6)\qquad \ell(x,t) = 0 \quad \text{on } R^{n+1}_+ \setminus (B(0, 1+\varepsilon) \times (0,\varepsilon)),$$

$$(3.7)\qquad |\ell(x,t)| \le C_{3.1} \min\left(t^a \|\psi\|_{\Lambda_a}, t^b \|\psi\|_{\Lambda_b}\right),$$

(3.8) $\operatorname{supp}\psi_j \subset B(x_j, \varepsilon)$ *for some* $x_j \in B(0, 1+\varepsilon)$, $j = 1, \ldots, L$,

(3.9)
$$\sum_{j=1}^{L} \|\psi_j\|_{\Lambda_a} \le \varepsilon' \|\psi\|_{\Lambda_a},$$

(3.10)
$$\sum_{j=1}^{L} \|\psi_j\|_{\Lambda_b} \le \varepsilon' \|\psi\|_{\Lambda_b}$$

and that

(3.11)
$$\psi(x) = \int \int_{R_+^{n+1}} (\varphi)_s(y-x) k(y,s)\, dy \frac{ds}{s} + \sum_{j=1}^{L} \psi_j(x),$$

where L is a positive integer depending only on ε *and n and where* $C_{3.1}$ *is a constant depending only on a, b,* ε, ε', φ *and n.*

Proof. Take $\eta \in \mathscr{D}(R^n)$ so that (3.1)–(3.3) hold. Let

(3.12)
$$\mu \in (0, \varepsilon/4).$$

Put

$$k(x,t) = \begin{cases} \psi * (\eta_-)_{t/\mu} * (\eta_+)_{t/\mu}(x) & \text{if } 0 < t \le \mu/e, \\ \psi * (\eta)_{t/\mu} * (\eta)_{t/\mu}(x) & \text{if } \mu/e < t \le \mu, \\ 0 & \text{if } t > \mu. \end{cases}$$

Then (3.6) is clear. By (3.5) we get (3.7) with the constant $C_{3.1}$ depending only on a, b, η and μ. [The constant μ will be determined later depending on a, b, ε, ε', η and φ.]

By (3.2) and by

$$(\eta)_{t/\mu} * (\eta)_{t/\mu} - (\eta)_{et/\mu} * (\eta)_{et/\mu} = (\eta_-)_{t/\mu} * (\eta_+)_{t/\mu},$$

which we learned from Garnett and Latter [3], we get

$$\begin{aligned} \psi(x) &= \lim_{\tau \to +0} \int_\tau^{e\tau} \psi * (\eta)_s * (\eta)_s(x) \frac{ds}{s} \\ &= \lim_{\tau \to +0} \left\{ \int_\tau^{\mu} \psi * (\eta)_{t/\mu} * (\eta)_{t/\mu}(x) \frac{dt}{t} \right. \\ &\qquad \left. - \int_\tau^{\mu/e} \psi * (\eta)_{et/\mu} * (\eta)_{et/\mu}(x) \frac{dt}{t} \right\} \\ &= \int_0^{\mu} k(x,t) \frac{dt}{t} \\ &= \int_0^{\mu} k(\cdot, t) * (\check{\varphi})_t(x) \frac{dt}{t} \\ &\quad + \left\{ \int_0^{\mu} k(x,t) - k(\cdot,t) * (\check{\varphi})_t(x) \frac{dt}{t} \right\} \end{aligned}$$

(continues)

$$= \cdots + \left\{ \int_0^1 k(x, \mu s) - k(\cdot, \mu s) * (\check{\varphi})_{\mu s}(x) \frac{ds}{s} \right\}$$

$$= \cdots + \left\{ \int_0^{1/e} \psi * (\eta_-)_s * (\eta_+)_s * (\delta - (\check{\varphi})_{\mu s})(x) \frac{ds}{s} \right.$$

$$\left. + \int_{1/e}^1 \psi * (\eta)_s * (\eta)_s * (\delta - (\check{\varphi})_{\mu s})(x) \frac{ds}{s} \right\}$$

$$= \cdots + \{(3.13)\},$$

where $\check{\varphi}(x) = \varphi(-x)$ and where δ denotes the Dirac measure concentrated at the origin. So it is enough to show that (3.13) can be written in the form $\sum_{j=1}^L \psi_j(x)$ with the properties (3.8)–(3.10).

Let $\{x_j\}_{j=1}^L \subset B(0, 1 + \varepsilon)$ be such that

$$\bigcup_{j=1}^{L} B(x_j, \varepsilon/2) \supset B(0, 1 + \varepsilon) \tag{3.14}$$

and that

$$|x_i - x_j| \geq \varepsilon/2 \quad \text{if } i \neq j. \tag{3.15}$$

For $j = 1, \dots, L$ put

$$\Omega_1 = B(x_1, \varepsilon/2), \tag{3.16}$$

$$\Omega_j = B\left(x_j, \frac{\varepsilon}{2}\right) \setminus \bigcup_{k=1}^{j-1} B\left(x_k, \frac{\varepsilon}{2}\right), \qquad j = 2, \dots, L,$$

and

$$\psi_j(x) = \int \int_{\Omega_j \times (0, 1/e]} \{\psi * (\eta_-)_s(y)\} \{(\eta_+ * (\delta - (\check{\varphi})_\mu))_s(x - y)\} \, dy \frac{ds}{s}$$

$$+ \int \int_{\Omega_j \times (1/e, 1]} \psi(y) \{(\eta * \eta * (\delta - (\check{\varphi})_\mu))_s(x - y)\} \, dy \frac{ds}{s}$$

$$= (3.17)_j + (3.18)_j.$$

Then

$$(3.13) = \sum_{j=1}^{L} \psi_j(x).$$

Since

$$\operatorname{supp} \eta_+ * (\delta - (\check{\varphi})_\mu) \subset B(0, \varepsilon/2) \quad \text{by (3.4) and (3.12),}$$

$$\| \eta_+ * (\delta - (\check{\varphi})_\mu) \|_{\Lambda_{b+1}} \le C\mu$$

and since

$$| \psi * (\eta_-)_t(x) | \le C \min\left(t^a \| \psi \|_{\Lambda_a}, t^b \| \psi \|_{\Lambda_b} \right)$$

by (3.4) and (3.5), we get

$$\operatorname{supp}(3.17)_j \subset B(x_j, \varepsilon), \tag{3.19}$$

$$\|(3.17)_j\|_{\Lambda_a} \le C\mu \| \psi \|_{\Lambda_a} \tag{3.20}$$

and

$$\|(3.17)_j\|_{\Lambda_b} \le C\mu \| \psi \|_{\Lambda_b} \tag{3.21}$$

by Lemma 2.1. Since

$$\operatorname{supp} \eta * \eta * (\delta - (\check{\varphi})_\mu) \subset B(0, \varepsilon/2)$$

by (3.1) and (3.12) and since

$$\| \eta * \eta * (\delta - (\check{\varphi})_\mu) \|_{\Lambda_b} \le C\mu,$$

we get

$$\operatorname{supp}(3.18)_j \subset B(x_j, \varepsilon) \tag{3.22}$$

and

$$\|(3.18)_j\|_{\Lambda_a} + \|(3.18)_j\|_{\Lambda_b} \le C\mu \| \psi \|_{L^\infty} \le C\mu \min\left(\| \psi \|_{\Lambda_a}, \| \psi \|_{\Lambda_b} \right). \tag{3.23}$$

The condition (3.8) follows from (3.19) and (3.22). The conditions (3.9) and (3.10) follow from (3.20), (3.21) and (3.23) by taking μ small enough. □

LEMMA 3.2. *Let a, b, ε, φ and ψ be as in the Main Lemma. Then there exists a measurable function $k(x, t)$ defined on R_+^{n+1} such that*

$$k(x, t) = 0 \quad \text{on } R_+^{n+1} \setminus (B(0, 1 + \varepsilon) \times (0, \varepsilon)), \tag{3.24}$$

$$|k(x, t)| \le C_{3.2} \min\left(t^a \| \psi \|_{\Lambda_a}, t^b \| \psi \|_{\Lambda_b} \right) \tag{3.25}$$

and (2.6) hold, where $C_{3.2}$ is a constant depending only on a, b, ε, φ and n.

Proof. We may assume

$$\varepsilon \in (0, 1).$$

Let $\varepsilon' \in (0,1)$. Applying Lemma 3.1 gives k, $\psi_1, \dots, \psi_L$ with the properties (3.6)–(3.11). Next, we apply Lemma 3.1 with dilation to each ψ_j and get k_j, $\psi_{j,1}, \dots, \psi_{j,L}$ such that

$$(3.6)' \qquad k_j(x,t) = 0 \quad \text{on } R_+^{n+1} \backslash \big(B\big(x_j, (1+\varepsilon)\varepsilon\big) \times (0, \varepsilon^2)\big)$$

(for x_j see (3.8)),

$$(3.7)' \qquad |k_j(x,t)| \le C_{3.1} \min\big(t^a \|\psi_j\|_{\Lambda_a}, t^b \|\psi_j\|_{\Lambda_b}\big),$$

$$(3.8)' \quad \operatorname{supp} \psi_{j,l} \subset B\big(x_{j,l}, \varepsilon^2\big) \quad \text{for some } x_{j,l} \in B\big(x_j, (1+\varepsilon)\varepsilon\big),$$

$$(3.9)' \qquad \sum_{l=1}^{L} \|\psi_{j,l}\|_{\Lambda_a} \le \varepsilon' \|\psi_j\|_{\Lambda_a},$$

$$(3.10)' \qquad \sum_{l=1}^{L} \|\psi_{j,l}\|_{\Lambda_b} \le \varepsilon' \|\psi_j\|_{\Lambda_b}$$

and

$$(3.11)' \qquad \psi_j(x) = \int \int_{R_+^{n+1}} (\varphi)_s(y-x) k_j(y,s)\, dy \frac{ds}{s} + \sum_{l=1}^{L} \psi_{j,l}(x).$$

Combining (3.6)–(3.11) and (3.6)′–(3.11)′ gives

$$(3.26) \quad \psi(x) = \int \int (\varphi)_s(y-x) \left\{ k(y,s) + \sum_{j=1}^{L} k_j(y,s) \right\} dy \frac{ds}{s} + \sum_{j=1}^{L} \sum_{l=1}^{L} \psi_{j,l}(x),$$

$$(3.27) \quad \sum_{j=1}^{L} k_j(x,t) = 0 \quad \text{on } R_+^{n+1} \backslash \big(B(0, (1+\varepsilon)(1+\varepsilon)) \times (0, \varepsilon^2)\big),$$

$$(3.28) \qquad \Big|\sum k_j(x,t)\Big| \le C_{3.1} \varepsilon' \min\big(t^a \|\psi\|_{\Lambda_a}, t^b \|\psi\|_{\Lambda_b}\big),$$

$$(3.29) \quad \operatorname{supp} \psi_{j,l} \subset B\big(x_{j,l}, \varepsilon^2\big) \quad \text{with } x_{j,l} \in B(0, (1+\varepsilon)(1+\varepsilon)),$$

$$(3.30) \qquad \sum_{j,l} \|\psi_{j,l}\|_{\Lambda_a} \le \varepsilon'^2 \|\psi\|_{\Lambda_a}$$

and

$$(3.31) \qquad \sum_{j,l} \|\psi_{j,l}\|_{\Lambda_b} \le \varepsilon'^2 \|\psi\|_{\Lambda_b}.$$

Repeating this process m times we get

$$\left\{ \ell_{j_1,\dots,j_l} \right\}_{l \in \{1,\dots,m-1\};\, j_1,\dots,j_l \in \{1,\dots,L\}}$$

and

$$\left\{ \psi_{j_1,\dots,j_m} \right\}_{j_1,\dots,j_m \in \{1,\dots,L\}}$$

such that

$$(3.26)' \quad \psi(x) = \int\int (\varphi)_s(y-x) \times \left\{ \ell(y,s) + \sum_{l=1}^{m-1} \sum_{j_1,\dots,j_l \in \{1,\dots,L\}} \ell_{j_1,\dots,j_l}(y,s) \right\} dy \frac{ds}{s} + \sum_{j_1,\dots,j_m \in \{1,\dots,L\}} \psi_{j_1,\dots,j_m}(x),$$

$$(3.27)' \quad \sum_{j_1,\dots,j_l} \ell_{j_1,\dots,j_l}(x,t) = 0 \quad \text{on } R_+^{n+1} \setminus \left(B\left(0, (1+\varepsilon)(1+\varepsilon+\varepsilon^2+\cdots+\varepsilon^{l-1})\right) \times (0,\varepsilon^l) \right),$$

$$(3.28)' \quad \left| \sum_{j_1,\dots,j_l} \ell_{j_1,\dots,j_l}(x,t) \right| \le C_{3.1} \varepsilon'^l \min\left(t^a \|\psi\|_{\Lambda_a}, t^b \|\psi\|_{\Lambda_b} \right),$$

$$(3.29)' \quad \operatorname{supp} \psi_{j_1,\dots,j_m} \subset B\left(x_{j_1,\dots,j_m}, \varepsilon^m\right) \quad \text{with } x_{j_1,\dots,j_m} \in B\left(0, (1+\varepsilon)(1+\varepsilon+\varepsilon^2+\cdots+\varepsilon^{m-1})\right),$$

$$(3.30)' \quad \sum_{j_1,\dots,j_m} \left\| \psi_{j_1,\dots,j_m} \right\|_{\Lambda_a} \le \varepsilon'^m \|\psi\|_{\Lambda_a}$$

and

$$(3.31)' \quad \sum_{j_1,\dots,j_m} \left\| \psi_{j_1,\dots,j_m} \right\|_{\Lambda_b} \le \varepsilon'^m \|\psi\|_{\Lambda_b}.$$

Put

$$k(x,t) = \ell(x,t) + \sum_{l=1}^{\infty} \sum_{j_1,\dots,j_l} \ell_{j_1,\dots,j_l}(x,t).$$

By (3.28)$'$ this converges everywhere and satisfies (3.25). Since the second term on the right-hand side of (3.26)$'$ goes to zero as $m \to \infty$ by (3.29)$'$, (3.30)$'$ and by ε, $\varepsilon' \in (0,1)$, we get (2.6) by letting $m \to \infty$ in (3.26)$'$.

Since

$$k(x,t)=0 \quad \text{on } R_+^{n+1}\backslash(B(0,(1+\varepsilon)/(1-\varepsilon))\times(0,\varepsilon))$$

by (3.27)′, we get (3.24) by taking $\varepsilon/3$ instead of ε. □

LEMMA 3.3. *Let a, b, ε, φ and ψ be as in the Main Lemma. Let $\varepsilon' \in (0,1)$. Let $\mathscr{E} \subset R_+^{n+1}$ be a measurable set such that*

$$\int\int_{\mathscr{E}\cap Q(x,t)} dy\,ds \le C_{3.3}t^{n+1} \tag{3.32}$$

for any $(x,t)\in R_+^{n+1}$. Then there exist a measurable function $\mathscr{k}(x,t)$ defined on R_+^{n+1} and $\psi_1,\dots,\psi_L \subset \Lambda_b(R^n)$ such that

$$\mathscr{k}(x,t)=0 \quad \text{on } \mathscr{E}, \tag{3.33}$$

(3.6),

$$|\mathscr{k}(x,t)| \le C_{3.4}\min\left(t^a\|\psi\|_{\Lambda_a},\, t^b\|\psi\|_{\Lambda_b}\right) \tag{3.7$''$}$$

and (3.8)–(3.11) *hold, where L is a positive integer depending only on ε and n and where $C_{3.3}$ and $C_{3.4}$ are positive constants depending only on a, b, ε, ε', φ and n.*

Proof. Applying Lemma 3.2 gives $k(x,t)$ that satisfies (3.24), (3.25) and (2.6). Take $\{\Omega_j\}_{j=1}^L$ as in (3.14)–(3.16). Put

$$\mathscr{k}(x,t)=k(x,t)\chi_{\mathscr{E}^c}(x,t)$$

and

$$\psi_j(x)=\int\int_{\mathscr{E}\cap(\Omega_j\times(0,\varepsilon))}(\varphi)_s(y-x)k(y,s)\,dy\,\frac{ds}{s}.$$

Then (3.33), (3.6), (3.7)″ and (3.11) are clear. Since

$$\int\int_{Q(x,t)\cap\mathscr{E}}|k(y,s)|\,dy\,\frac{ds}{s}\le CC_{3.3}\min\left(t^{n+a}\|\psi\|_{\Lambda_a},\, t^{n+b}\|\psi\|_{\Lambda_b}\right)$$

for any $(x,t)\in R_+^{n+1}$ by (3.25) and (3.32), we get

$$\|\psi_j\|_{\Lambda_a}\le CC_{3.3}\|\psi\|_{\Lambda_a}$$

and

$$\|\psi_j\|_{\Lambda_b}\le CC_{3.3}\|\psi\|_{\Lambda_b}$$

by Lemma 2.1. Thus by taking $C_{3.3}$ small enough depending on a, b, ε, ε' and n, we get (3.9) and (3.10). Since

$$\operatorname{supp} \psi_j \subset B(x_j, 2\varepsilon)$$

(for x_j see (3.16)), we get (3.8) by taking $\varepsilon/2$ instead of ε. □

Note that Lemmas 3.3 and 3.1 differ only by the condition (3.33). Similarly, the Main Lemma and Lemma 3.2 differ only by the condition (2.3). So the Main Lemma follows from Lemma 3.3 in exactly the same way as Lemma 3.2 followed from Lemma 3.1, using the fact that the set

$$\mathscr{E}_r = \{(rx, rt) \colon (x, t) \in \mathscr{E}\}$$

satisfies (3.32) for any $r > 0$ whenever $\mathscr{E}$ satisfies it. We do not repeat this argument.

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