

INTERTWINING OPERATORS BETWEEN HOLOMORPHICALLY INDUCED MODULES

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Let (G, K) be an irreducible hermitian symmetric pair of non-compact type. The authors study the spaces of covariant differential intertwining operators between the continuations of holomorphic discrete series for this pair. In particular, when $G = \mathrm{SU}(p, q)$, $\mathrm{Sp}(n, \mathbf{R})$, or $\mathrm{SO}^*(2n)$, an algorithm is provided to reduce the problem for regular integral infinitesimal character to that for semiregular representations. In case $G = \mathrm{SU}(p, q)$, this latter problem is solved using an equivalence of categories due to Enright-Shelton (providing a reduction to the regular case for a lower rank group).

In addition, an explicit description of the non-zero spaces of intertwining operators is given for the two exceptional cases, E_6 and E_7 . The remaining case, $G = \mathrm{SO}(2, n)$, has been treated previously. All results are obtained in the (dual) setting of homomorphisms between generalized Verma modules.

1. Introduction. Let (G, K) be an irreducible hermitian symmetric pair of non-compact type; that is, G is a real Lie group and K a maximal compact subgroup, such that G/K is a hermitian symmetric space. Associated to this pair is a family of holomorphically induced representations, the *continuation of holomorphic discrete series*. This family has been investigated from many points of view, by numerous people. In the present note, we study the spaces of covariant differential intertwining operators between these representations. In particular, it is known from a previous work [3] that, in the integral case, these spaces are at most one-dimensional; thus, the question is reduced to one of existence. Our main result is an algorithm for reducing the problem for regular integral infinitesimal character to that for semiregular modules, when $G = \mathrm{SU}(p, q)$, $\mathrm{Sp}(n, \mathbf{R})$, or $\mathrm{SO}^*(2n)$. In case $G = \mathrm{SU}(p, q)$, the latter problem is solved using an equivalence of categories due to Enright-Shelton. We also provide an explicit description of the non-zero spaces of intertwining operators in the exceptional cases, E_6 and E_7 (in the regular integral setting). A similar result for the remaining hermitian symmetric case, $G = \mathrm{SO}(2, n)$, was obtained previously in [2].

Associated to the pair (G, K) there is a maximal parabolic subalgebra \mathfrak{p} of \mathfrak{g} (unexplained notation follows [3]), $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{s}$, where \mathfrak{s} is the sum

of the positive noncompact root spaces of \mathfrak{g} . For $\lambda \in \mathfrak{h}^*$ regular integral on \mathfrak{k} , we can define the *generalized Verma module* $V(\lambda)$ of highest weight $\lambda - \rho$ induced from \mathfrak{p} , and the associated *Verma module* $M(\lambda)$ and irreducible quotient $L(\lambda)$. Then it is well known (cf. [10]) that the space of covariant differential operators intertwining a pair of continuations of holomorphic discrete series for G , is isomorphic to $\text{Hom}_{\mathfrak{g}}(V(\lambda), V(\lambda'))$, for appropriate λ, λ' . Our answer to the classification problem will be given in this generalized Verma module setting. More precisely, we work in the category $\mathcal{O}(\mathfrak{g}, \mathfrak{p}, \chi)$ of all finitely generated $U(\mathfrak{g})$ -modules which are \mathfrak{p} -finite, \mathfrak{k} -semisimple, and have generalized infinitesimal character χ .

The problem of classifying homomorphisms between generalized Verma modules in the hermitian symmetric setting was first studied systematically by the first author. In his Ph. D. thesis [1], he obtained all maps between so-called “scalar” modules (those induced from one-dimensional \mathfrak{p} -modules). This result was extended by Jakobsen [10] to include all maps whose domain *or* range is a scalar module. The current result completes the program in the regular integral case.

Crucial to all that follows is the structure of the *Hasse diagram* [7] W^K attached to (G, K) : this subset of W parametrizes the generalized Verma modules having a given regular integral infinitesimal character. Following [3], having fixed such an infinitesimal character, we associate to each $w \in W^K$ a generalized Verma module V_w . Recall the *arrow relation* on W^K : $y \rightarrow w$ means $l(w) = l(y) + 1$ and $w = ys_\alpha$ for some positive root α . It turns out that, in the hermitian symmetric case, we may in fact always choose α *simple*; moreover, concrete models for the posets W^K are known [7].

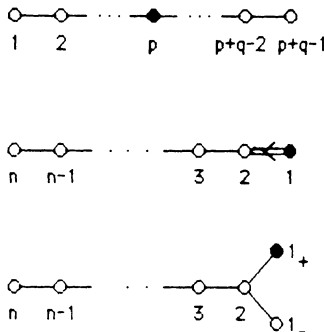


FIGURE 1

We now describe these models in the cases of interest. Number the simple roots as indicated in Figure 1. For $\mathfrak{g} = \mathfrak{su}(p, q)$,

$$W^K \cong \mathcal{A}^{p,q} = \{(a_1, a_2, \dots, a_q) : p \geq a_1 \geq a_2 \geq \dots \geq a_q \geq 0, a_i \in \mathbf{Z}\}.$$

The arrow relations in $\mathcal{A}^{p,q}$ have the form $(a_1, \dots, a_i - 1, \dots, a_q) \rightarrow (a_1, \dots, a_i, \dots, a_q)$; the indicated arrow corresponds to reflection in the simple root α_{a_i+q-i} .

For $\mathfrak{g} = \mathfrak{sp}(n, \mathbf{R})$ or $\mathfrak{so}^*(2(n+1))$,

$$W^K \cong \mathcal{C}^n = \{(a_1, a_2, \dots, a_n) : n \geq a_1 \geq a_2 \geq \dots \geq a_n \geq 0, \\ a_i > a_{i+1} \text{ if } a_i > 0, a_i \in \mathbf{Z}\}.$$

The arrow relations in \mathcal{C}^n also have the form $(a_1, \dots, a_i - 1, \dots, a_n) \rightarrow (a_1, \dots, a_i, \dots, a_n)$. If $a_i > 1$ or $\mathfrak{g} = \mathfrak{sp}(n, \mathbf{R})$, the indicated arrow corresponds to reflection in the simple root α_{a_i} . If $a_i = 1$ and $\mathfrak{g} = \mathfrak{so}^*(2(n+1))$, the reflection is in α_{1+} if $n - i$ is even, and in α_{1-} if $n - i$ is odd.

(1.1) DEFINITION. For $W^K \cong \mathcal{A}^{p,q}$ or \mathcal{C}^n , whenever $w' \equiv (\dots, a_i - 1, \dots) \rightarrow (\dots, a_i, \dots) \equiv w$ is effected by the simple root α (with $w, w' \in W^K$), we shall say that α acts on w' and w at coordinate i .

For \mathfrak{g} of type E_6 or E_7 , the partially ordered sets W^K are given in [4], and may also be distilled from Figures 2 and 3.

2. The classical cases. We first treat the ‘‘standard maps’’ [2]. Given $w \in W^K$, $w = (a_1, \dots, a_n)$, set $\mathcal{J} = \{1 \leq i \leq n : (a_1, \dots, a_i - 1, \dots, a_n) \in W^K\}$. Now if $x \in W^K$, $x = (b_1, \dots, b_n)$, we say x is *standard with respect to w* if, for $1 \leq i \leq n$, either $b_i = a_i$ or $b_i = a_i - 1$, with $b_i = a_i - 1$ only if $i \in \mathcal{J}$. By [8, Lemma 4.2] we have:

(2.1) LEMMA. *If x is standard with respect to w , then there is a non-zero standard homomorphism $V_x \rightarrow V_w$, which is unique up to scalar multiple.*

We now state some necessary conditions for the existence of non-zero maps in general.

(2.2) LEMMA [6]. *Suppose $w > ws_\alpha \equiv w'$ with $w, w' \in W^K$, α simple. Then there is a short exact sequence*

$$0 \rightarrow V_w \rightarrow \varphi_\alpha \psi_\alpha V_w \rightarrow V_{w'} \rightarrow 0$$

(where $\varphi_\alpha, \psi_\alpha$ are translation functors).

(2.3) COROLLARY (of (2.2), [3, (2.11)]). *For $x, w \in W^K$, $\text{Hom}_{\mathfrak{g}}(V_x, V_w) = 0$ if there is a simple root α such that either*

- (a) $w > ws_{\alpha} \in W^K$ but α does not act on x ,
 or (b) $x < xs_{\alpha} \in W^K$ but α does not act on w .

We can now state our main result.

(2.4) THEOREM. *Fix G to be one of $\text{SU}(p, q)$, $\text{Sp}(n, \mathbf{R})$, or $\text{SO}^*(2n)$, and fix a regular integral infinitesimal character. Suppose that $x, w \in W^K$, $x \leq w$, and assume that there is no simple root α satisfying condition (a) or (b) of Corollary 2.3. Then either (x, w) is a standard pair (so $\text{Hom}_{\mathfrak{g}}(V_x, V_w) \neq 0$ by Lemma 2.1), or there is a simple root α (which can be determined explicitly) such that $\text{Hom}_{\mathfrak{g}}(V_x, V_w) = \text{Hom}_{\mathfrak{g}}(\psi_{\alpha}V_x, \psi_{\alpha}V_w)$. In either case, non-zero homomorphisms are unique up to scalar multiple.*

We first prove some technical lemmas concerning the combinatorial aspects of W^K . While the statements of the lemmas are true for all G , we shall prove them and the theorem only for $G = \text{SU}(p, q)$. The arguments for the other cases are similar, requiring minor modifications only as dictated by the different structure of W^K . For the remainder of this section, we shall assume that $\mathfrak{g} = \mathfrak{su}(p, q)$.

(2.5) LEMMA. *Fix $w \in W^K$, and suppose that $\alpha(1)$, $\alpha(2)$ are simple roots acting on coordinates $i(1)$, $i(2)$ in w . Assume that $i(1) \leq i(2)$, and if $i(1) = i(2)$, then $ws_{\alpha(2)} < w < ws_{\alpha(1)}$. Then $\alpha(1) = \alpha_{j_1}$, $\alpha(2) = \alpha_{j_2}$, with $j_1 > j_2$ (in the ordering of Fig. 1).*

Proof. Let $w = (a_1, \dots, a_q)$. Then $j_1 = a_{i(1)} + q - i(1) + \delta_1$, $j_2 = a_{i(2)} + q - i(2) + \delta_2$, $\delta_1, \delta_2 \in \{0, 1\}$. Hence

$$j_1 - j_2 = (a_{i(1)} - a_{i(2)}) + (i(2) - i(1)) + (\delta_1 - \delta_2).$$

Now $i(1) \leq i(2)$ implies $a_{i(1)} \geq a_{i(2)}$. Moreover, if $a_{i(1)} = a_{i(2)}$ then $\delta_1 = 1$ and $\delta_2 = 0$. Thus $j_1 - j_2 > 0$. \square

(2.6) LEMMA. *Let $x, w \in W^K$, $x \leq w$. Let α be a simple root such that $xs_{\alpha}, ws_{\alpha} \in W^K$. Suppose α acts on coordinate i in x and coordinate j in w . Then $i \leq j$.*

Proof. Set $x = (b_1, \dots, b_q)$, $w = (a_1, \dots, a_q)$, with $b_k \leq a_k$, $1 \leq k \leq q$. We have

$$b_i + q - i + \delta_i = a_j + q - j + \delta_j \quad (\delta_i, \delta_j \in \{0, 1\}).$$

Suppose $i > j$. Then $b_i \leq b_j \leq a_j$. On the other hand, $b_i = a_j + (i - j) + (\delta_j - \delta_i)$, forcing $i = j + 1$, $\delta_j = 0$, $\delta_i = 1$, $b_{j+1} = a_j$. But $\delta_j = 0$ implies $a_{j+1} < a_j = b_{j+1}$, a contradiction. \square

(2.7) LEMMA. Let $x, w \in W^K$, $x \leq w$. Assume that, for each simple root α such that $x < xs_\alpha \in W^K$, α acts on the same coordinate in w as it does in x . Then x is standard with respect to w .

Proof. Let

$$(2.8) \quad x = (b_{i(1)}, \dots, b_{i(1)}, b_{i(2)}, \dots, b_{i(2)}, \dots, b_{i(m)}, \dots, b_{i(m)})$$

where $b_{i(1)} > b_{i(2)} > \dots > b_{i(m)}$, and $i(k)$ is the index of the first occurrence of $b_{i(k)}$. If $w = (a_1, \dots, a_q)$, then our hypothesis implies that $a_{i(k)} = b_{i(k)} + \delta_k$, $\delta_k \in \{0, 1\}$, $1 \leq k \leq m$; moreover, if $\delta_k = 1$ then $a_{i(k)+1} < a_{i(k)}$. But this, together with the assumption $x \leq w$, implies that

$$w = (b_{i(1)} + \delta_1, b_{i(1)}, \dots, b_{i(1)}, b_{i(2)} + \delta_2, b_{i(2)}, \dots, b_{i(2)}, \dots, b_{i(m)} + \delta_m, b_{i(m)}, \dots, b_{i(m)}).$$

Hence x is standard with respect to w . \square

Proof of Theorem 2.4. We continue to assume $\mathfrak{g} = \mathfrak{su}(p, q)$. Let x have the form (2.8). Denote by $\alpha(k)$ the simple root which acts on x by sending $b_{i(k)}$ to $b_{i(k)} + 1$. By hypothesis, each $\alpha(k)$ acts on w . By Lemma 2.7, we may assume there is some k for which $\alpha(k)$ does *not* act on coordinate $i(k)$ in w ; choose k minimal with this property. Set $\alpha = \alpha(k)$. By Lemma 2.6, α must act on a coordinate a_s in w with $s > i(k)$. Thus

$$(2.9) \quad w = (b_{i(1)} + \delta_1, b_{i(1)}, \dots, b_{i(1)}, \dots, b_{i(k-1)} + \delta_{k-1}, b_{i(k-1)}, \dots, b_{i(k-1)}, a_{i(k)}, a_{i(k)+1}, \dots, a_{s-1}, a_s, \dots)$$

where $a_s = b_{i(k)} + s - i(k) + \delta_k$, $\delta_j \in \{0, 1\}$, $1 \leq j \leq k$.

We claim that $a_{s-1} = a_s$. To see this, denote by $\beta(j)$ the simple root which acts on x by sending $b_{i(j)}$ to $b_{i(j)} - 1$ in coordinate $i(j) - 1$. For $1 \leq j < k$, it is clear from Lemma 2.6 and (2.9) that if $W^K \ni ws_{\beta(j)} \rightarrow w$, $\beta(j)$ must send $b_{i(j)}$ to $b_{i(j)} - 1$ in coordinate $i(j) - 1$. Thus if $a_{s-1} > a_s$, then the simple reflection β which sends a_{s-1} to $a_{s-1} - 1$ in w cannot be $\beta(j)$ for $1 \leq j < k$; it also is not $\alpha(j)$ for $1 \leq j < k$. But since β acts to the left of $\alpha(k)$ in w , Lemma 2.5 implies that β cannot act on x . This contradicts our hypothesis concerning Corollary 2.3(a).

Thus $a_{s-1} = a_s$, implying that $\delta_k = 1$. Hence $w' \equiv ws_{\alpha(k)} \rightarrow w$. But now $w's_\beta \rightarrow w'$ (where β is defined in the preceding paragraph), so again using Corollary 2.3(a), we have $\text{Hom}_{\mathfrak{g}}(V_x, V_{w'}) = 0$. Applying $\text{Hom}_{\mathfrak{g}}(V_x, -)$

to the exact sequence in Lemma 2.2 gives

$$\mathrm{Hom}_{\mathfrak{g}}(V_x, V_w) \cong \mathrm{Hom}_{\mathfrak{g}}(V_x, \varphi_\alpha \psi_\alpha V_w) \cong \mathrm{Hom}_{\mathfrak{g}}(\psi_\alpha V_x, \psi_\alpha V_w). \quad \square$$

(2.10) COROLLARY. *In case $G = \mathrm{SU}(p, q)$, the spaces of homomorphisms between generalized Verma modules having regular integral infinitesimal character are at most one-dimensional, and can be computed inductively. That is, if the result is known for all $\mathrm{SU}(p', q')$, $p' \leq p$, $q' \leq q$, with at least one inequality strict, then the result for $\mathrm{SU}(p, q)$ follows.¹*

Proof. This is a direct consequence of the theorem, and the following result.

(2.11) THEOREM (Enright-Shelton [5]). *There is a natural equivalence of categories between $\mathcal{O}(\mathfrak{su}(p-1, q-1), \mathfrak{p}', \mathrm{triv})$ and $\mathcal{O}(\mathfrak{su}(p, q), \mathfrak{p}, \lambda)$, where λ is the semi-regular weight obtained by pushing ρ to any one simple root wall.*

3. The exceptional cases. Let (G, K) correspond to an exceptional E_6 or E_7 case. We sketch the argument for computing all homomorphisms between the generalized Verma modules in this case.

Step 1. Given $w \in W^K$, compute the composition factors of V_w . In fact, we can compute a \mathfrak{g} -filtration of V_w with semisimple subquotients; for details, see [4] or [9]. In particular, as noted by Irving [9], V_w has an irreducible socle $L_{w^\#}$; recall, socle $V_w =$ largest semisimple submodule of V_w .

Step 2. Fix V_w and assume we understand $H^0(\mathfrak{s}, V_{w'})$ for all $w' < w$; i.e., we start from the irreducible V_e and “work up” the diagram. From step 1, we have

$$V_w = L_{y_1} + \cdots + L_{y_n} + L_{z_1} + \cdots + L_{z_m}$$

in the Grothendieck group. Furthermore, this indexing is set up so that L_{y_1}, \dots, L_{y_n} correspond to non-zero standard maps $V_{y_i} \rightarrow V_w$; these are settled as in Lemma 2.1. It remains to determine if L_{z_i} corresponds to a non-zero map $V_{z_i} \rightarrow V_w$. To do this we proceed case-by-case, in the z_i 's. First of all (as noted in step 1), socle $V_w = L_{w^\#}$. Obviously there is a non-zero map $V_{w^\#} \rightarrow V_w$. A key observation is:

(3.1) If $z_i \not\geq w^\#$, then there does *not* exist a non-zero map $V_{z_i} \rightarrow V_w$.

Using (3.1), we can typically resolve that several of the factors L_{z_i} do *not*

¹*Added in proof:* The first author and T. Enright have recently obtained a closed form description of the homomorphisms for $\mathrm{SU}(p, q)$.

correspond to maps. This leaves a few L_{z_i} 's to work with. These are handled using $\text{Hom}(V_{z_i}, -)$ applied to appropriate sequences, as in Lemma 2.2. Although we do not have results as in [5] at our disposal, the singular modules $\psi_\alpha V_{z_i}$ and $\psi_\alpha V_w$ are sufficiently simple to detect their \mathfrak{s} -invariants; typically, they have just two composition factors. We leave it to the (ambitious) reader to fill in the details from here.

We have now proved

(3.2) THEOREM. *The homomorphisms between generalized Verma modules for E_6 and E_7 are known and are tabulated in Figures 2 and 3, and Table 1.*

(3.3) *Explanation of figures and table.* In Figures 2 and 3, we have reproduced W^K for E_6 and E_7 . The elements of W^K are numbered sequentially. In Figure 2, all non-zero homomorphisms $V_x \rightarrow V_w$ are indicated. For E_7 , only the non-zero standard maps are indicated in Figure 3; the non-standard homomorphisms are listed in Table 1. The

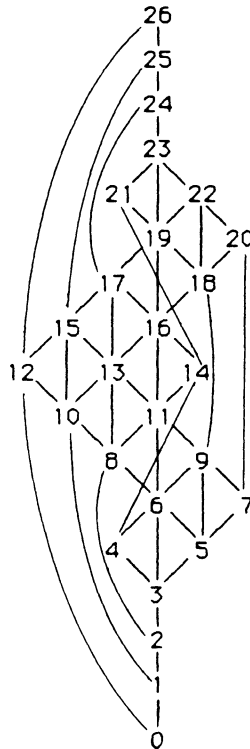


FIGURE 2

Maps between generalized Verma modules for E_6

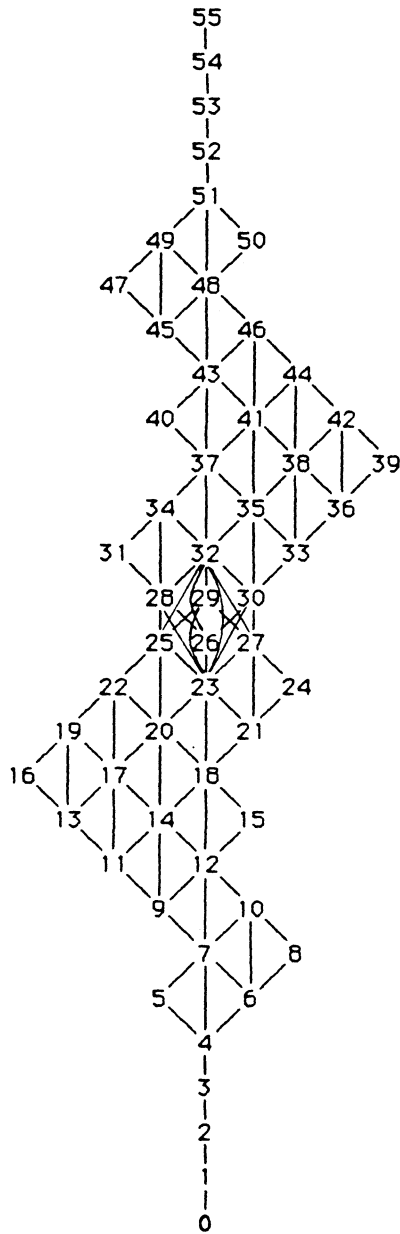


FIGURE 3

Standard maps between generalized Verma modules for E_7

<u>x</u>	<u>w</u>	<u>x</u>	<u>w</u>
0	16, 55	16	42
1	13, 54	17	36, 38
2	11, 53	18	37
3	9, 52	19	38
4	51	20	33, 35
5	15, 50	21	31, 34
6	49	22	35
7	48	24	34
8	24, 47	29	40
9	46	31	47
10	21, 45	34	45
11	44	39	55
12	43	40	50
13	39, 42	42	54
14	41	44	53
15	26, 40	46	52

TABLE 1
Non-standard maps for E_7 .

parametrization of Figure 3 is used. The first column gives x , and the second column all w , such that $\text{Hom}_{\mathfrak{g}}(V_x, V_w) \neq 0$.

4. Two classical examples. In this section, we present the results of applying the algorithm (2.4), in case $G = \text{SU}(3, 3)$ or $\text{SO}^*(10)$. The partially ordered sets W^K , together with all maps, are given by Figures 4 and 5. Note that W^K contains subdiagrams “at the bottom” isomorphic in an obvious fashion to W^K for lower rank groups: either $\text{SU}(p, q)$ with $p, q \leq 3$, or $\text{SO}^*(2n)$ with $n < 5$. Moreover, there are equivalences analogous to [3, (2.6)], which means that the maps for these lower rank groups can be read off from the corresponding subdiagrams. The same ideas apply to $\text{Sp}(4)$, leading to a diagram isomorphic to Figure 5.

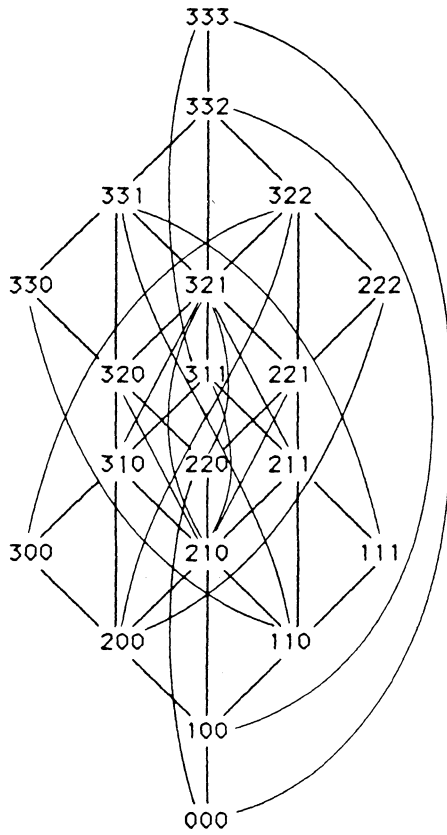


FIGURE 4

Maps between generalized Verma modules for $SU(3, 3)$

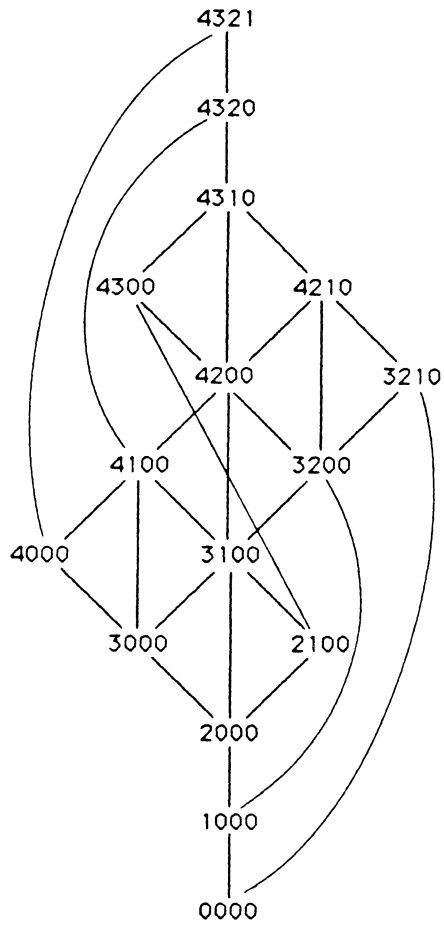


FIGURE 5

Maps between generalized Verma modules for $SO^*(10)$

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