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HOLOMORPHIC CONTINUATION IN SEVERAL COMPLEX VARIABLES

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This paper is mainly devoted to the question about the holomorphic extendability on a domain $D \subset\subset \mathbf{C}^n$ of the CR-functions defined on a relatively open connected subset $\partial D \setminus K$ of ∂D. Pursuing the investigation of our earlier paper proving that the $\mathcal{O}(\overline{D})$-convexity of K suffices, when $n \geq 2$, for the desired extendability, here we obtain some further results on this and similar matters, and a Hartogs' type theorem for certain domains in a Levi-flat hypersurface. All the results of this paper concern the case $n \geq 3$ and fail to be true in general for $n = 2$.

Introduction. Throughout this paper D and K will denote respectively a bounded open domain in $\mathbf{C}^n$, $n \geq 2$, and a compact subset of ∂D, such that $\partial D \setminus K$ is a connected real hypersurface of class C^1 in $\mathbf{C}^n \setminus K$. Moreover, $\mathrm{CR}(\partial D \setminus K)$ will denote the continuous CR-functions on $\partial D \setminus K$, i.e. the complex-valued continuous functions on $\partial D \setminus K$ which are solutions of the weak tangential Cauchy-Riemann equation.

We shall be concerned in the main with the following problem: under what conditions on D and K does every $f \in \mathrm{CR}(\partial D \setminus K)$ extend to a function $F \in \mathcal{O}(D) \cap C^0(\overline{D} \setminus K)$?

This problem is not a completely new one. A parallel problem in the setting of holomorphic functions was considered, for D pseudoconvex, by Stout [7], and some results relating to the problem itself have already been obtained by Tomassini and myself ([5], [8] and [6]).

In particular, a condition on D and K which turns out to be sufficient for the desired extendability property is that K is $\mathcal{O}(\overline{D})$-convex, i.e. that $\hat{K}_{\overline{D}} = K$, where $\hat{K}_{\overline{D}}$ denotes the $\mathcal{O}(\overline{D})$-hull of K ([6], Theorem 1. Cf. also [7], Theorem I.1(A)).

A noteworthy case in which the above condition holds is when there is a plurisubharmonic function ρ on some pseudoconvex open neighborhood of $\overline{D}$, so that $K \subset \{\rho = 0\}$ and $\overline{D} \setminus K \subset \{\rho > 0\}$; or, slightly more generally, when there is a family $\{U_i\}$ of open neighborhoods of K which are Runge in some pseudoconvex open neighborhood of $\overline{D}$, so that $\{U_i \cap \overline{D}\}$ is a neighborhood basis for K in $\overline{D}$.

On the other hand, the $\mathcal{O}(\overline{D})$-convexity of K need not hold when ρ is plurisubharmonic only on a pseudoconvex neighborhood of K, or, even more, when the U_i's are required only to be pseudoconvex. For example, the latter occurs in case K is contained in the boundary of a pseudoconvex domain Ω, such that $\overline{\Omega} \cap (\overline{D} \setminus K) = \varnothing$ and $\overline{\Omega}$ has a neighborhood basis of pseudoconvex open sets.

Hence, it seems natural to ask if the extendability property under consideration is still valid in the above situations and in similar other situations which the results on this problem obtained so far do not suffice to deal with.

In the present paper we wish to show that some partial positive answers to such question are possible for $n \geq 3$ complex variables.

In the first section of the paper we discuss an improvement, for $n \geq 3$, of the theorem of **[6]** quoted above; in the second section we obtain consequently some further results of a more geometrical character, including an improvement of Theorem I.1 (B) of [7] and a Hartogs' type theorem for certain domains in a Levi-flat hypersurface.

We point out that all the results of this paper relate to the case $n \geq 3$ and fail to be true in general for $n = 2$, in spite of the fact that our previous result of **[6]**—concerned with the $\mathcal{O}(\overline{D})$-convexity of K—is true for $n \geq 2$. Indeed, the dichotomy, in this kind of problems, between the cases $n \geq 3$ and $n = 2$ appears already in [7], and also in a part of the proof of the theorem of **[6]** itself.

1. If N is any subset of $\mathbf{C}^n$ with $K \subset N$, let us write

$$\tilde{K}_N = \bigcap_{\varphi \in \mathcal{O}(N)} \varphi^{-1}(\varphi(K)) = \bigcap_{U \supset N} \bigcap_{\varphi \in \mathcal{O}(U)} \varphi^{-1}(\varphi(K)),$$

where U ranges over the open neighborhoods of N. Thus $\tilde{K}_N$ is the complement in N of the set of all points $z \in N$ such that there is a $\varphi \in \mathcal{O}(N)$ with $\varphi(z) = 0$ and φ zero free on K. Plainly, $\tilde{K}_N \subset \hat{K}_N$ (the $\mathcal{O}(N)$-hull of K), and $\tilde{K}_N \subset \tilde{K}_{N'}$ whenever $N \subset N'$.

The object of this section is to prove the following theorem:

THEOREM 1.1. *Suppose $n \geq 3$. Every $f \in \mathrm{CR}(\partial D \setminus K)$ has a unique extension $F \in \mathcal{O}(D) \cap C^0(\overline{D} \setminus K)$ provided either of the following two conditions hold*:

(a) *There exists an open neighborhood N of K in $\overline{D}$ such that $\tilde{K}_N = K$ and every component of the hypersurface $\Sigma = (\partial D \setminus K) \cap N$ contains some peak points for $\mathcal{O}(\overline{D})$*;[1]

[1] We recall that a point $z^0 \in \partial D$ is said to be a peak point for $\mathcal{O}(\overline{D})$ in case there is a $\varphi \in \mathcal{O}(\overline{D})$ with $|\varphi(z^0)| > |\varphi(z)|$, for all $z \in \overline{D} \setminus z^0$.

(b) *There exists an open neighborhood N of K in $\overline{D}$ such that $\tilde{K}_N = K$; moreover $\hat{K}_{\overline{D}} \cap \partial D = K$.*

REMARKS. (i) This theorem is not true for $n = 2$, as the following simple example shows. Let $D = \{(z,w) \in \mathbf{C}^2;\ |z|^2 + |w|^2 < 2,\ |w| < 1\}$ and $K = \{(z,w) \in \mathbf{C}^2;\ |z|^2 + |w|^2 \leq 2,\ |w| = 1\}$. Then $\partial D \setminus K = \{(z,w) \in \mathbf{C}^2;\ |z|^2 + |w|^2 = 2,\ |w| < 1\}$ is a connected real hypersurface of class C^ω in $\mathbf{C}^n \setminus K$. Given any open neighborhood N of K in $\overline{D}$, let φ denote the restriction to N of the canonical projection $(z,w) \mapsto w$. Since $\varphi^{-1}(\varphi(K)) = K$, it follows that $\tilde{K}_N = K$; moreover, since $D \subset \mathbf{B}^4(\sqrt{2})$ (the ball with center the origin and radius $\sqrt{2}$) and $\partial D \setminus K \subset \partial \mathbf{B}^4(\sqrt{2})$, it follows that every point of $\partial D \setminus K$ is a peak point for $\mathcal{O}(\overline{D})$ and also, as a consequence, that $\hat{K}_{\overline{D}} \cap \partial D = K$. Hence D and K verify both the conditions (a) and (b) of Theorem 1.1. On the other hand, the function $f(z,w) = z^{-1}$, which is holomorphic on a neighborhood of $\partial D \setminus K$, has no holomorphic extension to D.

(ii) Since the condition (b) of Theorem 1.1 includes in particular the case when K is $\mathcal{O}(\overline{D})$-convex, it follows that Theorem 1.1 improves the result of [**6**] for $n \geq 3$.

(iii) The condition "$\tilde{K}_{\overline{D}} = K$" is equivalent to the "$\mathcal{M}(\overline{D})$-convexity" of K considered in [**7**].

Proof of Theorem 1.1. We use the same technique and follow the same lines as in [**6**]. Therefore, though all steps of the proof are discussed, many technical details are waived.

Let us first list some of the notations to be used in the following.

$\omega(\zeta) \in Z_{\bar\partial}^{n,n-1}(\mathbf{C}^n \setminus \zeta)$ is the Martinelli kernel-form relative to a point $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbf{C}^n$;

$L_\zeta(\varphi)$ is the level-set through ζ of a function $\varphi \in \mathcal{O}(U)$ (U open set containing ζ);

$\mathcal{O}_\varphi^n(U \times U)$ is the set of all holomorphic maps $h = (h_1, \ldots, h_n)$: $U \times U \to \mathbf{C}^n$ corresponding to a given $\varphi \in \mathcal{O}(U)$ in such a way that $\varphi(z) - \varphi(\zeta) = \sum_{\alpha=1}^n h_\alpha(z,\zeta)(z_\alpha - \zeta_\alpha)$, for every $(z,\zeta) \in U \times U$;

$\Phi_h(\zeta) \in A^{n,n-2}(U \setminus L_\zeta(\varphi))$ is the canonical $\bar\partial$-primitive of $\omega(\zeta)$ associated to a given $h \in \mathcal{O}_\varphi^n(U \times U)$ (cf. [**6**], §1).

Now, let $\{D_s\}_{s=1}^\infty$ be an increasing sequence of subdomains of D with the following properties: for each s, $\overline{D \setminus D_s} \subset N$ and $\partial D_s = \Gamma_s \cup K_s$, where Γ_s and K_s are compact hypersurfaces with boundary, of class C^1, such that $\Gamma_s \cap K_s = \partial \Gamma_s = \partial K_s$, $\Gamma_s \subset \partial D \setminus K$ and $K_s \setminus \partial K_s \subset N \cap D$; moreover $D = \bigcup_{s=1}^\infty D_s$ and $\partial D \setminus K = \bigcup_{s=1}^\infty \Gamma_s$. Such a sequence $\{D_s\}$, which clearly exists, in particular allows one to express the assumption "$\tilde{K}_N = K$" as follows:

$$(1.2) \qquad N \setminus K \subset \bigcup_{U \supset N} \bigcup_{\varphi \in \mathcal{O}(U)} \bigcup_{s=1}^{\infty} \left[U \setminus \varphi^{-1}\left(\varphi\left(\overline{D \setminus D_s}\right)\right)\right],$$

where U ranges over the open neighborhoods of N.

Next, corresponding to any given $f \in \mathrm{CR}(\partial D \setminus K)$, we introduce the following family $\mathcal{F} = \{\tilde{f}_h^s\}$ of complex-valued functions: for every open neighborhood U of N, $\varphi \in \mathcal{O}(U)$, $h \in \mathcal{O}_\varphi^n(U \times U)$ and positive integer s, $\tilde{f}_h^s$ is defined on $[U \setminus \varphi^{-1}(\varphi(\overline{D \setminus D_s}))] \setminus \partial D$ by

$$\tilde{f}_h^s(\zeta) = \int_{\Gamma_s} f\omega(\zeta) - \int_{\partial \Gamma_s} f\Phi_h(\zeta).$$

It is then possible to prove the following facts.

1.3. A necessary condition in order that f be the boundary values of a function $F \in \mathcal{O}(D) \cap C^0(\overline{D} \setminus K)$ is that $F = \tilde{f}_h^s$ on $[U \setminus \varphi^{-1}(\varphi(\overline{D \setminus D_s}))] \cap D$.

1.4. The family $\mathcal{F}$ is coherent, i.e. any two functions $\tilde{f}_h^s$ and $\tilde{f}_{h'}^{s'}$ of $\mathcal{F}$ coincide on the intersection of their sets of definition.

1.5. Every $\tilde{f}_h^s \in \mathcal{F}$ is holomorphic.

The proofs of these facts proceed as in **[6]** (cf. Propositions 2.4 and 2.5): 1.3 derives from the Martinelli integral representation applied to D_s; 1.4 and 1.5 from the general properties of the $\bar{\partial}$-primitives of $\omega(\zeta)$ which enter (cf. the remark at the end of this section).

Now, let us denote by $\tilde{f}$ the union of the coherent family $\mathcal{F}$. Then $\tilde{f}$ is a holomorphic function on the open set $E \setminus \partial D$—where $E = \bigcup_{U \supset N} \bigcup_{\varphi \in \mathcal{O}(U)} \bigcup_{s=1}^{\infty} [U \setminus \varphi^{-1}(\varphi(\overline{D \setminus D_s}))]$—which, by (1.2), includes $N \cap D = \mathring{N}$. Moreover, in view of 1.3, if an extension $F \in \mathcal{O}(D) \cap C^0(\overline{D} \setminus K)$ of f actually exists, it has to coincide with $\tilde{f}$ on $\mathring{N}$, which implies in particular the uniqueness of such an extension.

Therefore, it follows that what is to be proved now is that $\tilde{f}|_{\mathring{N}}$ has boundary values f on $\Sigma = (\partial D \setminus K) \cap N$, i.e. that, for each point $z^0 \in \Sigma$, we have:

$$(1.6) \qquad \lim_{\zeta \in \mathring{N},\, \zeta \to z^0} \tilde{f}(\zeta) = f(z^0).$$

Consequently, it is plain that the extension F of f will be given by

$$F(\zeta) = \begin{cases} \int_{\Gamma_s} f\omega(\zeta) + \int_{K_s} \tilde{f}\omega(\zeta) & \text{for } \zeta \in D_s, \\ \tilde{f}(\zeta) & \text{for } \zeta \in D \setminus D_s, \end{cases}$$

where s is any positive integer, and so the proof of the theorem will be concluded.

For the proof of (1.6) the assumption "$\tilde{K}_N = K$" alone is no longer sufficient, and hence we have to distinguish between the two conditions (a) and (b).

In the first place let us prove the following:

1.7. Assume that the condition (a) holds. Let Σ' be a connected component of Σ and V an open neighborhood of Σ' such that $V \setminus \Sigma' = V_+ \cup V_-$, where V_+, V_- are connected separated open sets and $V_+ \subset \mathring{N}$, $V_- \subset E \setminus \overline{D}$. Then it follows that $\tilde{f} = 0$ on V_-.

Let $z^0 \in \Sigma'$ be a peak point for $\mathcal{O}(\overline{D})$. Then there exist an open neighborhood U^0 of $\overline{D}$ and a function $\varphi^0 \in \mathcal{O}(U^0)$ such that $|\varphi^0(z^0)| > |\varphi^0(z)|$ for all $z \in \overline{D} \setminus z^0$. Let $h^0 \in \mathcal{O}^n_{\varphi^0}(U^0 \times U^0)$ and consider the function $\tilde{f}^1_{h^0}$. As in [6] (Proposition 2.6), one shows that $\tilde{f}^1_{h^0} = 0$ on $W = \{\zeta \in U^0;\ |\varphi^0(\zeta)| > |\varphi^0(z^0)| = \max_{\overline{D}}|\varphi^0|\}$. Since $\tilde{f}^1_{h^0}$ belongs to the family $\mathscr{F}$, it follows that $\tilde{f} = 0$ on W. Finally, since—by the maximum principle—$W \cap V_- \neq \varnothing$, and V_- is connected, we conclude that $\tilde{f} = 0$ on V_- as well.

Now, on account of 1.7, the proof of (1.6) proceeds as in [6] (Proposition 2.7), with some minor changes, using a known potential-theoretic property of the Martinelli kernel-form.

It remains to prove the validity of (1.6) under the condition (b). Actually 1.7 is still true in this case, but a direct proof would be more involved, and we can dispense with it. Indeed we can prove the following, from which (1.6) follows at once (as well as 1.7):

1.8. Assume that the condition (b) holds. Then f has a unique extension $f' \in \mathcal{O}(D \setminus \hat{K}_{\overline{D}}) \cap C^0(\overline{D} \setminus \hat{K}_{\overline{D}})$ and $f' = \tilde{f}$ on $(D \setminus \hat{K}_{\overline{D}}) \cap N$.

The existence and uniqueness of f' follows from Theorem 2 of [6]; moreover f' can be obtained as the union of a coherent family $\mathscr{F}'$ of holomorphic functions of the same kind as those of $\mathscr{F}$ (but defined by means of the holomorphic functions on the open neighborhoods of $\overline{D \setminus \hat{K}_{\overline{D}}}$). Now, by the same argument which proves 1.4, it is possible to show that also the family $\mathscr{F} \cup \mathscr{F}'$ is coherent, and hence f' and $\tilde{f}$ coincide on the intersection of their sets of definition. □

REMARK. Let us point out that 1.4, which is a crucial point in the above proof, derives from the fact that, for $n \geq 3$, the difference $\Phi_h(\zeta) - \Phi'_{h'}(\zeta)$ of two $\bar{\partial}$-primitives $\Phi_h(\zeta) \in A^{n,n-2}(U \setminus L_\zeta(\varphi))$ and $\Phi'_{h'}(\zeta) \in A^{n,n-2}(U' \setminus L_\zeta(\varphi'))$ of $\omega(\zeta)$ is $\bar{\partial}$-cohomologous to zero in $(U \setminus L_\zeta(\varphi)) \cap (U' \setminus L_\zeta(\varphi'))$. This implies that, for $f \in \mathrm{CR}(\partial D \setminus K)$, $f(\Phi_h(\zeta) - \Phi'_{h'}(\zeta))$ is exact in $(\partial D \setminus K) \cap (U \setminus L_\zeta(\varphi)) \cap (U' \setminus L_\zeta(\varphi'))$ (in the sense of continuous regular forms), and hence, if $\partial\Gamma_s \subset (U \setminus L_\zeta(\varphi)) \cap (U' \setminus L_\zeta(\varphi'))$, it follows that:

$$\int_{\partial\Gamma_s} f\big(\Phi_h(\zeta) - \Phi'_{h'}(\zeta)\big) = 0. \tag{1.9}$$

On the contrary, for $n = 2$ $\Phi_h(\zeta) - \Phi'_{h'}(\zeta)$ is a holomorphic 2-form and $f(\Phi_h(\zeta) - \Phi'_{h'}(\zeta))$ in general is only closed, so that (1.9) need not be true.

Nevertheless (1.9) turns out to be true also for $n = 2$ in case U and U' contain the whole of $\partial D \setminus K$ and $|\varphi(\zeta)| > \max_{\partial\Gamma_s}|\varphi|$, $|\varphi'(\zeta)| \geq \max_{\partial\Gamma_s}|\varphi'|$; which is the reason why Theorem 1 of [6] holds for $n = 2$ too.

2. Let us say that *K has a pseudoconvex neighborhood basis relative to $\bar{D}$* in case there exists a family $\{U_i\}$ of pseudoconvex open neighborhoods of K such that the family $\{U_i \cap \bar{D}\}$ is a neighborhood basis for K in $\bar{D}$. In particular this is the case if K has a neighborhood basis in $\mathbf{C}^n$ of pseudoconvex open sets, and, less trivially, if there exist finitely many pseudoconvex open domains $\Omega_j \subset\subset \mathbf{C}^n$, $j = 1, \ldots, m$, with pairwise disjoint closures, so that $K \subset \bigcup_{j=1}^m \partial\Omega_j$, $(\bar{D} \setminus K) \cap \bigcup_{j=1}^m \bar{\Omega}_j = \varnothing$ and each $\bar{\Omega}_j$ has a neighborhood basis of pseudoconvex open sets.[2]

Now we prove:

THEOREM 2.1. *Suppose $n \geq 3$. Every $f \in \mathrm{CR}(\partial D \setminus K)$ has a unique extension $F \in \mathcal{O}(D) \cap C^0(\bar{D} \setminus K)$ provided the following two conditions hold*:

(i) *There exists a C^2-bounded strictly pseudoconvex open domain $\Omega \subset\subset \mathbf{C}^n$ such that $D \subset \Omega$ and, for a sufficiently small neighborhood V of K, $(\partial D \setminus K) \cap V \subset \partial\Omega$;*

[2] For example, as is well known, the latter occurs when $\partial\Omega_j$ is C^2-smooth and strictly Levi-convex; and also when $\partial\Omega_j$ is real-analytic and weakly Levi-convex (cf. Diederich-Fornaess [1]).

(ii) *There exists an open neighborhood N of K in D such that $\tilde{K}_N = K$, or else K has a pseudoconvex neighborhood basis relative to $\overline{D}$.*

Proof. It is well known that every point of $\partial\Omega$ is a peak point for $\mathcal{O}(\overline{\Omega})$ (cf. Gunning-Rossi [2], Corollary IX.C.7) and hence the condition (i) implies that every point of $(\partial D \setminus K) \cap V$ is a peak point for $\mathcal{O}(\overline{D})$. In view of Theorem 1.1, this already suffices to conclude the proof under the former case of (ii), for we may assume that $N \subset V$, so that the condition (a) of Theorem 1.1 is fulfilled. To deal with the latter case of (ii), we consider first the particular situation when there is a plurisubharmonic function ρ on a pseudoconvex open neighborhood U of K such that $K \subset \{\rho = 0\}$ and $(\overline{D} \setminus K) \cap U \subset \{\rho > 0\}$. Then, since the $\mathcal{O}(U)$-hull $\hat{K}_U$ of K coincides with the hull $\hat{K}_U^P$ of K with respect to the plurisubharmonic functions (cf. Hörmander [4], Theorem 4.3.4), it follows that $K = \tilde{K}_N = \hat{K}_N$, for any open neighborhood N of K in $\overline{D}$ with $N \subset U$; and so we fall back into the previous case. Finally, let us consider the latter case of (ii) in full generality. Given a pseudoconvex open neighborhood U of K, we can find a C^∞-strictly plurisubharmonic exhaustion function u for U such that $K \subset U_0 = \{z \in U;\ u(z) < 0\}$. Then, let $D' = D \setminus \overline{U_0}$ and $K' = \overline{D} \cap \partial U_0$. Clearly, if $U \cap \overline{D}$ is small enough, we are again in the above situation, with D' and K' in place of D and K. Since we may choose U so that $U \cap \overline{D}$ is as small as we please, we conclude that the theorem is true for D and K as well.[3] □

Our next theorem is concerned with holomorphic functions, rather than with CR-functions, and improves a result of Stout [7] (Theorem I.1.(B)).

THEOREM 2.2. *Suppose $n \geq 3$. Let $\Omega \subset D$ be an open set such that $\Omega \cup (\partial D \setminus K)$ is a neighborhood in $\overline{D}$ of $\partial D \setminus K$. For every $f \in \mathcal{O}(\Omega)$ there exists an $F \in \mathcal{O}(D)$ with $F = f$ on Ω near $\partial D \setminus K$ provided the following two conditions hold*:

(i) *D is pseudoconvex*;

(ii) *There exists an open neighborhood Nof K in $\overline{D}$ such that $\tilde{K}_N = K$, or else K has a pseudoconvex neighborhood basis relative to $\overline{D}$.*[4]

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[3] The role of the condition (i) of this theorem is merely to ensure that, for each component K_1 of K, there are peak points for $\mathcal{O}(\overline{D})$ on $\partial D \setminus K$ as close to K_1 as we please. Indeed the theorem is still true under this weaker condition in place of (i).

[4] Here we may dispense with the assumption, made in Introduction, that $\partial D \setminus K$ is C^1-smooth.

Proof. In view of (i) we may find an increasing sequence $\{D_s\}_{s=1}^{\infty}$ of C^{∞}-bounded strictly pseudoconvex subdomains of D such that $D = \bigcup_{s=1}^{\infty} D_s$. We consider first the former case of (ii). Let $P \subset N$ be a closed neighborhood of K in $\overline{D}$ and let $\tilde{P}_N = \bigcap_{\varphi \in \mathcal{O}(N)} \varphi^{-1}(\varphi(P))$. Clearly, since $\tilde{K}_N = K$, we may assume that $\tilde{P}_N \setminus K$ is as small as we please, provided P is small enough. Now let us set $D_s' = D_s \setminus \tilde{P}_N$, $K_s' = \overline{D}_s' \cap \tilde{P}_N$ and $N_s' = \overline{D}_s' \cap N$. Then $\widetilde{(K_s')}_{N_s'} = K_s'$, $\partial D_s' \setminus K_s' \subset \partial D_s$ and further, for all sufficiently large s, $\partial D_s' \setminus K_s' \subset \Omega$; and hence, on account of Theorem 2.1, it follows that there exists an $F \in \mathcal{O}(D \setminus \tilde{P}_N)$ with $F = f$ on Ω near $\partial D \setminus \tilde{P}_N$. This concludes the proof under the former case of (ii), since $\tilde{P}_N \setminus K$ may be as small as we please. The proof under the latter case of (ii) is similar and even simpler: we just need to replace $\tilde{P}_N$ by the closure of the open set U_0 considered in the proof of Theorem 2.1 □

From the theorem above we can readily derive the following further result, relating to our main problem, which may be considered as a partial improvement of the classical Hans Lewy extension theorem:

COROLLARY 2.3. *Suppose $n \geq 3$. Every $f \in \mathrm{CR}(\partial D \setminus K)$ has a unique extension $F \in \mathcal{O}(D) \cap C^0(\overline{D} \setminus K)$ provided the following three conditions hold*:

(i) *D is pseudoconvex*;

(ii) *There exists an open neighborhood N of K in $\overline{D}$ such that $\tilde{K}_N = K$, or else K has a pseudoconvex neighborhood basis relative to $\overline{D}$*;

(iii) *$\partial D \setminus K$ is C^2-smooth and its Levi form (with respect to D) has at least one positive eigenvalue at each point.*

Proof. By a refined version of Hans Lewy extension theorem (cf. Harvey-Lawson [**3**], Theorem 10.2), we know that, for every $f \in \mathrm{CR}(\partial D \setminus K)$, there exists an open neighborhood T of $\partial D \setminus K$ in $\overline{D} \setminus K$ such that f has a unique extension $f_1 \in \mathcal{O}(T \cap D) \cap C^0(T)$. Hence our thesis follows at once from Theorem 2.2, for $\Omega = T \cap D$. □

Our next—and last—result is of a rather different kind from the previous ones, as being a Hartogs' type theorem for a suitable class of relatively open domains in a Levi-flat hypersurface.

THEOREM 2.4. *Let M be a C^2-smooth Levi-flat real hypersurface in an open set $U \subset \mathbf{C}^n$, $n \geq 3$. Let $\Omega \subset\subset U$ be a pseudoconvex open domain such that the following two conditions hold*:

(i) *$A = M \cap \Omega$ is an open domain in M with connected boundary $\partial A = M \cap \partial\Omega$*;

(ii) $\Omega \setminus A = \Omega^1 \cup \Omega^2$, *with* Ω^1 *and* Ω^2 *being separated open domains. Then every* $f \in \mathcal{O}(\partial A)$ *has a unique extension* $F \in \mathcal{O}(A)$.

Proof. In the first place we observe that, since Ω is pseudoconvex and M is Levi-flat, Ω^1 and Ω^2 are both pseudoconvex as well. Hence we may consider three increasing sequences of C^∞-bounded strictly pseudoconvex domains $\{\Omega_s\}_{s=1}^\infty$, $\{\Omega_s^1\}_{s=1}^\infty$ and $\{\Omega_s^2\}_{s=1}^\infty$, such that $\Omega = \bigcup_{s=1}^\infty \Omega_s$, $\Omega^1 = \bigcup_{s=1}^\infty \Omega_s^1$ and $\Omega^2 = \bigcup_{s=1}^\infty \Omega_s^2$. Now, let $f \in \mathcal{O}(\partial A)$ be given. Plainly, there is an integer t large enough so that $f \in \mathcal{O}(M \cap \partial\Omega_s)$ for $s \geq t$, and therefore it suffices to prove that there is an $F \in \mathcal{O}(M \cap \overline{\Omega}_t)$ with $F = f$ on a neighborhood of $M \cap \partial\Omega_t$. Writing $D_s = \Omega_t \setminus (\overline{\Omega}_s^1 \cup \overline{\Omega}_s^2)$ and $K_s = \overline{\Omega}_t \cap (\partial\Omega_s^1 \cup \partial\Omega_s^2)$, it is readily seen that $M \cap \Omega_t = \bigcap_{s=1}^\infty D_s$ and $M \cap \partial\Omega_t = \bigcap_{s=1}^\infty (\partial D_s \setminus K_s)$, and hence $f \in \mathcal{O}(\partial D_s \setminus K_s)$ for all sufficiently large s. Since D_s and K_s satisfy the conditions of Theorem 2.1, it follows that there is exactly an $F \in \mathcal{O}(D_s) \cap C^0(\overline{D}_s \setminus K_s)$ with $F = f$ on $\partial D_s \setminus K_s$, which yields the desired conclusion. □

REMARK. None of the results of this section extends in general to the case $n = 2$. As regards Theorem 2.1, Theorem 2.2 and Corollary 2.3, this is shown by the same example which disproves Theorem 1.1 for $n = 2$, discussed in the first remark of §1. In fact that example satisfies also all the conditions of Theorem 2.1, Theorem 2.2, and Corollary 2.3 (both cases of the condition (ii)). As regards Theorem 2.4, let $M = \{(z, w) \in \mathbf{C}^2;\ |w| = 1\}$, which is a real-analytic Levi-flat hypersurface in $\mathbf{C}^2$, and $\Omega = \mathbf{B}^4(\sqrt{2})$. Then M and Ω satisfy the two conditions of Theorem 2.4, with $A = \{(z, w) \in \mathbf{C}^2;\ |z| < 1,\ |w| = 1\}$ and $\partial A = \{(z, w) \in \mathbf{R}^2,\ |z| = |w| = 1\}$, but plainly the thesis of that theorem is not true in this case.

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