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OSCILLATORY PROPERTIES OF SYSTEMS OF FIRST ORDER LINEAR DELAY DIFFERENTIAL INEQUALITIES

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Sufficient conditions are obtained for the nonexistence of eventually positive bounded solutions of the system of delay differential inequalities

$$\frac{dx_i(t)}{dt} + \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij}) \leq 0; \qquad i = 1, 2, \ldots, n$$

and for the nonexistence of eventually negative bounded solutions of

$$\frac{dx_i(t)}{dt} + \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij}) \geq 0; \qquad i = 1, 2, \ldots, n.$$

As a corollary to the above we obtain sufficient conditions for all bounded solutions of

$$\frac{dx_i(t)}{dt} + \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij}) = 0; \qquad i = 1, 2, \ldots, n$$

to be oscillatory.

1. Introduction. The oscillatory and asymptotic behaviour of scalar delay differential equations and inequalities has been the subject of numerous investigations. For a recent survey of results we refer to Zhang [**20**]. First order differential inequalities with delayed arguments have been discussed by Ladas and Stavroulakis [**9**] and Stavroulakis [**18**]. The purpose of this brief article is to derive a set of sufficient conditions for all bounded solutions of a linear system of the type

$$\frac{dx_i(t)}{dt} + \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij}) = 0 \qquad i = 1, 2, \ldots, n;\ t > t_0 \tag{1.1}$$

to be "oscillatory" by considering the twin systems of inequalities

$$\frac{dx_i(t)}{dt} + \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij}) \leq 0 \qquad i = 1, 2, \ldots, n;\ t > t_0 \tag{1.2}$$

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and

$$(1.3) \quad \frac{dx_i(t)}{dt} + \sum_{j=1}^{n} a_{ij} x_j(t - \tau_{ij}) \geq 0 \qquad i = 1, 2, \ldots, n;\ t > t_0.$$

The literature concerned with the oscillation and nonoscillation of scalar systems of differential equations with and without deviating arguments is quite extensive. It appears that vector systems such as (1.1) where n is any positive integer have not received much attention with respect to their oscillation and nonoscillation characteristics especially when $n \geq 3$. Oscillation and nonoscillation of mostly two dimensional systems with deviating arguments have been considered by some authors (Kitamura and Kusano [**5–8**], Bulgakov and Sergeev [**1**], Bykov [**2**], Foltynska and Werbowski [**3**], Izyumova and Mirzov [**4**], Mirzov [**11–16**], Varekh and Shevelo [**19**], Marusiak [**10**], Shevelo [**17**]).

We recall that it is customary to define a real valued continuous function x defined on a half-line $[t_0, \infty)$ to be oscillatory if there exists a sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that $x(t_m) = 0$ for each t_m. Such a definition has been adequate for analysing the oscillatory and nonoscillatory characteristics of scalar systems of delay differential equations and inequalities. In vector systems such as (1.1)–(1.3), it is advantageous to use the following:

DEFINITION 1. A real valued differentiable function u defined on a half-line $[t_0, \infty)$ is said to be oscillatory if there exists a sequence $\{t_m\} \to \infty$ as $m \to \infty$ such that $t_m \in (t_0, \infty)$ and $u(t_m)\dot{u}(t_m) = 0$ for each $t_m \in (t_0, \infty)$ where $\dot{u}(t_m) = du/dt$ at t_m; u is said to be nonoscillatory on $[t_0, \infty)$ if there exists a $t^* > t_0$ such that $u(t)\dot{u}(t) \neq 0$ for $t \geq t^*$.

Using the above definition we now define oscillation and nonoscillation of $\mathbf{R}^n$-valued functions as follows:

DEFINITION 2. An $\mathbf{R}^n$-valued function $x(t) = \{x_1(t), x_2(t), \ldots, x_n(t)\}$ defined on a half-line $[t_0, \infty)$ with differentiable components is said to be oscillatory if at least one component of x is oscillatory in the sense of Definition 1; a vector x: $(t_0, \infty) \to \mathbf{R}^n$ with differentiable components is said to be nonoscillatory if every component of x is nonoscillatory as in Definition 1.

DEFINITION 3. The system (1.1) is said to be oscillatory if every solution of (1.1) defined on a half-line (t_0, ∞) is oscillatory in the sense of Definition 2; the system (1.1) is said to be nonoscillatory if (1.1) has at least one solution defined on a half-line which is nonoscillatory.

We remark that definitions of oscillatory and nonoscillatory $\mathbf{R}^n$-valued functions are varied in the literature. Our Definitions 2 and 3 above provide one of several possible ways of generalising the corresponding notions of oscillation and nonoscillation of real valued functions to the case of $\mathbf{R}^n$-valued functions.

DEFINITION 4. An $\mathbf{R}^n$-valued function $x(t) = \{x_1(t), \ldots, x_n(t)\}$ defined on a half-line (t_0, ∞) with differentiable components is said to be eventually positive if x is nonoscillatory on (t_0, ∞) and there exists a $t^* > t_0$ such that $x_j(t) > 0$ for $t \geq t^*$ and $j = 1, 2, 3, \ldots, n$. An $\mathbf{R}^n$-valued eventually negative function is defined analogously.

2. Delay induced oscillations. We consider the systems (1.1)–(1.3) together with the following assumptions.

(A_1) a_{ij}, τ_{ij} $(i, j = 1, 2, \ldots, n)$ are real constants such that

$$\text{(2.1)} \qquad \text{(i)} \quad a_{ii} > 0; \; \tau_{ii} > 0; \; i = 1, 2, 3, \ldots, n;$$

$$\text{(2.2)} \qquad \text{(ii)} \quad \tau_{ij} \geq 0; \; i, j = 1, 2, 3, \ldots, n;$$

$$\tau_{ii} \geq \tau_{ji}; \; i, j = 1, 2, \ldots, n.$$

$$\text{(2.3)} \qquad (\mathrm{A}_2) \quad e\left[\min_{1 \leq i \leq n} \{\tau_{ii}\}\right]\left[\min_{1 \leq i \leq n}\left(a_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ji}|\right)\right] > 1.$$

THEOREM. *If the assumptions* (A_1) *and* (A_2) *hold, then the system* (1.2) *cannot have a nonoscillatory eventually positive bounded solution on* $[0, \infty)$.

Proof. Our strategy of proof is to show that the existence of an eventually positive and bounded nonoscillatory solution of (1.2) contradicts the condition (2.3). Let us then suppose that (1.2) has a nonoscillatory bounded eventually positive solution $u(t) = \{u_1(t), u_2(t), \ldots, u_n(t)\}$ on $[0, \infty)$. There exists a $t_1 > 0$ such that

$$\text{(2.4)} \qquad u_i(t) > 0 \quad \text{for } t \geq t^*; \qquad i = 1, 2, \ldots, n$$

and

$$\text{(2.5)} \qquad \frac{du_i(t)}{dt} \leq -a_{ii}u_i(t - \tau_{ii}) - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}u_j(t - \tau_{ij}).$$

It follows from the boundedness, nonoscillation and eventual positivity of $u_1, u_2, \ldots, u_n$ that $u_i(t)$ converges as $t \to \infty$. We let

$$\lim_{t \to \infty} u_i(t) = c_i \geq 0, \qquad i = 1, 2, \ldots, n. \tag{2.6}$$

We claim that $c_i = 0$, $i = 1, 2, \ldots, n$; suppose not. Then the nonoscillation of $u_1, u_2, \ldots, u_n$ and the eventual positivity of $u_1, u_2, \ldots, u_n$ shows that the convergence in (2.6) is monotonic in t eventually and hence there exists a $t_2 > t_1 + \tau$ ($\tau = \max(\tau_{ij};\ i, j = 1, 2, \ldots, n)$) such that

$$\begin{aligned} u_i(t) &< c_i + \varepsilon \quad \text{for } t > t_2, \\ u_i(t) &> c_i - \varepsilon \quad i = 1, 2, \ldots, n, \end{aligned} \tag{2.7}$$

where ε is any arbitrary positive number. We have from (2.5) that

$$\begin{aligned} \frac{d}{dt}\left(\sum_{i=1}^{n} u_i(t)\right) &\leq -\sum_{i=1}^{n} a_{ii} u_i(t - \tau_{ii}) \\ &\quad + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| u_j(t - \tau_{ij}), \qquad t > t_2 + \tau \end{aligned} \tag{2.8}$$

$$\leq -\sum_{i=1}^{n} \left\{ a_{ii}(c_i - \varepsilon) - \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|(c_j + \varepsilon) \right\} \tag{2.9}$$

$$\begin{aligned} &\leq -\sum_{i=1}^{n} \left(a_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ji}| \right) c_i + \varepsilon \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \left(a_{ii} + \sum_{j=1}^{n} |a_{ji}| \right) \right] \\ &\leq -m \sum_{i=1}^{n} c_i + M\varepsilon; \qquad t \geq t_2 = \tau, \end{aligned} \tag{2.10}$$

where

$$m = \min_{1 \leq i \leq n} \left(a_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ji}| \right), \qquad M = \sum_{i=1}^{n} a_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ji}|. \tag{2.11}$$

The assumption (A_2) implies that $m > 0$. Now if $\sum_{i=1}^{n} c_i > 0$ then choosing ε small enough one can show that there exists a positive number μ such that

$$\frac{d}{dt}\left(\sum_{i=1}^{n} u_i(t)\right) \leq -\mu \quad \text{for } t_2 + \tau \tag{2.12}$$

which leads to

$$(2.13)\qquad \sum_{i=1}^{n} u_i(t) \leq -\mu(t - t_2 - \tau) + \sum_{i=1}^{n} u_i(t_2 + \tau) \qquad t > t_2 + \tau$$

implying that $\sum_{i=1}^{n} u_i(t)$ can become negative for large enough t; but this is impossible. Thus we have $\sum_{i=1}^{n} c_i = 0$ and hence $c_i = 0$, $i = 1, 2, \ldots, n$; thus

$$(2.14)\qquad \lim_{t\to\infty} u_i(t) = 0, \qquad i = 1, 2, \ldots, n,$$

and the convergence in(2.14) is monotonic in t eventually due to the nonoscillatory nature of $u(t) = \{u_1(t), \ldots, u_n(t)\}$. It follows from (2.8) that

$$(2.15)\qquad \frac{d}{dt}\left\{\sum_{i=1}^{n} u_i(t)\right\} \leq -\sum_{i=1}^{n} a_{ii}u_i(t - \tau_{ii}) - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ji}|u_i(t - \tau_{ji})$$

which on using

$$(2.16)\qquad t_{ii} \geq \tau_{ji} \Rightarrow t - \tau_{ii} \leq t - \tau_{ji} \Rightarrow u_i(t - \tau_{ii}) \geq u_i(t - \tau_{ji}), \qquad i, j = 1, 2, \ldots, n$$

leads to

$$(2.17)\qquad \frac{d}{dt}\left(\sum_{i=1}^{n} u_i(t)\right) \leq -\sum_{i=1}^{n}\left(a_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ji}|\right) u_i(t - \tau_{ii}) \qquad t > t_2 + \tau$$

$$(2.18)\qquad \leq -m\sum_{i=1}^{n} u_i(t - \tau_{ii}); \qquad t > t_2 + \tau$$

$$(2.19)\qquad \leq -m\sum_{i=1}^{n} u_i(t - \sigma); \qquad t > t_2 + \tau$$

where

$$(2.10)\qquad m = \min_{1\leq i\leq n}\left\{a_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ji}|\right\}, \qquad \sigma = \min\{\tau_{11}, \tau_{22}, \ldots, \tau_{nn}\}.$$

Note that $\sigma > 0$ due to (2.3) and we have used the eventual monotonic convergence of $u_i(t - \tau_{ii})$ to zero as $t \to \infty$ $(i = 1, 2, \ldots, n)$ in the derivation of (2.19) from (2.18).

Now if we let

$$(2.21) \qquad y(t) = \sum_{i=1}^{n} u_i(t); \qquad t > t_2 + \tau,$$

then we have from (2.19) that

$$(2.22) \qquad dy(t)/dt \leq -my(t-\sigma); \qquad t > t_2 + \tau$$

and y is an eventually positive solution of the scalar delay differential inequality (2.22) in which the constants m and σ satisfy

$$(2.23) \qquad e\sigma m > 1$$

as a consequence of (A_2) and (2.20).

It is well known that the scalar delay differential inequality (2.22) cannot have an eventually positive solution (Ladas and Stavroulakis [**9**]) when (2.23) holds; this contradiction shows that (1.2) cannot have a bounded nonoscillatory eventually positive solution and this completes the proof.

COROLLARY 1. *Assume that* (A_1) *and* (A_2) *hold. Then the system of inequalities* (1.3) *cannot have an eventually negative bounded nonoscillatory solution.*

Proof. The conclusion follows from the result of the above Theorem since an eventually negative bounded solution of (1.3) is an eventually positive nonoscillatory bounded solution of (1.2).

COROLLARY 2. *Assume that* (A_1) *and* (A_2) *hold. Then all bounded solutions of* (1.1) *are oscillatory.*

Proof. The assertion is a consequence of the fact that (1.1) cannot have nonoscillatory bounded solutions which are eventually positive or which are eventually negative.

We conclude with the remark that further generalisation of our result to nonautonomous systems (with variable coefficients and variable delays) and to nonlinear systems is of some interest for applications.

REFERENCES

[1] A. I. Bulgakov and B. A. Sergeev, *Oscillation properties of solutions of second order system of ordinary differential equations*, Differential Equations, **20** (1984), 158–164.

[2] Ya. V. Bykov, *On a class of systems of ordinary differential equations*, Differential Equations, **1** (1965), 1139–1159.

[3] I. Foltynska and J. Werbowski, *On the oscillatory behaviour of solutions of systems of differential equations with deviating arguments in "Qualitative theory of differential equations"*, Colloquia Mathematica Soc. Janos Bolyai, 30, Szeged (Hungary) (1979), 243–256.

[4] D. V. Izyumova and D. D. Mirzov, *Oscillation properties of solutions of nonlinear differential systems*, Differential Equations, **12** (1976), 838–842.

[5] Y. Kitamura and T. Kusano, *On the oscillation of a class of nonlinear differential systems with deviating argument*, J. Math. Anal. Appl., **66** (1978), 20–36.

[6] _____, *Vanishing oscillations of a class of differential systems with retarded argument*, Accad. Nazionale Slincei Rend., **62** (1977), 325–334.

[7] _____, *Oscillation and a class of nonlinear differential systems with general deviating arguments*, Nonlinear Anal., **2** (1978), 537–551.

[8] _____, *Asymptotic properties of solutions of two dimensional differential systems with deviating argument*, Hiroshima Math. J., **8** (1978), 305–326.

[9] G. Ladas and I. P. Stavroulakis, *On delay differential inequalities of first order*, Funkcialaj Ekvacioj, **25** (1982), 105–113.

[10] P. Marusiak, *On the oscillation of nonlinear differential systems with retarded arguments*, Math. Slovaca, **34** (1984), 73–88.

[11] D.D. Mirzov, *Oscillatory properties of solutions of a system of nonlinear differential equations*, Differential Equations, **9** (1973), 447–449.

[12] _____, *Ability of solutions of a system of nonlinear differential equations to oscillate*,, Math. Notes of the Acad. Sciences USSR, **16** (1974), 932–935.

[13] _____, *Oscillation of solutions of a system of differential equations*, Math. Notes of the Acad. Sciences USSR, **23** (1978), 218–220.

[14] _____, *Asymptotic properties of solutions of two dimensional differential systems*, Differential Equations, **13** (1977), 1527–1535.

[15] _____, *Zeros of solutions of a nonlinear system*, Differential Equations, **20** (1984), 1237–1242.

[16] _____, *On some analogs of Sturm's and Kneser's theorems for nonlinear systems*, J. Math. Anal. Appl., **53** (1976), 418–425.

[17] V. N. Shevelo, *Oscillations of solutions of differential equations with deviating arguments*, Naukov Dumka, Kiev, (1978), (Russian).

[18] I. P. Stavroulakis, *Nonlinear delay differential inequalities*, Nonlinear Anal., **6** (1982), 389–396.

[19] N. V. Varekh and V. N. Shevelo, *Certain properties of solutions of differential equations with retarded argument*, Ukr. Math. J., **34** (1982), 1–7.

[20] B. G. Zhang, *A survey of the oscillation of solutions to first order differential equations with deviating arguments*, in the Proceedings of the VI International Conference on "Trends in the Theory and Practice of Nonlinear Analysis" held at the University of Texas Al-Arlington, June 18–22, 1984, Ed. Lakshmikantham, North Holland, Amsterdam, 1984.

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