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s-SMITH EQUIVALENT REPRESENTATIONS OF DIHEDRAL GROUPS

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Dihedral groups of order 2^m, for sufficiently large m, have nonisomorphic but s-Smith equivalent representations. That is, these groups can act smoothly and semilinearly on a homotopy sphere with two fixed points such that the isotropy representations at the fixed points are distinct.

1. Introduction. Let G be a finite group. If G acts smoothly on a closed homotopy sphere with exactly two fixed points, then the isotropy (or tangential) representations at these points are said to be Smith equivalent [**P2**]. If, in addition, the homotopy sphere Σ is semilinear, (i.e. the fixed set Σ^H is a homotopy sphere for every subgroup H of G [**R**]), then the isotropy representations are called s-Smith equivalent [**P2**]. The main result of this paper is

THEOREM A. *Dihedral groups of order 2^m, m sufficiently large, have nonisomorphic but s-Smith equivalent representations.*

Theorem A is a consequence of Theorem 3.4 which gives a sufficient condition for representations of dihedral groups to be s-Smith equivalent. Theorem 3.4 is followed by explicit examples of nonisomorphic but s-Smith equivalent representations of dihedral groups D_{2^m} of order 2^{m+1} for $m > 10$.

REMARK. At present, we know a class of cyclic groups [**P2**], certain abelian groups [**Su**], and generalized quaternion groups of order high powers of 2 [**Ch**] have nonisomorphic but s-Smith equivalent representations.

Following is a brief description of the general technique given by Petrie in [**P2**] and in [**PR**], which we will apply.

Let G be a finite group. Let V and W be representations of G satisfying certain conditions, which will be discussed in detail in §3 (see Theorem 3.4). Let Y be the unit sphere $S(V+R)$ of the representation $V+R$, where R is the trivial one dimensional real representation of G. If the fixed set V^G is $\{0\}$, then the fixed set Y^G consists of two points

$p = (0, 1)$ and $q = (0, -1)$ and the isotropy representations T_pY and T_qY are isomorphic to V.

Suppose there exist G-vector bundles E_+ and E_- over Y with $\mathrm{Iso}(E_+) = \mathrm{Iso}(E_-) = \mathrm{Iso}(V)$ and a G-map h between them. Here, $\mathrm{Iso}(V)$ is the set of all isotropy subgroups of V. Suppose, also that the map $h: E_+ \to E_-$ is a proper equivariant fiber preserving map such that the fiber degree of $h^H: E_+^H \to E_-^H$ is 1 for all subgroups H of G.

Suppose h is transverse to the zero section Y in E_-. Then $X = h^{-1}(Y)$ is a G-manifold and the restriction $f = h \mid X$ of h on X is a G-map.

The degree of f^H is 1 for every $H \subseteq G$ and the tangent bundle TX is stably isomorphic to $f^*(TY + E)$, where $E = E_+ - E_-$.

The pair (X, f) with additional bundle data is called a normal map (precise definition in (3.4)).

The normal map (X, f) is converted, by applying equivariant surgery, into another normal map (X', f'), where $f': X' \to Y$ is a G-homotopy equivalence. The tangent bundle TX' is stably isomorphic to $f'^*(TY + E)$ and $f'^G: X'^G \to Y^G$ is a bijection. Hence,

$$T_pX' = T_pY + E_p \quad \text{and} \quad T_qX' = T_qY + E_q.$$

Here, the fixed set X'^G is identified with $\{p, q\}$. Thus if $i^*E = (0, W - V)$, where $i: Y^G \to Y$ is the inclusion, then $T_pX' = V$ and $T_qX' = W$, i.e. V and W are s-Smith equivalent.

When G is a cyclic group, the G-vector bundles E_+ and E_- over Y and the G-fiber homotopy equivalences can be found by applying results from representation theory, equivariant K-theory, equivariant J-homomorphism, and Adam's operation **[P2]**. One can extend them for cases where G is not cyclic, by applying induction construction on the equivariant vector bundles and the equivariant bundle maps. The induction construction on equivariant vector bundles is a map of $K_H(Y)$ into $K_G(Y)$ which is a generalization of the usual induction on representations

$$\mathrm{ind}_K^G: R(H) \to R(G),$$

where H is a subgroup of G **[ChSu]**.

The normal maps we construct fail to satisfy the gap hypothesis required in the definition of normal map given by Petrie in **[PR]** (definition given in §3). The difficulties that arise from this fact in the process of equivariant surgery are avoided by Lemma 3.2.

The surgery obstructions are shown to vanish by applying Hambleton and Milgram's results on surgery obstruction groups for finite 2-groups **[HM]**.

For the convenience of the readers, some of the necessary results are reviewed in §2. And the main result is proved in §3.

Historical Remark. Under certain conditions either on the acting groups or on the representations, Smith equivalent or s-Smith equivalent representations are isomorphic, as proved in **[AB]**, **[M]**, **[B]**, and **[Sa]**. The first example of nonisomorphic Smith equivalent representations was found by Petrie **[P1]**. Cappell and Shaneson gave the first example of nonisomorphic Smith-action equivalent representations of Z_{4n}, $n > 1$ **[CS2]**. Here, the definition of Smith-action equivalence is stronger than that of Smith equivalence, it requires the action to be of Smith type (i.e., each fixed set of the sphere be either discrete or connected). In fact, the example is also s-Smith equivalent (called semilinear-action equivalent in **[CS2]**). More examples of nontrivial Smith equivalent or s-Smith equivalent representations were found by Dovermann **[D]**, Siegel **[Si]**, Petrie **[P2]**, Dovermann-Petrie **[DP]**, Suh **[Su]**, and Cho **[Ch]**. Recently, Dovermann and Washington showed an infinite class of small odd order cyclic groups with nonisomorphic Smith equivalent representations **[DW]**. For more background and results, see papers by Masuda-Petrie **[MP]** and Dovermann-Petrie-Schultz **[DPS]**, and also a book by Petrie-Randall **[PR]**.

2. This section reviews (a) an induction construction in equivariant K-theory from **[ChSu]**, and (b) a result on surgery obstruction groups for finite 2-groups from **[HM]**.

(a) Let G be a finite group and K a subgroup of G. G can be viewed as a K-space via group multiplication on the left. Given K-space X, the set $(G,X)_K$ of all K-maps from G to X equipped with the compact open topology is a G-space if we define an action of G by $g: f \to (gf: g' \to f(g'g))$ where g and g' are elements of G. (Since G is finite, the compact open topology on $(G,X)_K$ is the relative topology induced from the product topology on X^G.)

If Y is a G-manifold and (E,Y,p) a K-vector bundle over Y, then $((G,E)_K, (G,Y)_K, p')$ is a G-vector bundle over the space $(G,Y)_K$, where the projection p' is defined by $p'(f) = p \cdot f$ for f in $(G,E)_K$.

Let $F: Y \to (G,Y)_K$ be a map given by $(F(y))(g) = g(y)$. Then F is a G-map, and the pullback $F^*(G,E)_K$ is a G-vector bundle over Y, which we call the induced vector bundle of E and denote by $\mathrm{ind}_K^G E$.

The set $\mathrm{Vect}_G(Y)$ of all G-vector bundles over Y is a semigroup under Whitney sum, and the map ind_K^G is a homomorphism of the semigroup $\mathrm{Vect}_K(Y)$ of all K-vector bundles over Y to the semigroup $\mathrm{Vect}_G(Y)$. Therefore, ind_K^G naturally extends to a group homomorphism of $K_K(Y)$ into $K_G(Y)$, which is also denoted by ind_K^G.

If $h\colon E \to F$ is a fiber preserving K-map, then the map defined by

$$(\mathrm{ind}_K^G h)(y, f) = (y, h \cdot f)$$

for (y, f) in $\mathrm{ind}_K^G E$ is a G-map and preserves fiber. (Recall f is a K-map of G into E, by the definition of induced bundle.)

For p in Y^G, the fiber $(\mathrm{ind}_K^G E)_p$ as a representation of G is isomorphic to $\mathrm{ind}_K^G(E_p)$ where E is a K-vector bundle over Y. In particular, $\mathrm{ind}_K^G\colon K_K(Y) \to K_G(Y)$ and $\mathrm{ind}_K^G\colon R(K) \to R(G)$ coincide when Y is a point.

(b) In [HM], Hambleton and Milgram determined the oriented surgery obstruction groups $L_*^h(G)$ for G any finite 2-group. We will apply the fact that the surgery obstruction group $L_0^h(G)$ is torsion-free, when G is a dihedral group of order power of 2.

It is the consequence of the fact that, when $G = D_{2^n}$, $L_0^p(G)$ is isomorphic to a direct sum of Z by Theorem A of [HM], and the terms $H^*(Z/2; K_0(G))$ in the Ranicki-Rothenberg exact sequence

$$\to L_{k+1}^h(G) \to L_{k+1}^p(G) \to H^k(Z/2; K_0(G)) \to L_k^h(G) \to$$

vanish.

3. Main result and the proof. Throughout this section, G is the dihedral group D_{2d}, with $2d = 2^k$, i.e. the group generated by two generators x and y with relations

$$x^{2d} = y^2 = 1, \quad y^{-1}xy = x^{-1},$$

K is the subgroup generated by x, and H the subgroup generated by x^2. The cyclic subgroup K is also viewed as a subgroup of the multiplicative group C^* of nonzero complex numbers by identifying the generator x with a primitive $2d$th root of unity.

The representation ring $R(K)$ is isomorphic to the polynomial ring $z[t]/(t^{2d} - 1)$, where t is the representation with the underlying vector space the complex plane C on which K acts via complex multiplication.

Let

$$V' = a_1 t^1 + a_2 t^2 + \cdots + a_{2d-1} t^{2d-1}$$

be a complex representation of K such that

$$V'^K = 0 \quad \text{and} \quad \mathrm{Iso}(V') = \{1, H, K\}. \tag{3.1}$$

The condition (3.1) is equivalent to the explicit condition on the coefficients a_i's of V'

$$a_d \neq 0 \quad \text{and} \quad a_r = 0 \quad \text{when } r \text{ is even but } r \neq d. \tag{3.1'}$$

Let V be $\mathrm{ind}_K^G V'$, the representation of G induced from V', and Y the unit sphere $S(V+R)$ of $V+R$, where R is the trivial representation of G. It is easily checked that $V^G = V^K = 0$ and $Y^G = Y^K = \{p, q\}$ where $p = (0, 1)$ and $q = (0, -1)$.

A representation V of G is called stable if for each M in $\mathrm{Iso}(V)$, the set of all isotropy subgroups of V, and each nontrivial representation χ of M, either the multiplicity $m_\chi(V)$ of χ in V is zero or

$$d_\chi m_\chi(V) \geq m_1(V) = \dim_R V^M.$$

Here d_χ is the real dimension of D_χ, the algebra of real endomorphisms of χ [P2].

A G-space X is said to satisfy the gap hypothesis if either $X^A = X^B$ or the dimension of X^A is less than half the dimension of X^B for all subgroups A and B of G with $B \subseteq A$.

Definition IV.2.1 [PR]. A normal map rel. C, (X, f, b), where $f\colon X \to Y$, and Y satisfies the gap hypothesis, consists of

1. A closed manifold X with $\mathrm{Dim}(X) = \mathrm{Dim}(Y)$, where $\mathrm{Dim}(X)$ is a map of $\Pi(X) =$ the set of components of fixed point sets into Z given by $\mathrm{Dim}(X)(a) =$ the dimension of the component X_a of X^M for some subgroup.
2. $f\colon X \to Y$ is a G-map such that $f\colon \Pi(X) \to \Pi(Y)$ is bijective and $\deg f^M = 1$ for all $M \subseteq G$.
3. There is a stable G-vector bundle isomorphism $b\colon sTX \to f^*s(TY + E)$ for some virtual G-vector bundle E over Y.
4. f^M is a homotopy equivalence for every M in C.
5. $\dim Y^M$ is either 0, 1 or greater than 4 for all subgroups M of G.

To construct a normal map in Theorem 3.1 we need following condition (3.2)

(3.2) The virtual representation $z' = W' - V'$ belongs to $R(K; V'')$, where $V'' = V' + t^d V'$, and belongs to $2^d I(H)$. Here $I(H)$ is the kernel of the map

$$\mathrm{res}_H \times \mathrm{fix}_H\colon R(K) \to R(H) \times R(K/H)$$

where Δ is a numerical function depending upon

$$I = I(S) = \mathrm{Ker}(\mathrm{res}_H \times \mathrm{fix}_s)$$

and V' given by

$$(3.3) \qquad \begin{aligned} \Delta(I, V') &= a(V') + b(I) + c(V') + d(V'), \quad \text{where} \\ a(V') &= \sum_{j=1} \langle A_{2^j} 2^{j-k+1} \rangle, \\ b(I) &= 2, \\ c(V') &= 0, \\ d(V') &= 2a_d + 1. \end{aligned}$$

Here, $A_r = \sum_{(s,2d)=r} a_s$ for $r < d$,

$$A_d = a_d - 1,$$

and $\langle n \rangle$ is the least integer greater than or equal to n.

REMARK. The formula (3.3) is chosen to satisfy the condition of the Theorem 5.19 of **[P2]**, where Δ is given more generally in (5.12) as follows:

$$a(V') = \sum_{r|d} \left\langle \frac{A_r}{\mu(d/r)} \right\rangle.$$

(Here $\mu(x)$ is the highest power of 2 dividing x.)

$$\begin{aligned} b(I) &= 1 \text{ if } |K/S| = 0 \pmod 8, \\ &= 2 \text{ if } 2 < |K/S| \text{ and } |K/S| \neq 0 \pmod 8. \\ c(V') &= 0 \text{ if the dimension of } V' \text{ is odd}, \\ &= 1 \text{ if the dimension of } V' \text{ is even}. \\ d(v') &= \text{ whichever of } \dim V'^H + 1 \text{ or } \dim V'^H \text{ is odd.} \end{aligned}$$

(For the discussion of the formula, see 4.2, 4.3, and 4.4 of **[P2]**.)

THEOREM 3.1. *Let V' be a representation of K satisfying the condition* (3.1) *and W' a representation of K satisfying the condition* (3.2). *If V is stable, then there is a normal map (X, f) rel. C, where $f: X \to Y$, such that $T_pX = \mathrm{ind}_K^G V'$ and $T_qX = \mathrm{ind}_K^G W'$.*

Here C is a closed family of subgroups of G containing H.

Proof. Theorem 5.19 of **[P2]** proves that if z' satisfies the condition (3.2), there exists an element E' in $K_K(Y; V'')$ with $J_{V''}(E') = 0$

and $i^*\rho(E') = (0, z')$, where ρ is the obvious map of $K_K(Y; V'')$ into $K_K(Y)$.

Here, $K_K(Y)$ is the Grothendieck group associated to the semigroup $\mathrm{Vect}_K(Y)$ of all K-vector bundles on Y under Whitney sum. And $K_K(Y; V'')$ is the Grothendieck group associated to the semigroup $\mathrm{Vect}_K(Y; V'')$ of all K-vector bundles E over Y with $E_y < V''$ at every point y of Y. $E_y < V''$ means E_y is a subrepresentation of nV'' for some positive integer n. This is equivalent to the fact that every irreducible real representation of K_y occurring in E_y occurs in V''. And i^* is the map induced from the inclusion $i\colon Y^K \to Y$.

For our case, extra technical conditions required in Theorem 5.19 of **[P2]** are trivially satisfied because of our choice of V'' and of the group K being of order power of 2.

Now apply the induction construction on E' to find

$$E = \mathrm{ind}_K^G E' \quad \text{in } K_G(Y; \tilde{V})$$

such that

$$J_{\tilde{V}}(E) = 0, \qquad i_{\tilde{V}}^* E = (0, 2^{\Delta} \mathrm{ind}_K^G z') = (0, 2^{\Delta} z).$$

Here $\tilde{V}$ is $\mathrm{ind}_K^G V''$.

The normal map (X, f) with $f\colon X \to Y$ is constructed by the transversality argument as in Theorem 5.19 of **[P2]**. □

The normal map (X, f) constructed in Theorem 3.1 fails to satisfy the gap hypothesis as required by the definition of the normal map. It does satisfy the gap hypothesis at every level, but not at $X^L < X^H$ and at $X^M < X^H$, where L is the group generated by x^2 and y and M the group generated by x^2 and xy. In fact, we have

$$\dim X^L = 1/2 \dim X^H \quad \text{and} \quad \dim X^M = 1/2 \dim X^H.$$

Since the purpose of the gap hypothesis is to provide imbeddings when a series of surgeries is to be performed, if the normal map (X, f) can be arranged so that no surgery inside or on X^H is necessary then (X, f) does not have to satisfy the gap hypothesis at $X^A < X^H$ for any subgroup $A > H$. The arrangement is possible by the following lemmas, which will allow us to assume $X^H = Y^H$ and the map $f^H = f \mid X^H$ is the identity map.

Let $n = \dim_R V'^H$, and z' and E' are related by

$$i^*\rho(E') = (0, z')$$

as in the proof of Theorem 3.1.

LEMMA 1.2 [CH]. *If* $z'^H = 0$ *then the element* $2^n E'^H$ *of*

$$K_{K/H}(Y^H; V''^H)$$

is zero.

REMARK. To compare with the original statement of the lemma in [Ch], notice $\dim_R V'^H = (1/2)\dim_R(\mathrm{ind}_K^G V')^H$.

Since $J_{V''}(E') = 0$, there is a K-fiber homotopy equivalence

$$\Theta\colon E'_+ \to E'_-$$

where $E' = E'_+ - E'_-$.

Let Θ_1 be $2^n\Theta\colon 2^n E'_+ \to 2^n E'_-$. Then by the previous lemma, we may write

$$2^n E'^H_+ = 2^n E'^H_- = Y^H \times A$$

for some A in $R(K/H)$.

Hence Θ_1^H is a K/H-map of $Y^H \times A$ into itself and can be viewed as an element of $\omega^0_{K/H}(Y^H)$. By definition,

$$\omega_G^{-1}(X, Y) = [(S^i \wedge X^+, S^i \wedge Y^+), (F, \mathrm{id})]^G.$$

Here, $[A, B]^G$ denotes the set of G-homotopy classes of base point preserving G-maps of A into B, X^+ is the one point compactification of X, $F = \lim_{\to W \in R(G)} M(W)$, and $M(W)$ is the identity component of the space of self maps of the unit sphere $S(W)$. $M(W)$ is a G-space via

$$g\colon f \to gfg^{-1}$$

for f in $M(W)$ and g in G [P3].

Let R_- denote the real one dimensional nontrivial representation of K/H. (Recall that K/H is isomorphic to Z_2.)

LEMMA 3.2. Θ_1^H *is in* $\omega^0_{K/H}(Y^H, S(R_-))$ *and* $2^{2n}\Theta_1^H = 0$.

Proof. Since Θ_1 is a K-fiber homotopy equivalence, the fiber degree of Θ_1^A is 1 for every subgroup A of K. $S(R_-)$ is contained in Y^H because R_- is a subrepresentation of V'^H. Θ_1^H restricted on the fibers over $S(R_-)$ is K/H-homotopic to the identity, i.e. it lies in $\omega^0_{K/H}(Y^H, S(R_-))$. The second part follows from the Mayer-Vietoris sequence arising from the decomposition of Y^H into the union of upper and lower hemisperes

$$\begin{aligned}\to \omega^{-1}_{K/H}(S(V'^H), S(R_-)) \to \omega^0_{K/H}(Y^H, S(R_-)) \to \\ \to \omega^0_{K/H}(D_+(V'^H), S(R_-)) \oplus \omega^0_{K/H}(D_-(V'^H), S(R_-)) \to \\ \to \omega^0_{K/H}(S(V'^H), S(R_-)) \to\end{aligned}$$

and the fact that

$$\omega^0_{K/H}(S(R_-)) = \omega^{-1}(\text{point}) = \pi_{i+1}(S^i) = Z_2$$

and that every element in

$$\omega^{-1}_{K/H}(S(V'^H), S(R_-))$$

is killed by 2^{n-1} (Lemma 3.9 **[P2]**). □

By previous two lemmas, we may assume $E'^H_+ = E'^H_-$ and $\Theta\colon E'^H_+ \to E'^H_-$ is the identity by replacing E' by $2^{3n}E'$ and Θ by $2^{3n}\Theta$.

Now it follows easily from the definition of the induction that

$$(\operatorname{ind}^G_K E'_+)^H = (\operatorname{ind}^G_K G'_-)^H$$

and $(\operatorname{ind}^G_K \Theta)^H$ is the identity map.

The normal map (X, f) rel. C is G-normally cobordant rel. C to (X', f') rel. $C \cup \{1\}$ if and only if the obstruction $\sigma_1(f)L^h_m(G, w)$ vanishes **[P2]**. Here $m = \dim X$ and $w\colon G \to Z_2$ is trivial. By Theorem A of **[HM]** reviewed in §2, the group $L^h_m(G, w)$ is torsion free, and $\sigma_1(f)$ is detected by the map Sign, i.e.

$$\sigma_1(f) = 0 \quad \text{if and only if } \operatorname{Sign}(\sigma_1(f)) = 0.$$

LEMMA 3.3. $\operatorname{Sign}(\sigma_1(f)) = 0$.

Proof. By the definition of the induction on equivariant vector bundles, $\operatorname{res}_H E' = 0$ implies $\operatorname{res}_M E = 0$ and $\operatorname{res}_L E = 0$, where $L = \langle H, y\rangle$, $M = \langle H, xy\rangle$, and $E = \operatorname{ind}^G_K E'$. Since

$$TX = f^*(TY + F) \quad \text{and} \quad \operatorname{res}_L F = \operatorname{res}_M F = 0.$$

$\operatorname{Sign}(g, X) = \operatorname{Sign}(g, Y)$ for g in $L \cup M$.

If g is not in $L \cup M$, g must be an odd power of x, hence it generates K. Since $Y^g = Y^K = \{p, q\}$ and both T_pX and T_qX contain t^d,

$$\operatorname{Sign}(g, X) = (T_pX)(g) + (T_qX)(g) = 0.$$

The number $(T_pX)(g)$ is given by

$$\prod_s \left(\frac{1+t^s}{1-t^s}\right)^{m^s}$$

when $T_pX = \sum m_s t^s$.

THEOREM 3.4. *Let G be the dihedral group D_{2d}, $2d = 2^k$, K the cyclic subgroup of index 2, and H the index 2 subgroup of K. Let V' and W' be representations of K such that*

(3.4) *V' satisfies* (3.1) *and $z' = V' - W'$ belongs to $2^{\Delta'}I(H)$ where $I(H) = \mathrm{Ker}(\mathrm{res}_H \times \mathrm{fix}_H)$,*

$$\Delta' = \Delta + 3\dim_R V'^H \quad \textit{and} \quad \Delta = \Delta(I, V')$$

defined in (3.3).

If $V = \mathrm{ind}_K^G V'$ is stable, then V and $W = \mathrm{ind}_K^G W'$ are s-Smith equivalent.

Proof. By Theorem 3.1 and the following lemmas, there exists a normal map (X, f, b) such that $X^H = Y^H$, $f^H : X^H \to Y^H$ is the identity, and $T_pX = V$, $T_qX = W$. It is obvious that $\mathrm{Iso}(Y) = \mathrm{Iso}(V)$ is contained in $C \cup \{1\}$, where C is a closed family of subgroups of G which contain H. Since $\sigma_1(f)$ vanishes (Lemma 3.3), the normal map (X, f) is cobordant rel. C to a normal map (X', f') rel. $C \cup \{1\}$, i.e. a G-homotopy equivalence $f' : X' \to Y$. □

When the order 2^{k+1} of the group G is sufficiently large, then there are representations V' and W' of K satisfying the conditions of Theorem 3.4 such that the induced representations V and W are distinct. Hence our main result Theorem A given in §1. The dimension of V (and of W) and the order of G are large due to the divisibility condition in (3.4) involving Δ which depends upon the order of G and the representation.

The following examples of nonisomorphic s-Smith equivalent representations of D_{2^k} are some of the simplest kinds that can be found by our method.

EXAMPLE. 1. Let $G = D_{2d}$, $d = 2^{10}$. (Note $|G| = 2^{12}$.)
Let

$$V' = dt^r + t^d,$$
$$W' = dt^{d+r} + t^d, \quad \text{where } r \text{ is odd.}$$

Then the induced representations

$$V = d\,\mathrm{ind}_K^G t^r + \mathrm{ind}_K^G t^d,$$
$$W = d\,\mathrm{ind}_K^G t^{d+r} + \mathrm{ind}_K^G t^d$$

are s-Smith equivalent. Obviously, V and W are not isomorphic. Let $T^r = \mathrm{ind}_K^G t^r$ and $A + B = \mathrm{ind}_K^G t^d$.

In matrix form they are as follows

$$T^r(x) = \begin{pmatrix} e^{2\pi ri/2d} & 0 \\ 0 & e^{-2\pi ri/2d} \end{pmatrix},$$
$$T^r(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$A(x) = A(y) = -1 \quad \text{and} \quad B(x) = -1 \quad B(y) = 1.$$

EXAMPLE 2. Let $G = D_{2d}$, $d = 2^{11}$.

Let

$$V' = 2^{10}(t^r + t^s) + t^d,$$
$$W' = 2^{10}(t^{r+d} + t^{s+d}) + t^d$$

where r and s are odd. Then the induced representations

$$V = 2^{10}(T^r + T^s) + A + B,$$
$$W = 2^{10}(T^{r+d} + T^{s+d}) + A + B$$

are s-Smith equivalent and $V \neq W$.

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