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THE n-DIMENSIONAL ANALOGUE OF THE CATENARY: EXISTENCE AND NON-EXISTENCE

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We study "heavy" n-dimensional surfaces suspended from some prescribed $(n-1)$-dimensional boundary data. This leads to a mean curvature type equation with a non-monotone right hand side. We show that the equation has no solution if the boundary data are too small, and, using a fixed point argument, that the problem always has a smooth solution for sufficiently large boundary data.

Consider a material surface M of constant mass density which is suspended from an $(n-1)$-dimensional surface Γ in $\mathbb{R}^n \times \mathbb{R}^+$, $\mathbb{R}^+ = \{t > 0\}$, and hangs under its own weight. If M is given as graph of a regular function $u\colon \Omega \to \mathbb{R}^+$ on a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, then u provides an equilibrium for the potential energy $\mathscr{E}$ under gravitational forces,

$$\mathscr{E}(u) = \int_\Omega u\sqrt{1+|Du|^2}.$$

Thus u solves the Dirichlet problem

$$(1)\qquad \begin{aligned} \operatorname{div}\left\{\frac{u \cdot Du}{\sqrt{1+|Du|^2}}\right\} &= \sqrt{1+|Du|^2} &&\text{in } \Omega,\\ u &= \varphi &&\text{on } \partial\Omega \end{aligned}$$

The corresponding variational problem

$$(2)\qquad \int_\Omega u\sqrt{1+|Du|^2} + \frac{1}{2}\int_{\partial\Omega} |u^2 - \varphi^2|\, d\mathscr{H}_{n-1} \to \min$$

in the class

$$BV_2^+(\Omega) := \{u \in L_2(\Omega)\colon u \geq 0,\ u^2 \in BV(\Omega)\}$$

has been solved by Bemelmans and Dierkes in **[BD]**. It was shown in [**BD**, Theorem 7] that the coincidence set $\{u = 0\}$ of a minimizer u is non-empty provided that

$$(3)\qquad |\varphi|_{\infty,\partial\Omega} < \frac{|\Omega|}{\mathscr{H}_{n-1}(\partial\Omega)},$$

where $|\Omega|$ denotes the Lebesgue measure of Ω and $\mathscr{H}_n$ denotes n-dimensional Hausdorff measure.

We want to show here that (1) has *no* solution in case (3) holds, whereas (1) has *always* a solution for sufficiently large boundary data. More precisely we prove the following existence–non-existence result.

THEOREM. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain of class $C^{2,\alpha}$, $\alpha > 0$, with non-negative (inward) mean curvature. Suppose $\varphi \in C^{2,\alpha}(\overline{\Omega})$ satisfies*

$$k_0 := \inf_{\partial\Omega} \varphi \geq c(n)\left(1 + \sqrt{2^{n+1}}\right)^2 |\Omega|^{1/n}, \tag{4}$$

where $c(n) = n^{-1}\omega_n^{-1/n}$ is the isoperimetric constant. Then the Dirichlet problem (1) has a global regular solution $u \in C^{2,\alpha}(\overline{\Omega})$. Moreover, if $u \in C^{0,1}(\overline{\Omega})$ is a weak positive solution of (1) with Lipschitz constant L, then we have

$$h := \sup_{\partial\Omega} \varphi \geq (1 + L^{-2})^{1/2} \frac{|A|}{\mathscr{H}_{n-1}(\partial A)} \tag{5}$$

for every Caccioppoli set $A \subset \Omega$.

Since $c(n)$ is the isoperimetric constant, we have

$$c(n)|\Omega|^{1/n} \geq \frac{|\Omega|}{\mathscr{H}_{n-1}(\partial\Omega)}$$

and therefore it is an interesting question whether our existence result remains true if we replace (4) with an inequality of the form

$$k_0 \geq \text{const.} \frac{|\Omega|}{\mathscr{H}_{n-1}(\partial\Omega)}.$$

The proof of the theorem is based on a priori bounds for solutions to the related problem

$$\Delta u - \frac{D_i u D_j u}{1 + |Du|^2} D_i D_j u = f^{-1},$$

which enable us to apply a fixed point argument. Notice that the operator

$$\Delta - \frac{D_i u D_j u}{1 + |Du|^2} D_i D_j = (1 + |Du|^2) \cdot \Delta_M$$

where Δ_M is the Laplace-Beltrami operator on $M = \operatorname{graph} u$.

Let us make some comments on the literature. For two dimensional parametric surfaces in $\mathbb{R}^3$ the existence problem has been investigated by Böhme, Hildebrandt and Tausch **[BHT]**. To our knowledge the first existence result for the Dirichlet problem (1), in case $n = 2$, is due to Dierkes **[D1]**. The variational problem (2) is solved in **[BD]**. It is shown in **[D2]** that minima u of (2) are regular up to the boundary provided only the boundary is mean curvature convex. A non-existence result of a different type has been proved by J. C. C. Nitsche in **[N]**.

Proof. We consider regular solutions $u_f \in C^{2,\alpha}(\overline{\Omega})$ of the related problem

$$(6) \qquad \sqrt{1+|Du|^2}\operatorname{div}\frac{Du}{\sqrt{1+|Du|^2}} = f^{-1} \quad \text{in } \Omega, \qquad u = \varphi \quad \text{on } \partial\Omega,$$

where $f \in C^{1,\alpha}(\overline{\Omega})$ and $0 < d \le f$. As a first step we establish a priori bounds for $\sup_\Omega u$ and $\inf_\Omega u$.

LEMMA. *Let $u_f \in C^{2,\alpha}(\overline{\Omega})$ be a solution to the Dirichlet problem* (6). *If*

$$f \ge d \ge \left(1+\sqrt{2^{n+1}}\right) c(n)|\Omega|^{1/n}$$

and

$$k_0 = \inf\nolimits_{\partial\Omega}\varphi \ge \left(1+\sqrt{2^{n+1}}\right)^2 c(n)|\Omega|^{1/n},$$

then we have $h \ge u_f \ge d$.

Proof of the Lemma. The first inequality follows immediately from the maximum principle since f is positive. To prove the second relation we chose $\delta \ge -k_0$ and put $w = \min(u+\delta, 0)$, $A(\delta) = \{x \in \Omega\colon u < -\delta\}$. Multiplying (6) with w, integrating by parts and using $w|_{\partial\Omega} = 0$, we obtain

$$\int_\Omega \frac{|Dw|^2}{\sqrt{1+|Dw|^2}} = \int_{A(\delta)} \frac{|w|}{f\sqrt{1+|Du|^2}}, \quad \text{hence}$$
$$\int_\Omega |Dw| \le |A(\delta)| + d^{-1}\int_{A(\delta)} |w|.$$

We use Sobolev's inequality on the left and Hölder's inequality on the right hand side and get with $c(n) = n^{-1}\omega_n^{-1/n}$

$$|w|_{n/n-1}\cdot\{c^{-1}(n) - d^{-1}|\Omega|^{1/n}\} \le |A(\delta)|,$$

where $|w|_{n/n-1}$ stands for the $L_{n/n-1}$-norm of w. Another application of Hölder's inequality yields

$$(\delta_1 - \delta_2)|A(\delta_1)| \le \left\{ \frac{c(n)d}{d - c(n)|\Omega|^{1/n}} \right\} |A(\delta_2)|^{1+1/n}$$

for all $\delta_1 \ge \delta_2 \ge -k_0$. In view of a well-known lemma due to Stampacchia, **[St**, Lemma 4.1], this is easily seen to imply

$$|A(-k_0 + 2^{n+1} \cdot c_1 |A(-k_0)|^{1/n})| = 0, \quad \text{where}$$
$$c_1 = \frac{c(n)d}{d - c(n)|\Omega|^{1/n}}.$$

Clearly this means that

$$u \ge k_0 - \frac{2^{n+1} d c(n)|\Omega|^{1/n}}{d - c(n)|\Omega|^{1/n}}.$$

Since $k_0 \ge (1 + \sqrt{2^{n+1}})d$ and $d \ge (1 + \sqrt{2^{n+1}})c(n)|\Omega|^{1/n}$ we finally obtain $u \ge d$. □

To derive a gradient estimate at the boundary, we rewrite (6) into

$$(7) \qquad (1 + |Du|^2)\Delta u - D_i u D_j u D_i D_j u = f^{-1}(1 + |Du|^2).$$

We can then apply the results of Serrin **[Se1]**, see also **[GT**, Chapter 14.3]. Equation (7) satisfies the structure condition (14.41) in **[GT]** and the RHS is $\mathscr{O}(|Du|^2)$. So we obtain a gradient estimate on the boundary which is independent of $|Df|$:

$$\sup_{\partial\Omega} |Du_f| \le c_2 = c_2(n, \Omega, h, |\varphi|_{2,\Omega}),$$

provided only that $\partial\Omega$ has non-negative (inward) mean curvature.

It is not possible to derive interior gradient estimates independent of $|Df|$, but we can prove

$$(8) \qquad \sup_{\Omega} |Du_f| \le \max \left\{ 2, \frac{1}{4} \sup_{\Omega} |Df|, 2e^{4(hd^{-1}-1)} \sup_{\partial\Omega} |Du_f| \right\},$$

which will be sufficient for our fixed point argument. Estimate (8) can be obtained from a careful analysis of the structure conditions in **[GT**, Chapter 15]. Here we present a selfcontained proof, using the geometric nature of equation (6). For a similar procedure we refer to **[K]**.

In the following let $v = (1 + |Du|^2)^{1/2}$ and denote by H and Δ the mean curvature and the Laplace-Beltrami operator on $M = \operatorname{graph} u$ respectively. Then equation (6) takes the form

$$v^2 \Delta u = f^{-1} \Leftrightarrow H = f^{-1} v^{-1}. \tag{9}$$

Let $\tau_1, \tau_2, \dots, \tau_n, \nu$ be an adapted local orthonormal frame on M, such that ν is the upper unit normal and

$$\nabla_i \nu = -h_{il} \tau_l, \qquad \nabla_i \tau_j = h_{ij} \nu,$$

where ∇_i is the tangential derivative with respect to τ_i and h_{il} is the second fundamental form. Then we get for $v = (1 + |Du|^2)^{1/2} = \langle \nu, e_{n+1} \rangle^{-1}$ the Jacobi-Codazzi equation

$$\begin{aligned} \Delta v &= \nabla_i \nabla_i \langle \nu, e_{n+1} \rangle^{-1} = \nabla_i (v^2 \langle h_{il} \tau_l, e_{n+1} \rangle) \\ &= |A|^2 v + 2v^{-1} |\nabla v|^2 + v^2 \langle \nabla H, e_{n+1} \rangle, \end{aligned}$$

where $|A|^2 = h_{il} h^{il}$. Now (9) implies

$$\Delta v = |A|^2 v + 2v^{-1} |\nabla v|^2 - f^{-2} v \langle \nabla f, e_{n+1} \rangle - f^{-1} \langle \nabla v, e_{n+1} \rangle. \tag{10}$$

If we now extend all functions from M to $\mathbb{R}^{n+1}$ by

$$f(\hat{x}, x_{n+1}) = f(\hat{x})$$

such that

$$\begin{gathered} \nabla f = Df - \nu \langle Df, \nu \rangle, D_{n+1} f = 0 \quad \text{and} \\ \langle \nabla f, e_{n+1} \rangle = -v^{-1} \langle Df, \nu \rangle \end{gathered} \tag{11}$$

then we derive from (10) and (11)

$$\Delta v \geq 2v^{-1} |\nabla v|^2 - f^{-1} \langle \nabla v, e_{n+1} \rangle - f^{-2} |Df|. \tag{12}$$

Next we compute for $\alpha > 0$ and $g = e^{\alpha u} \cdot v$ the inequality

$$\begin{aligned} \Delta g \geq e^{\alpha u} \{ & 2v^{-1} |\nabla v|^2 - f^{-1} \langle \nabla v, e_{n+1} \rangle - f^{-2} |Df| \\ & + 2\alpha \nabla_i v \nabla_i u + \alpha v \Delta u + v \alpha^2 |\nabla u|^2 \}. \end{aligned}$$

Using again the equation (9) and

$$\nabla_i g = \nabla_i v e^{\alpha u} + \alpha v e^{\alpha u} \nabla_i u$$

we obtain

$$\begin{aligned} \Delta g \geq {} & 2v^{-1} \nabla_i v \nabla_i g - f^{-1} \langle \nabla g, e_{n+1} \rangle + \alpha f^{-1} e^{\alpha u} v \langle \nabla u, e_{n+1} \rangle \\ & - f^{-2} |Df| e^{\alpha u} + \{ v^{-1} \alpha f^{-1} + v \alpha^2 |\nabla u|^2 \} e^{\alpha u}. \end{aligned}$$

In view of relation (11) we finally conclude

$$\Delta g \geq 2v^{-1}\nabla_i v \nabla_i g - f^{-1}\langle \nabla g, e_{n+1}\rangle + \{\alpha^2|\nabla u|^2 - \alpha f^{-1} - v^{-1}f^{-2}|Df|\}g$$

Now let again $d \leq f \leq h$ and choose $\alpha = 4d^{-1}$. Then, since

$$|\nabla u|^2 = \frac{|Du|^2}{1+|Du|^2} \geq \frac{1}{2} \quad \text{for } |Du| \geq 1,$$

we see that g cannot have an interior maximum if

$$v \geq \max\left\{2, \tfrac{1}{4}\sup_{\Omega}|Df|\right\}.$$

Therefore we get the estimate

$$\sup_{\Omega} v \leq \max\left\{2, \tfrac{1}{4}\sup_{\Omega}|Df|, e^{4(hd^{-1}-1)}\sup_{\partial\Omega} v\right\}$$

yielding (8).

To prove existence of a solution to equation (1) we now define the set

$$\mathscr{M} := \left\{f \in C^{1,\alpha}(\overline{\Omega}) \colon d \leq f \leq h, \sup_{\Omega}|Df| \leq M\right\}$$

for $M > 0$ large and consider the operator

$$\begin{aligned} T\colon \mathscr{M} &\to C^{1,\alpha}(\overline{\Omega}), \\ f &\to u_f. \end{aligned}$$

In view of our estimates for u_f and $|Du_f|$ we may then choose M so large that

$$T(\mathscr{M}) \subset \mathscr{M}.$$

Moreover, standard theory ensures that T is continuous and $T(\mathscr{M})$ is precompact. So we can apply Schauder's fixed point theorem, see e.g. ([**GT**], Cor. 11.2) to obtain the existence of a regular $u \in C^{2,\alpha}(\overline{\Omega})$ satisfying (1).

To prove the necessary conditions (5) we proceed similarly as in [**G**]. To this end let $A \Subset \Omega$ have finite perimeter $\mathbb{M}(\partial A)$. There exists a sequence of positive functions $\varphi_k \in C_c^1(\Omega)$ such that $\varphi_k \to \varphi_A$ in $L_1(\Omega)$, and

$$\int_{\Omega} |D\varphi_k| \to \mathbb{M}(\partial A),$$

where φ_A denotes the characteristic function of the set A.

We test (1) with φ_k and integrate,

$$\int_{\Omega}\left\{\frac{uDuD\varphi_k}{\sqrt{1+Du|^2}} + \varphi_k\sqrt{1+|Du|^2}\right\} dx = 0. \tag{13}$$

Now, since $u \in \mathrm{Lip}(\overline{\Omega})$ it follows from standard regularity theory that $u \in C^\infty(\Omega)$ and therefore

$$\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} \geq 0 \quad \text{on } \Omega, \text{ whence } u \leq h.$$

Using this in (13) we get

$$\int_\Omega \varphi_k \, dx \leq \frac{h \cdot L}{\sqrt{1+L^2}} \int_\Omega |D\varphi_k|$$

and, letting $k \to \infty$,

$$|A| \leq \frac{h \cdot L}{\sqrt{1+L^2}} \mathbb{M}(\partial A), \quad \text{or}$$
$$h \geq \{1 + L^{-2}\}^{1/2} \frac{|A|}{\mathbb{M}(\partial A)}.$$

The general case follows by an approximation argument, using the fact that

$$\mathbb{M}(\partial[A \cap \Omega_\varepsilon]) \to \mathbb{M}(\partial A) \quad \text{as } \varepsilon \to 0,$$

where

$$\Omega_\varepsilon := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

This completes the proof of the theorem.

Remark. With the same method we could as well deal with the integral

$$\int_\Omega u^\gamma \sqrt{1+|Du|^2}, \qquad \gamma > 0,$$

the Euler equation of which is given by

$$\operatorname{div} \frac{Du}{\sqrt{1+|Du|^2}} = \frac{\gamma}{u\sqrt{1+|Du|^2}}.$$

Clearly, in this case the constants appearing in the theorem would depend on γ too, however we shall not dwell on this.

References

[BD] J. Bemelmans and U. Dierkes, *On a singular variational integral with linear growth* I: *existence and regularity of minimizers*, Arch. Rat. Mech. Anal., **100** (1987), 83–103.

[BHT] R. Böhme, S. Hildebrandt and E. Tausch, *The two dimensional analogue of the catenary*, Pacific J. Math., **88** (1980), 247–278.

[D1] U. Dierkes, *A geometric maximum principle, Plateau's problem for surfaces of prescribed mean curvature, and the two-dimensional analogue of the catenary*, in: *Partial Differential Equations and Calculus of Variations*, pp. 116–141, Springer Lecture Notes in Mathematics **1357** (1988).

[D2] ——, *Boundary regularity for solutions of a singular variational problem with linear growth*, Arch. Rat. Mech. Anal., **105** (1989), 286–298.

[G] M. Giaquinta, *On the Dirichlet problem for surfaces of prescribed mean curvature*, Manuscripta Math., **12** (1974), 73–86.

[GT] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin-Heidelberg-New York, 1977.

[K] N. Korevaar, *An easy proof of the interior gradient bound for solutions to the prescribed mean curvature equation*, Proc. Symp. Pure Math., **45** (1986), 81–89.

[LU] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Local estimates for the gradients of solutions of non-uniformly elliptic and parabolic equations*, CPA M, **23** (1970), 677–703.

[N] J. C. C. Nitsche, *A non-existence theorem for the two-dimensional analogue of the catenary*, Analysis, **6** No. 2/3 (1986), 143–156.

[SE1] J. Serrin, *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Phil. Trans. Roy. Soc. London, Ser. A, **264** (1969), 413–496.

[SE2] *Gradient estimates for solutions of nonlinear elliptic and parabolic equations*, in: *Contributions to nonlinear functional analysis*, p. 565–602. Academic Press, New York 1971.

[St] G. Stampacchia, *Equations elliptiques du second ordre à coefficients discontinus*, Les Presse de l'Université, Montréal 1966.

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