PACIFIC JOURNAL OF MATHEMATICS
Vol. 142, No. 1, 1990

UNITARY BORDISM OF CLASSIFYING SPACES OF QUATERNION GROUPS

ABDESLAM MESNAOUI

Let Γ_k be the generalized quaternion group of order 2^k. In this article we determine a set of generators for the $U_*(pt)$-module $\widetilde{U}_*(B\Gamma_k)$ and give all linear relations between them. Moreover their orders are calculated.

0. Introduction. In this article we first study the case $\Gamma_k = \Gamma$ the quaternion group of order 8. We recall that

$$\Gamma = \{\pm 1, \pm i, \pm j, \pm k\}, \qquad i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = ij.$$

Γ acts on S^{4n-3} by using $(n+1)\eta$ where η denotes the following unitary irreducible representation of Γ: $i \to \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $j \to \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and we get the element $w_{4n+3} = [S^{4n+3}/\Gamma, q] \in \widetilde{U}_{4n+3}(B\Gamma)$, q being the natural embedding: $S^{4n+3}/\Gamma \subset B\Gamma$. In **[6]** we have defined three elements of $\widetilde{U}^2(B\Gamma)$ denoted by A, B, C as Euler classes for MU of irreducible representations of Γ of dimension 1 over $\mathbb{C}$. Let $u_{4n+1} \in \widetilde{U}_{4n+1}(B\Gamma)$, $v_{4n+1} \in \widetilde{U}_{4n+1}(B\Gamma)$ be respectively $A \cap w_{4n+3}$ and $B \cap w_{4n+3}$. Our first result is:

THEOREM 2.2. *The set $\{u_{4n+1}, v_{4n+1}, w_{4n+3}\}_{n\geq 0}$ is a system of generators for the $U_*(pt)$-module $\widetilde{U}_*(B\Gamma)$.*
Their orders are given by:

THEOREM 2.6. *We have*: $\operatorname{ord} w_{4n+3} = 2^{2n+3}$.

THEOREM 2.8. *We have*: $\operatorname{ord} u_{4n+1} = \operatorname{ord} v_{4n+1} = 2^{n+1}$.

Now let Ω_* be $U^*(pt)[[Z]]$ graded by taking $\dim Z = 4$. If $P(Z) = \sum_{i\geq r} \alpha_i Z^i \in \Omega_n$ and $\alpha_r \neq 0$ then we denote $\nu(P) = 4r$. Let W, V_1, V_2 be the submodules of $\widetilde{U}_*(B\Gamma)$ generated respectively by $\{w_{4n+3}\}_{n\geq 0}$, $\{u_{4n+1}\}_{n\geq 0}$, $\{v_{4n+1}\}_{n\geq 0}$. The following result gives the $U_*(pt)$-module structure of $\widetilde{U}_*(B\Gamma)$ and uses the elements $T(Z) \in \Omega_4$, $J(Z) \in \Omega_0$ as defined in **[6]**, Section II.

THEOREM 2.4. (a) $\widetilde{U}_*(B\Gamma) = W \oplus V_1 \oplus V_2$.

(b) *In* $\widetilde{U}_{2p+1}(B\Gamma)$ *we have* $0 = a_0 w_3 + a_1 w_7 + \cdots + a_n w_{4n+3} = b_0 u_1 + \cdots + b_m u_{4m+1}$ *iff there are homogeneous polynomials* $M(Z), M_2(Z)$ *and homogeneous formal power series* $N(Z), N_1(Z)$ *of* Ω_* *satisfying*: $b_m Z + b_{m-1} Z^2 + \cdots + b_0 Z^{m+1} = M(Z)(2 + J(Z)) + N(Z)$, $a_n Z + a_{n-1} Z^2 + \cdots + a_0 Z^{n+1} = M_1(Z) T(Z) + N_1(Z)$, $\nu(N) > 4(n+1)$, $\nu(N_1) > 4(n+1)$. *Moreover* $b_0 u_1 + \cdots + b_m u_{4m+1} = 0$ *iff* $b_0 v_1 + \cdots + b_m v_{4m+1} = 0$.

In Section III we consider $\widetilde{U}_*(B\Gamma_k)$, $k \geq 4$. The generalized quaternion group Γ_k is generated by u, v with $u^t = v^2$, $t = 2^{k-2}$, $uvu = v$. Γ_k acts on S^{4n+3} by means of the irreducible unitary representation η_1 of Γ_k:

$$u \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad v \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

ω being a primitive 2^{k-1}th root of unity. We get:

$$w'_{4n+3} = [S^{4n+3}/\Gamma_k, q'] \in \widetilde{U}_{4n+3}(B\Gamma_k), \quad q' \colon S^{4n+3}/\Gamma_k \subset B\Gamma_k.$$

Now we use the elements $B'_k = B_k + G_k(D_k) \in \widetilde{U}^2(B\Gamma_k)$, $C'_k = C_k + G_k(D_k) \in \widetilde{U}^2(B\Gamma)$ (see **[6]**, Theorem 3.14) to define $u'_{4n+1} = B'_k \cap w'_{4n+3} \in \widetilde{U}_{4n+1}(B\Gamma_k)$, $v'_{4n+1} = C'_k \cap w'_{4n+3} \in \widetilde{U}_{4n+1}(B\Gamma_k)$. Then we have Theorems 3.1, 3.2 identical respectively to the above Theorems 2.2, 2.4 where $w_{4n+3}, u_{4n+1}, v_{4n+1}$ are replaced by $w'_{4n+3}, u'_{4n+1}, v'_{4n+1}$. However:

THEOREM 3.4. *We have*: $\operatorname{ord} w'_{4n+3} = 2^{2n+k}$, $n \geq 0$.

THEOREM 3.5. *We have*: $\operatorname{ord} u'_{4n+1} = \operatorname{ord} v'_{4n+1} = 2^{n+1}$, $n \geq 0$, *which are therefore independent of* k.

The layout is as follows:

I Preliminaries and notations.
II Calculations in $\widetilde{U}_*(B\Gamma)$: generators, orders and relations.
III $\widetilde{U}_*(B\Gamma_k)$, $k \geq 4$: generators, orders and relations.

We assume that the reader is acquainted with the notations and results of **[6]**.

I. Preliminaries and notations. The notation U_*-AHSS will be used for the Atiyah-Hirzebruch spectral sequence corresponding to the homology theory determined by MU; μ and μ' denote the edge homomorphisms $U^*(X) \to H^*(X)$ and $U_*(X) \to H_*(X)$ obtained from the

U_*-AHSS for a CW complex X. We have the following well-known result:

THEOREM 1.1. *Suppose X a CW-complex such that:*

(a) *The U_*-AHSS for X collapses.*

(b) *For each $n \geq 0$ there is a system (a_{in}) generating the group $H_n(X)$.*

Then for each $n \geq 0$ there is a system (A_{in}) such that:

(a) *$A_{in} \in U_n(X)$, $\mu'(A_{in}) = a_{in}$ for every (i, n).*

(b) *The system (A_{in}) generates $U_*(X)$ as a $U_*(pt)$-module.*

Moreover, (b) *is valid for every system (A_{in}) such that $\mu'(A_{in}) = a_{in}$.* □

Consider the map of ring spectra $f\colon MU \to H$ (see **[1]**); by naturality of spectral sequences it follows that if X is a CW-complex then $f^{\#}(X) = \mu$ and $f_{\#}(X) = \mu'$ where $f^{\#}(X)\colon U^*(X) \to H^*(X)$, $f_{\#}(X)\colon U_*(X) \to H_*(X)$ denote the maps induced by f.

PROPOSITION 1.2. *If X is a CW-complex then the following diagram commutes:*

$$\begin{array}{ccc} U^m(X) \otimes U_n(X) & \xrightarrow{\cap} & U_{n-m}(X) \\ {\scriptstyle \mu\otimes\mu'}\downarrow & & \downarrow{\scriptstyle \mu'} \\ H^m(X) \otimes H_n(X) & \xrightarrow{\cap} & H_{n-m}(X) \end{array} \text{ commutes.}$$

Proof. Take $E = MU$. The cap product is the composite:

$$\widetilde{E}_m(X^+) \otimes \widetilde{E}_n(X^+) \xrightarrow{1\otimes\Delta_*} \widetilde{E}^m(X^+) \otimes \widetilde{E}_n(X^+ \wedge X^+) \xrightarrow{\ \backslash\ } \widetilde{E}_{n-m}(X^+),$$

$\backslash$ being the slant product and $\Delta(x) = [x, x]$. Since Δ_* commutes with $f_{\#}(-)$ we have to prove that the diagram:

$$\begin{array}{ccc} \widetilde{E}^m(X^+) \otimes \widetilde{E}_n(X^+ \wedge X^+) & \xrightarrow{\ \backslash\ } & \widetilde{E}_{n-m}(X^+) \\ \downarrow{\scriptstyle f^{\#}(-)\otimes f_{\#}(-)} & & \downarrow{\scriptstyle f_{\#}(-)} \\ \widetilde{H}^m(X^+) \otimes \widetilde{H}_n(X^+ \wedge X^+) & \xrightarrow{\ \backslash\ } & \widetilde{H}_{n-m}(X^+) \end{array} \text{ commutes.}$$

More generally the diagram

$$\begin{array}{ccc} \widetilde{E}^m(Y) \otimes \widetilde{E}_n(Y \wedge Z) & \xrightarrow{\ \backslash\ } & \widetilde{E}_{n-m}(Z) \\ \downarrow{\scriptstyle f^{\#}(-)\otimes f_{\#}(-)} & & \downarrow{\scriptstyle f_{\#}(-)} \\ \widetilde{H}^m(Y) \otimes \widetilde{H}_n(Y \wedge Z) & \xrightarrow{\ \backslash\ } & \widetilde{H}_{n-m}(Z) \end{array} \text{ commutes if } Y, Z$$

are pointed CW-complexes: indeed let x and y be any elements of $\widetilde{E}^m(Y)$ and $\widetilde{E}_n(Y \wedge Z)$ respectively represented by $g\colon Y \to \sum^m E$, $h\colon S^n \to E \wedge Y \wedge Z$. Then $f^{\#}(-)(x)$ is represented by the composite

$$g_1\colon Y \xrightarrow{g} \textstyle\sum^m E \xrightarrow{\sum^m f} \sum^m H \quad \text{and} \quad f_{\#}(-)(y)$$

by the composite:

$$h_1\colon S^n \xrightarrow{h} E \wedge Y \wedge Z \xrightarrow{f\wedge 1\wedge 1} H \wedge Y \wedge Z.$$

If we denote by T the transposition and k, k' the ring-spectra products then the diagram pictured on the next page commutes. Since the top line represents $x\backslash y$ and the bottom line

$$f^{\neq}(-)(x)\backslash f_{\neq}(-)(y)$$

we have $f_{\neq}(-)(x\backslash y) = f^{\neq}(-)(x)\backslash f_{\neq}(-)(y)$. □

Let X be any CW-complex and ξ a complex vector bundle of $\mathbb{C}$-dimension n over X. If h denotes a map: $X \to BU(n)$ classifying ξ and $M(\xi)$ the Thom space of ξ, then $M(h)\colon M(\xi) \to MU(n)$ determines an element $t_0(\xi) \in U^{2n}(M(\xi))$ which is a particular Thom class for ξ called the canonical Thom class for ξ. Moreover if $j\colon X \to M(\xi)$ is the zero section we have $j^*(t_0(\xi)) = cf_n(\xi)$, the highest Conner-Floyd characteristic class of ξ; $j^*(t_0(\xi))$ is also called the Euler class $e(\xi)$ of ξ.

Fundamental classes for a U-manifold M^n for $E = MU$ or H may be obtained in the following manner: M^n can be embedded in S^{n+2k} for some large k and the normal bundle τ can be given a $U(k)$-structure; let N be a tubular neighbourhood of M^n, which we identify with the total space of the normal disk bundle $D(\tau)$; we have the map $\pi\colon S^{n+2k} \to M(\tau)$ defined as follows: if $x \in N$ then $\pi(x)$ is the image of x by the projection $D(\tau) \to M(\tau)$ and if $x \in S^{n+2k} - \overset{\circ}{N}$, then $\pi(x) = *$ the base point of $M(\tau)$; let t be a Thom class of ξ for E; we have the Thom-isomorphism $\phi_t\colon E_{2k+r}(M(\tau)) \to E_r(M^n)$ such that $\phi_t(x) = p_*(t \cap x)$, p being the projection $D(\tau) \to M^n$; let $u\colon S^0 \to E$ be the unit of E; the map u is a map of spectra and is therefore a collection of maps $u_m\colon S^m \to E_m$ satisfying well-known axioms; then by **[8]**, page 333, if $[u_{n+2k}]$ is the element of $\widetilde{E}_{n+2k}(S^{n+2k})$ corresponding to u_{n+2k}, then the element $c(M) = \phi_t(\pi_*([u_{n+2k}])) \in E_n(M^n)$ is a fundamental class for M^n. Evidently the same method produces fundamental classes for the homology theory defined by the spectrum H.

$$
\begin{tikzcd}[column sep=small]
S^{n-m} \arrow[r, "\sum^{-m} h"] \arrow[d, equal] & \left(\sum^{-m} E\right) \wedge Y \wedge Z \arrow[r, "T\wedge 1"] \arrow[d, "\sum^{-m} f\wedge 1\wedge 1"'] & Y \wedge \sum^{-m} E \wedge Z \arrow[r, "g\wedge 1\wedge 1"] \arrow[d, "1\wedge \sum^{-m} f\wedge 1"'] & \sum^{m} E \wedge \sum^{-m} E \wedge Z \arrow[r, phantom, "\cong"] \arrow[d, "\sum^{m} f\wedge \sum^{-m} f\wedge 1"'] & E \wedge E \wedge Z \arrow[r, "k\wedge 1"] \arrow[d, "f\wedge f\wedge 1"'] & E \wedge Z \arrow[d, "f\wedge 1"'] \\
S^{n-m} \arrow[r, "\sum^{-m} h_1"] & \left(\sum^{-m} H\right) \wedge Y \wedge Z \arrow[r, "T\wedge 1"] & Y \wedge \sum^{-m} H \wedge Z \arrow[r, "g_1\wedge 1\wedge 1"] & \sum^{m} H \wedge \sum^{-m} H \wedge Z \arrow[r, phantom, "\cong"] & H \wedge H \wedge Z \arrow[r, "k'\wedge 1"] & H \wedge Z
\end{tikzcd}
$$

From **[8]**, page 335, §14-45, we have:

PROPOSITION 1.3. *If M^n is a closed U-manifold then $[M^n, 1] \in U_n(M^n) = E_n(M^n)$ is a fundamental class for M^n deduced from the canonical Thom class $t_0(\tau)$, τ being the normal bundle of an embedding $M^n \subset S^{n+2k}$, k large.* □

PROPOSITION 1.4. *Let M^n be a closed U-manifold; then*

$$f_\#(-)([M^n, 1]) \in H_n(M^n)$$

is a fundamental class for M^n.

Proof. From 1.3 we have

$$[M^n, 1] = \phi_{t_0}(\pi_*[u_{n+2k}]) = c(M);$$

then

$$\begin{aligned} f_\#(-)(c(M)) &= f_\#(-)[\phi_{t_0}(\pi_*([u_{n+2k}]))] = f_\#(-)[p_*(t_0 \cap \pi_*([u_{n+2k}]))] \\ &= p_*[f_\#(-)(t_0 \cap \pi_*([u_{n+2k}]))] \\ &= p_*[f^\#(-)(t_0) \cap f_\#(-)(\pi_*([u_{n+2k}]))] \\ &= p_*[f^\#(-)(t_0) \cap \pi_*(f(-)([u_{n+2k}]))]. \end{aligned}$$

Since f is a map of spectra the unit of H is the composite $v\colon S^0 \xrightarrow{u} MU \xrightarrow{f} H$ and hence $f_\#(-)([u_{n+2k}]) = [v_{n+2k}]$. Now $f^\#(-)(t_0)$ is a Thom class t_1 for H and therefore

$$\begin{aligned} f_\#(-)(c(M)) &= p_*[t_1 \cap \pi_*([v_{n+2k}])] \\ &= \phi_{t_1}(\pi_*([v_{n+2k}])) = c_1(M^n) \in H_n(M^n) \end{aligned}$$

is a fundamental class for M^n. □

The notation $c(M^n)$ will be for the fundamental class $[M^n, 1] \in U_n(M^n)$ and $c_1(M^n) \in H_n(M^n)$ will be the fundamental class $\mu'(c(M^n))$.

If PD or PD_1 denotes the Poincaré duality then we have:

PROPOSITION 1.5. *The following diagram commutes*

$$\begin{array}{ccc} U^m(M^n) & \xrightarrow{\text{PD}} & U_{n-m}(M^n) \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ H^m(M^n) & \xrightarrow{\text{PD}_1} & H_{n-m}(M^n) \end{array}$$

Proof. We have

$$\begin{aligned}\mu'(\mathrm{PD}(x)) &= \mu'(x \cap c(M^n)) = \mu(x) \cap \mu'(c(M^n)) \\ &= \mu(x) \cap c_1(M^n) = (\mathrm{PD})_1(\mu(x))\end{aligned}$$

by 1.2. □

Let N^m be a closed U-submanifold of a closed U-manifold M^n, and i the inclusion $N^m \subset M^n$; then the normal bundle τ of N^m in M^n is a complex-vector-bundle if $(n-m)$ is even and we have:

PROPOSITION 1.6. *If $(n-m)$ is even then $(\mathrm{PD})^{-1}([N^m, i])$ is represented by:*

$$M^n \to M^n/(M^n - \mathring{N}) \simeq D(\tau)/S(\tau) = M(\tau) \xrightarrow{M(h)} MU(\tfrac{1}{2}(n-m)),$$

where h is a map classifying τ and N a tubular neighborhood of N^m homeomorphic to $D(\tau)$ (see **[3]**, **[7]**). □

The generalized quaternion group Γ_k, $k \geq 4$, is generated by u, v subject to the relations $u^t = v^2$, $t = 2^{k-2}$, $uvu = v$. Consider the irreducible unitary representation η_1 of Γ_k: $u \to \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-1}\end{smallmatrix}\right)$, $v \to \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$, ω being a primitive 2^{k-1}th-root of unity. The group Γ_k acts on S^{4n+3} by means of $(n+1)\eta_1$ as a group of U-diffeomorphisms and we get a canonical U-structure on S^{4n+3}/Γ_k and a natural injection $S^{4n+3}/\Gamma_k \subset B\Gamma_k = \bigcup_{n\geq 0} S^{4n+3}/\Gamma_k$ (see **[3]**, **[10]**, page 508).

Let α be the complex vector bundle: $S^{4n+3} \times_{\Gamma_k} \mathbb{C}^2 \to S^{4n+3}/\Gamma_k$ where Γ_k acts on S^{4n+3} and $\mathbb{C}^2$ respectively by means of $(n+1)\eta_1$ and η_1: if $a \in \Gamma_k$ and $(x, v) \in S^{4n+3} \times \mathbb{C}^2$ we have $a(s, w) = (as, av) = (sa^{-1}, av)$ and $S^{4n+3} \times_{\Gamma_k} \mathbb{C}^2 = (S^{4n+3} \times C^2)/\Gamma_k$. Then by a result of R. H. Szczarba (**[9]**) we have $T(S^{4n+3}/\Gamma_k) + 1 = (n+1)\alpha$ where $T(S^{n+3}/\Gamma_k)$ denotes the tangent bundle of $S^{4n=3}/\Gamma_k$. As an easy consequence we have:

PROPOSITION 1.7. *If i denotes the embedding $S^{4n+3}/\Gamma_k \subset S^{4n+7}/\Gamma_k$ such that*

$$i([z_1, z_2, \ldots, z_{2n+2}]) = [z_1, z_2, \ldots, z_{2n+2}, 0, 0],$$

then the normal bundle of S^{4n+3}/Γ_k in S^{4n+7}/Γ_k is isomorphic to the complex vector bundle α. □

We shall give a proof of the next result which can be found in [7]:

PROPOSITION 1.8. *If i denotes the embedding $S^{4n+3}/\Gamma_k \subset S^{4n+7}/\Gamma_k$ then $i^* \circ (\mathrm{PD})^{-1}([S^{4n+3}/\Gamma_k, i]) = e(\alpha)$.*

Proof. Denote by τ the normal bundle of S^{4n+3}/Γ_k in S^{4n+7}/Γ_k and by h a classifying map: $S^{4n+3}/\Gamma_k \to BU(2)$ for τ. Then by 1.6, $(\mathrm{PD})^{-1}([S^{4n+3}/\Gamma_k, i])$ is represented by the composite:

$$\begin{aligned} S^{4n+7}/\Gamma_k &\to (S^{4n+7}/\Gamma_k) \,/\, (S^{4n+7}/\Gamma_k - \overset{\circ}{N}) \\ &\simeq \frac{D(\tau)}{S(\tau)} = M(\tau) \xrightarrow{M(h)} MU(2), \end{aligned}$$

N being a tubular neighbourhood of S^{4n+3}/Γ_k homeomorphic to $D(\tau)$. Since the composite:

$$\begin{aligned} S^{4n+3}/\Gamma_k \xrightarrow{i} S^{4n+7}/\Gamma_k &\to S^{4n+7}/\Gamma_k \,/\, (S^{4n+7}/\Gamma_k - \overset{\circ}{N}) \\ &\simeq \frac{D(\tau)}{S(\tau)} = M(\tau) \end{aligned}$$

is the zero section: $S^{4n+3}/\Gamma_k \to M(\tau)$, it follows that

$$i^* \circ (P(D)^{-1})([S^{4n+3}/\Gamma_k, i]) = e(\tau).$$

Since τ and α are isomorphic as complex vector bundles by 1.7 the proposition is proved. □

In Section III we shall use the following Euler classes for MU (see [6]):

$$\begin{aligned} A_k &= e(\xi_1) \in \widetilde{U}^2(B\Gamma_k), & B_k &= e(\xi_2) \in \widetilde{U}^2(B\Gamma_k), \\ C_k &= e(\xi_3) \in \widetilde{U}^2(B\Gamma_k), & D_k &= e(\eta_1) \in \widetilde{U}^4(B\Gamma_k) \end{aligned}$$

where $\xi_1, \xi_2, \xi_3, \eta_1$ are the complex vector bundles corresponding to the irreducible unitary representations $\xi_1 \colon u \to 1, v \to -1$, $\xi_2 \colon u \to -1, v \to 1$, $\xi_3 \colon k \to -1, v \to -1$ and η_1 as defined above.

In order to calculate $U_*(B\Gamma_k)$ we first consider the case $k = 3$: $\Gamma_3 = \Gamma$, the quaternion group of order 8. We recall that $\Gamma = \{\pm 1, \pm i, \pm j, \pm k\}$ subject to the relations $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$. The irreducible unitary representations of Γ are $1 \colon i \to 1, j \to 1$, $\xi_i \colon i \to 1, j \to -1$, $\xi_j \colon i \to -1, j \to 1$, $\xi_k \colon i \to -1, j \to -1$ and $\eta \colon i \to \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), j \to \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The character table of Γ is drawn on the next page.

The group Γ acts on S^{4n+3} by means of $(n+1)\eta$ as a group of U-diffeomorphisms; as with Γ_k we get a U-manifold $S^{4n+3}/\Gamma \subset B\Gamma = \bigcup_{n \geq 0} S^{4n+3}/\Gamma$. There will be no ambiguity if we use the same notation conjugacy classes

	1	-1	$\pm i$	$\pm j$	$\pm k$
1	1	1	1	1	1
ξ_i	1	1	1	1	-1
ξ_j	1	1	-1	1	-1
ξ_k	1	1	-1	-1	-1
η	2	2	0	0	0

α as for Γ_k for the complex vector bundle $S^{4n+3} \times_\Gamma \mathbb{C}^2 \to S^{4n+3}/\Gamma$. Evidently the Propositions 1.6 and 1.7 are valid if Γ_k is replaced by Γ.

In Section II the following Euler class for MU will be of fundamental importance (see [**6**]):

$$A = e(\xi_i) \in \widetilde{U}^2(B\Gamma), \qquad B = e(\xi_j) \in \widetilde{U}^2(B\Gamma),$$
$$C = e(\xi_k) \in \widetilde{U}^2(B\Gamma) \quad \text{and} \quad D = e(\eta) \in \widetilde{U}^4(B\Gamma).$$

II. Calculation of $\widetilde{U}_*(B\Gamma)$: generators, orders and relations. The reduced homology groups $\widetilde{H}_*(B\Gamma)$ are such that:

$$\widetilde{H}_{2n}(B\Gamma) = 0, \quad \widetilde{H}_{4n+1}(B\Gamma) = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \widetilde{H}_{4n+3}(B\Gamma) = \mathbb{Z}_8, \quad n \geq 0.$$

The $\widetilde{U}_*$-AHSS of $B\Gamma$ collapses and we have a filtration of $\widetilde{U}_n(B\Gamma)$:

$$J_{-1,n+1} = 0 \subset J_{0,n} \subset \cdots \subset J_{p,n-p} \subset \cdots \subset J_{n,0} = \widetilde{U}_n(B\Gamma)$$

with $J_{p,q} = \operatorname{Im}(\widetilde{U}_{p+q}(X^p) \to \widetilde{U}_{p+q}(B\Gamma))$, X^p being the p-skeleton of $B\Gamma$. Moreover $J_{p,q}/J_{p-1,q+1} = \widetilde{H}_p(B\Gamma, U_q(pt))$.

PROPOSITION 2.1. (a) $\widetilde{U}_{2n}(B\Gamma) = 0$, $\widetilde{U}_{2n+1}(B\Gamma) = U_{2n+1}(B\Gamma)$, $U_{2n}(B\Gamma) = U_{2n}(pt)$.

(b) $\operatorname{Ord}(\widetilde{U}_{4n+3}(B\Gamma)) = 2^r$,

$$r = 3\left(\sum_{i=0}^{n} \operatorname{Rank} U_{4i}(pt)\right) + 2\left(\sum_{i=0}^{n} \operatorname{Rank} U_{4i+2}(pt)\right); \qquad \operatorname{Ord}(\widetilde{U}_{4n+1}(B\Gamma)) = 2^s,$$

$$s = 3\left(\sum_{i=0}^{n-1} \operatorname{Rank} U_{4i+2}(pt)\right) + 2\left(\sum_{i=0}^{n} \operatorname{Rank} U_{4i}(pt)\right).$$

Proof. (a) From the filtration $J_{-1,2n+1} = 0 \subset J_{0,2n} \subset \cdots \subset J_{p,2n-p} \subset \cdots \subset J_{2n,0}$, and $J_{p,2n-p}/J_{p-1,2n-p+1} = H_p(B\Gamma, U_{2n-p}(pt)) = 0$ it follows that $\widetilde{U}_{2n}(B\Gamma) = 0$. Hence $U_{2n}(B\Gamma) = U_{2n}(pt)$ and $\widetilde{U}_{2n+1}(B\Gamma) = U_{2n+1}(B\Gamma)$ because $U_{2n+1}(pt) = 0$.

(b) The orders are easy consequences of:

$$\begin{aligned} J_{4p+3,2q}/J_{4p+2,2q+1} &= H_{4p+3}(B\Gamma, U_{2q}(pt)) \\ &= \mathbb{Z}_8 \otimes U_{2q}(pt) = U_{2q}(pt)/8.U_{2q}(pt), \\ J_{4p+2,2q+1}/J_{4p+1,2q+2} &= 0, \\ J_{4p+1,2q+2}/J_{4p,2q+3} & \\ &= U_{2q+2}(pt)/2U_{2q+2}(pt) \oplus U_{2q+2}(pt)/2U_{2q+2}(pt), \\ J_{4p,2q+3}/J_{4p-1,2q+4} &= 0. \end{aligned}$$

□

Let $w_{4n+3} \in \widetilde{U}_{4n+3}(B\Gamma)$ be $[S^{4n+3}/\Gamma, q]$, q being the inclusion $S^{4n+3}/\Gamma \subset B\Gamma$, $u_{4n+1} = A \cap w_{4n+3} \in \widetilde{U}_{4n+1}(B\Gamma)$, $v_{4n+1} = B \cap w_{4n+3} \in \widetilde{U}_{4n+1}(B\Gamma)$.

THEOREM 2.2. *The set* $\{u_{4n+1}, v_{4n+1}, w_{4n+3}\}_{n\geq 0}$ *is a system of generators for the* $U_*(pt)$*-module* $\widetilde{U}_*(B\Gamma)$.

Proof. Since the U_*-AHSS for $B\Gamma$ collapses we can use 1.1. If μ' denotes the edge homomorphism it is enough to prove that $\mu'(w_{4n+3})$, $\{\mu'(u_{4n+1}), \mu'(v_{4n+1})\}$ are systems of generators respectively for $\widetilde{H}_{4n+3}(B\Gamma)$ and $\widetilde{H}_{4n+1}(B\Gamma)$.

(a) Consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{U}_{4n+3}(S^{4n+3}/\Gamma) & \xrightarrow{q_*} & \widetilde{U}_{4n+3}(B\Gamma) \\ {\scriptstyle\mu'}\downarrow & & \downarrow{\scriptstyle\mu'} \\ \widetilde{H}_{4n+3}(S^{4n+3}/\Gamma) & \xrightarrow{q_*} & \widetilde{H}_{4n+3}(B\Gamma). \end{array}$$

We have $\mu'([S^{4n+3}/\Gamma, 1]) = c_1(S^{4n+3}/\Gamma)$, where $c_1(S^{4n+3}/\Gamma)$ denotes the fundamental class of S^{4n+3}/Γ (for H). Since $c_1(S^{4n+3}/\Gamma)$ is a generator of $\widetilde{H}_{4n+3}(B\Gamma)$ it follows that $q_*(c_1(S^{4n+3}/\Gamma))$ is a generator of $\widetilde{H}_{4n+3}(B\Gamma)$ because S^{4n+3}/Γ is the $(4n+3)$-skeleton of $B\Gamma$. Now $q_*([S^{4n+3}/\Gamma, 1]) = [S^{4n+3}/\Gamma, q]$ and then $\mu'([S^{4n+3}/\Gamma, q])$ is a generator of $\widetilde{H}_{4n+3}(B\Gamma)$.

(b) By [6], Section II, $\mu(A)$ and $\mu(B)$ generate the group $H^2(B\Gamma)$ and then if $A_1 = q^*(A) \in U^2(S^{4n+3}/\Gamma)$, $B_1 = q^*(B) \in U^2(S^{4n+3}/\Gamma)$, then the elements $\mu(A_1), \mu(B_1)$ generate $H^2(S^{4n+3}/\Gamma)$ because the following diagram commutes:

$$\begin{array}{ccc} U^2(B\Gamma) & \xrightarrow{q^*} & U^2(S^{4n+3}/\Gamma) \\ \mu\downarrow & & \downarrow\mu \\ H^2(B\Gamma) & \xrightarrow{q^*} & H^2(S^{4n+3}/\Gamma) \end{array}$$

and the bottom line is an isomorphism. Consider $t_{4n+3} = [S^{4n+3}/\Gamma, 1] \in U_{4n+3}(S^{4n+3}/\Gamma)$; then $\mu'(t_{4n+3}) = c_1(S^{4n+3}/\Gamma)$. Since the diagram:

$$\begin{array}{ccc} U^2(S^{4n+3}/\Gamma) & \xrightarrow{-\cap t_{4n+3}} & U_{4n+1}(S^{4n+3}/\Gamma) \\ \mu\downarrow & & \downarrow\mu' \\ H^2(S^{4n+3}/\Gamma) & \xrightarrow{-\cap c_1(S^{4n+3}/\Gamma)} & H_{4n+1}(S^{4n+3}/\Gamma) \end{array}$$

commutes by 1.5 and since the bottom line is an isomorphism it follows that $\mu'(A_1 \cap t_{4n+3})$ and $\mu'(B_1 \cap t_{4n+3})$ generate the group $H_{4n+1}(S^{4n+3}/\Gamma)$. Now by using the commutative diagram:

$$\begin{array}{ccc} U_{4n+1}(S^{4n+3}/\Gamma) & \xrightarrow{q_*} & U_{4n+1}(B\Gamma) \\ \downarrow\mu' & & \downarrow\mu' \\ H_{4n+1}(S^{4n+3}/\Gamma) & \xrightarrow{q_*} & H_{4n+1}(B\Gamma) \end{array}$$

we see that $q_*(A_1 \cap t_{4n+3})$ and $q_*(B_1 \cap t_{4n+3})$ generate the group $H_{4n+1}(B\Gamma)$. Since $q_*(A_1 \cap t_{4n+3}) = q_*(q^*(A) \cap t_{4n+3}) = A \cap q_*(t_{4n+3}) = A \cap w_{4n+3}$ and $q_*(B_1 \cap t_{4n+3}) = B \cap w_{4n+3}$ the assertion (b) has been proved. □

(1) *Relations between the generators.* We first recall the definition of the pull back transfer. Let M^n be a closed U-manifold, N^m a closed U-submanifold of M^n with $(n-m)$ even and i the inclusion $N^m \subset M^n$. If $[V^r, f] \in U_r(M^n)$, then there is a weakly complex representative map $g\colon V^r \to M^n$ transversal to N^m. Hence $g^{-1}(N^m)$ is a smooth closed submanifold of V^r and $\dim g^{-1}(N^m) = r+m-n$. Since N^m is a U-submanifold of M^n the normal vector bundle τ of N^m is in fact a complex vector bundle and by transversality we have $T(W^{r+m-n}) + g_1^*(\tau) = j^*(T(V^r))$ (1) where $W^{r+m-n} = g^{-1}(N^m)$, $g_1 = g|g^{-1}(N^m)$, $j\colon W^{r+m-n} \subset V^r$ and $T(-)$ being the tangent vector

bundle. Since V^r is a U-manifold the stable tangent bundle of V^r has a complex structure and the above relation (1) determines a unique complex structure on the stable tangent bundle of W^{r+m-n} (see [5], page 16). Then we define $i!\colon U_r(M^n) \to U_{r+m-n}(N^m)$ by $i!([V^r, f]) = [W^{r+m-n}, g_1]$. Moreover, the following diagram is commutative:

$$\begin{array}{ccc} U^k(M^n) & \xrightarrow{i^*} & U^k(N^m) \\ {\scriptstyle \mathrm{PD}}\downarrow & & \downarrow{\scriptstyle \mathrm{PD}} \\ U_{n-k}(M^n) & \xrightarrow{i!} & U_{m-k}(N^m) \end{array}$$

PD being the Poincaré duality (see [2], [7]).

Now, there is a map $\Delta\colon \widetilde{U}_*(B\Gamma) \to \widetilde{U}_*(B\Gamma)$ defined by $\Delta(x) = D \cap x$, with $D = e(\eta)$, the Euler class of η. The map Δ is a homomorphism of graded $U_*(pt)$-modules of degree -4.

PROPOSITION 2.3. *We have*

$$\Delta(w_{4n+3}) = w_{4(n-1)+3}, \qquad \Delta(u_{4n+1}) = u_{4(n-1)+1},$$
$$\Delta(v_{4n+1}) = v_{4(n-1)+1}, \qquad n \geq 0.$$

Proof. Let p, r, s be respectively the inclusions $S^{4(n-1)+3}/\Gamma \subset S^{4n+3}/\Gamma$, $S^{4n+3}/\Gamma \subset S^{4n+7}/\Gamma$, $S^{4n+7}/\Gamma \subset B\Gamma$. Then

$$[S^{4n+3}/\Gamma, r] \in U_{4n+3}(S^{4n+7}/\Gamma).$$

We have the pull back transfer

$$r!\colon U_{4n+3}(S^{4n+7}/\Gamma) \to U_{4(n-1)+3}(S^{4n+3}/\Gamma)$$

and the commutative diagram:

$$\begin{array}{ccc} U^4(S^{4n+7}/\Gamma) & \xrightarrow{r^*} & U^4(S^{4n+3}/\Gamma) \\ {\scriptstyle \mathrm{PD}}\downarrow & & \downarrow{\scriptstyle \mathrm{PD}} \\ U_{4n+3}(S^{4n+7}/\Gamma) & \xrightarrow{r!} & U_{4(n-1)+3}(S^{4n+3}/\Gamma). \end{array}$$

The element $r!([S^{4n+3}/\Gamma, i])$ is $[g^{-1}(S^{4n+3}/\Gamma), g|g^{-1}(S^{4n+3}/\Gamma)]$ where g is the map: $S^{4n+3}/\Gamma \to S^{4n+7}/\Gamma$ defined by $g([z_1, z_2, \ldots, z_{2n+2}]) = [z_1, z_2, \ldots, z_{2n}, 0, 0, z_{2n+2}]$ because g is homotopic to r and transversal to S^{4n+3}/Γ. But $g^{-1}(S^{4n+3}/\Gamma) = S^{4(n-1)+3}/\Gamma$ and $g|g^{-1}(S^{4n+3}/\Gamma) = p$. It is easily seen that

$$r!([S^{4n+3}/\Gamma, r]) = [S^{4(n-1)+3}/\Gamma, p] \in U_{4(n-1)+3}(S^{4n+3}/\Gamma),$$

the U-structure on $S^{4(n-1)+3}/\Gamma$ being the canonical one (this result can be found in [7], Lemma 2.5, page 145). Now by 1.8 we have $r^* \circ (PD)^{-1}([S^{4n+3}/\Gamma, r]) = e(\alpha)$, α being $\mathbb{C}$-vector bundle $S^{4n+3} \times_\Gamma \mathbb{C}^2 \to S^{4n+3}/\Gamma$, Γ acting on S^{4n+3} and $\mathbb{C}^2$ respectively by using $(n+1)\eta$ and η (see Section I). Since $\alpha = (s \circ r)^*(\eta)(\eta\colon E \times_\Gamma \mathbb{C}^2 \to B\Gamma)$, we have $e(\alpha) = (s \circ r)^*(D)$ and then $r^* \circ (\mathrm{PD})^{-1}([S^{4n+3}/\Gamma, r]) = (s \circ r)^*(D)$. From the above diagram it follows that $(s \circ r)^*(D) = (\mathrm{PD})^{-1}([S^{4(n-1)+3}/\Gamma, p])$. The fundamental class of S^{4n+3}/Γ for MU involved in the Poincaré duality being $[S^{4n+3}/\Gamma, 1] \in U_{4n+3}(S^{4n+3}/\Gamma)$ (see 1.3) we have:

$$(s \circ r)^*(D) \cap [S^{4n+3}/\Gamma, 1] = [S^{4(n-1)+3}/\Gamma, p]$$

and consequently

$$\begin{aligned} w_{4(n-1)+3} &= (s \circ r)_*([S^{4(n-1)+3}/\Gamma, p]) \\ &= (s \circ r)_*[(s \circ r)^*(D) \cap [S^{4n+3}/\Gamma, 1]] \\ &= D \cap (s \circ r)_*([S^{4n+3}/\Gamma, 1]) \\ &= D \cap [S^{4n+3}/\Gamma, s \circ r] = D \cap w_{4n+3} = \Delta(w_{3n+3}). \end{aligned}$$

We have

$$\begin{aligned} \Delta(u_{4n+1}) &= \Delta(A \cap w_{4n+3}) = (D \cdot A) \cap (w_{4n+3}) \\ &= A \cap [D \cap w_{4n+3}] = A \cap w_{(n-1)+3} = u_{4(n-1)+1}. \end{aligned}$$

Similarly $\Delta(v_{4n-1}) = v_{4(n-1)+1}$. □

REMARK. The homomorphism Δ is sometimes called the Smith-homomorphism.

We recall from [6], Lemma 2.11 and Theorem 2.12, that if Λ_* denotes the $U^*(pt)$-graded algebra $U^*(pt)[[X, Y, Z]]$, $\dim X = \dim Y = 2$, $\dim Z = 4$ and Ω_* the sub-$U^*(pt)$-algebra $U^*(pt)[[Z]]$ then there is $T(Z) = 8Z + 2\lambda_2 Z^2 + \sum_{i \geq 3} \lambda_i Z^i \in \Omega_4$, $\lambda_2 \notin 2U^*(pt)$, such that: $M(D) = 0$ ($M(Z) \in \Omega_*$) iff $M(Z) \in T(Z)\Omega_*$. Moreover by [6], Lemmas 2.13, 2.15, there is

$$J(Z) = \mu_1 Z + \sum_{i \geq 2} \mu_i Z^i \in \Omega_0, \qquad \mu_1 \notin 2U^*(pt),$$

such that: $E(D) + AM(D) + BN(D) = 0$ iff $M(Z), N(Z)$ belong to $(2 + J(Z))\Omega_*$ and $E(Z)$ to $T(Z)\Omega_*$ ($M(Z)$, $N(Z)$, $E(Z)$ are elements of Ω_*). We also recall the following notation: if $M(Z) = \sum_{i \geq r} a_i Z^i \in \Omega_{2n}$ with $a_r \neq 0$ then $\nu(M) = 4r$. Let W, V_1, V_2 be the $U_*(pt)$-submodules of $\widetilde{U}_*(B\Gamma)$ generated respectively by $\{W_{4n+3}\}_{n \geq 0}$, $\{u_{4n+1}\}_{n \geq 0}$, $\{v_{4n+1}\}_{n \geq 0}$.

THEOREM 2.4. (a) $\widetilde{U}_*(B\Gamma) = W \oplus V_1 \oplus V_2$.

(b) *In* $\widetilde{U}_{2p+1}(B\Gamma)$ *we have* $0 = a_0w_3 + a_1w_7 + \cdots + a_nw_{4n+3} = b_0u_1 + \cdots + b_mu_{4m+1}$ *iff there are homogeneous polynomials* $M(Z), M_1(Z)$ *and homogeneous formal power series* $N(Z), N_1(Z)$ *of* Ω_* *satisfying*: $b_mZ + b_{m-1}Z^2 + \cdots + b_0Z^{m+1} = M(Z)(2 + J(Z)) + N(Z)$, $a_nZ + a_{n-1}Z^2 + \cdots + a_0Z^{n+1} = M_1(Z)T(Z) + N_1(Z)$, $\nu(N) > 4(m+1)$, $\nu(N_1) > 4(n+1)$. *Moreover*: $b_0u_1 + \cdots + b_mu_{4m+1} = 0$ *iff* $b_0v_1 + \cdots + b_mv_{4m+1} = 0$.

Proof. (a) Suppose that $(a_0w_3 + \cdots + a_nw_{4n+3}) + (b_0u_1 + \cdots + b_mu_{4m+1}) + (c_0v_1 + \cdots + c_rv_{4r+1}) = 0$. Then a proof similar to that of Lemma 2.14 of **[6]** shows that $b_m = 2d_m$, $d_m \in U_*(pt)$. Consider $H(Z) = b_mZ + b_{m-1}Z^2 + \cdots + b_0Z^{m+1}$; we have: $H(Z) - d_mZ(2 + J(Z)) = b'_{m-1}Z^2 + \cdots + b'_0Z^{m+1} + F(Z)$, $\nu(F) > 4(m+1)$. Then $AH(D) = A[b'_{m-1}D^2 + \cdots + b'_0D^{m+1}] + AF(D)$ and by taking the cup product by w_{4m+7} we obtain $b_0u_1 + \cdots + b_mu_{4m+1} = b'_0u_1 + \cdots + b'_{m-1}u_{4(m-1)+1}$. As seen before, we have: $b'_{m-1} = 2d'_{m-1}$, $d'_{m-1} \in U_*(pt)$. We repeat the same process and after a finite number of operations we get $b_mZ + \cdots + b_0Z^{m+1} = M(Z)(2 + J(Z)) + N(Z)$, $M(Z)$ being a homogeneous polynomial and $N(Z)$ a homogeneous formal power series such that $\nu(N) > 4(m+1)$. Hence $b_0u_1 + \cdots + b_mu_{4m+1} = M(D)A(2 + J(D)) \cap w_{4m+7} = 0$. Similarly $c_0v_1 + \cdots + c_rv_{4r+1} = 0$ which ends the proof of part (a).

(b) Suppose that $a_0w_3 + \cdots + a_nw_{4n+3} = 0$. As in Proposition 2.6 of **[6]** we have $a_n = 8e_n$, $e_n \in U_*(pt)$. We form $a_nZ + \cdots + a_0Z^{n+1} - e_nT(Z) = a'_{n-1}Z^2 + \cdots + a'_0Z^{n+1} + F_1(Z)$, $\nu(F_1) > 4(n+1)$ and by taking the cup-product by w_{4n+7} we obtain: $a_0w_3 + \cdots + a_nw_{4n+3} = a'_0w_3 + a'_{n-1}w_{4(n-1)+3}$. As before, we have $a'_{n-1} = 8e'_{n-1}$, $e'_{n-1} \in U_*(pt)$. We repeat the same process with $a'_{n-1}Z^2 + a'_{n-2}Z^3 + \cdots + a'_0Z^{n+1}$ and after a finite number of operations we get: $a_nZ + \cdots + a_0Z^{n+1} = M_1(Z)T(Z) + N_1(Z)$, $\nu(N_1) > 4(n+1)$, $M_1(Z)$ being a homogeneous polynomial and $N_1(Z)$ a homogeneous formal power series. The proof of part (a) shows that $b_mZ + \cdots + b_0Z^{n+1} = M(Z)(2 + J(Z)) + N(Z)$, $\nu(N) > 4(m+1)$. The remaining part of (b) is evident. □

(2) *Orders of the Generators.*

LEMMA 2.5. *Suppose* $t \in \widetilde{U}_{2n+1}(B\Gamma)$, $t \neq 0$. *If* $a \in U_4(pt)$ *is such that* $a \notin 2U_4(pt)$ *then* $a \cdot t \neq 0$.

Proof. Since $t \neq 0$ there is an integer $q \geq 0$ such that $t \in J_{q,2n+1-q}$ and $t \notin J_{q-1,2n+1-(q-1)}$. We have either $q = 4s + 3$ or $q = 4s + 1$.

Suppose $q = 4s + 3$. We have the following commutative diagram:

$$\begin{array}{ccc} U_4(pt) \otimes J_{4s+3,2n+1-(4s+3)} & \xrightarrow{\chi} & J_{4s+3,2(n-2s+1)} \\ {\scriptstyle 1\otimes h}\downarrow & & \downarrow{\scriptstyle h} \\ U_4(pt) \otimes U_{2(n-2s-1)}(pt) \otimes \mathbb{Z}_8 & \xrightarrow{\times\otimes 1} & U_{2(n-2s+1)} \otimes \mathbb{Z}_8 \end{array}$$

where h is the canonical map: $J_{**} \to E^{\infty}_{**} = H_*(B\Gamma, U_*(pt)) = U_*(pt)\otimes H_*(B\Gamma)$. It is enough to prove that in $U_*(pt) = \mathbb{Z}[x_1, x_2, \ldots, x_4, \ldots]$ if $a \in U_4(pt)$, $a \notin 2U_4(pt)$, $b \in U_{2k}(pt)$, $b \notin 8U_{2k}(pt)$ then $ab \notin 8U_{2(k+2)}(pt)$; we may suppose that a and b are monomials and then the assertion is clear. The case $q = 4s + 1$ is similar. □

THEOREM 2.6. *We have* $\operatorname{ord} w_{4n+3} = 2^{2n+3}$.

Proof. (a) $\operatorname{ord} w_3 = 2^3$.
We have $0 = T(D) = 2^3 D + H(D)D^2$ and $0 = T(D) \cap w_7 = 2^2 w_3 + H(D) \cap (D^2 \cap w_7) = 2^3 w_3$ because $D^2 \cap w_7 \in U_{-1}(B\Gamma) = 0$. Then by using the edge homomorphism $\mu'\colon \widetilde{U}_3(B\Gamma) \to \widetilde{H}_3(B\Gamma) = \mathbb{Z}_8$ we see that $2^2 w_3 \neq 0$. Hence $\operatorname{ord} w_3 = 2^3$.

(b) *Suppose* $\operatorname{ord} w_{4i+3} = 2^{2i+3}$, $0 \le i \le n-1$.
We have $0 = T(D) = 2^3 D + 2\lambda_2 D^2 + \lambda_3 D^3 + \cdots + \lambda_{n+1} D^{n+1} + H(D)D^{n+2}$, $\lambda_2 \in U^{-4}(pt) = U_4(pt)$, $\lambda_2 \notin 2U_4(pt)$. Take the cup-product by $w_{4n+7}\colon 2^3 w_{4n+3} + 2\lambda_2 w_{4(n-1)+3} + \lambda_3 w_{4(n-2)+3} + \cdots + \lambda_{n+1} w_3 = 0$ and after multiplication by 2^{2n-1} we get: $2^{2n+2} w_{4n+3} + \lambda_2 2^{2n} w_{4(n-1)+3} = 0$; since $\operatorname{ord} w_{4(n-1)+3} = 2^{2n+1}$ we have $2^{2n} w_{4(n-1)+3} \neq 0$ and by 2.5 $\lambda_2 2^{2n} w_{4(n-1)+3} \neq 0$ because $\lambda_2 \notin 2U_4(pt)$. Hence $2^{2n+2} w_{4n+3} \neq 0$. Now we have: $2^{2n+3} w_{4n+3} = -\lambda_2 2^{2n+1} w_{4(n-1)+3} = 0$. It follows that $\operatorname{ord} w_{4n+3} = 2^{2n+3}$ which ends the proof of 2.6. □

LEMMA 2.7. *If* $G_n = U_{4n-2}(pt)w_3 + U_{4n}(pt)u_1 + U_{4n}(pt)v_1$, $G'_n = U_{4n}(pt)w_3 + U_{4n+2}(pt)u_1 + U_{4n+2}(pt)v_1$ *then we have the exact sequences*:

$$0 \to G_n \to \widetilde{U}_{4n+1}(B\Gamma) \xrightarrow{\Delta} \widetilde{U}_{4(n-1)+1}(B\Gamma) \to 0$$

$$0 \to G'_n \to \widetilde{U}_{4n+3}(B\Gamma) \xrightarrow{\Delta} \widetilde{U}_{4(n-1)+3}(B\Gamma) \to 0$$

Proof. We wish to show that the sequence:

$$0 \to G_n \to \widetilde{U}_{4n+1}(B\Gamma) \xrightarrow{\Delta} \widetilde{U}_{4(n-1)+1}(B\Gamma) \to 0$$

is exact. It follows by 2.3 that Δ is surjective and $G_n \subset \ker\Delta$. Suppose $0 = aw_3 + bu_1 + cv_1$, $a \in U_{4n-2}(pt)$, $b \in U_{4n}(pt)$, $c \in U_{4n}(pt)$. Then $a \cdot w_3 \in J_{3,4n-2}$ and since $bu_1 + cv_1 \in J_{1,4n}$ we have a $w_3 \in J_{2,4n-1} \supset J_{1,4n}$. If h denotes the quotient map: $J_{3,4n-2} \to J_{3,4n-2}/J_{2,4n-1} = H^3(B\Gamma, U_{4n-2}(pt)) = U_{4n-2}(pt)/8U_{4n-2}(pt)$, it follows that $h(aw_3) = 0$ and consequently $a = 2^3a'$. Hence $aw_3 = a'2^3w_3 = 0$ and then $bu_1 + cv_1 = 0$. Similarly we have $b = 2b'$, $c = 2c'$ which means that $0 = aw_3 + bu_1 + cv_1$ ($a \in U_{4n-2}(pt)$, $b \in U_{4n}(pt)$, $c \in U_{4n}(pt)$) if and only if $a = 2^3a'$, $b = 2b'$, $c = 2c'$. Hence $\operatorname{ord} G_n = 2^k$, $k = 3\operatorname{Rank} U_{4n-2}(pt) + 2\operatorname{Rank} U_{4n}(pt)$. Now, we have $\operatorname{ord}\ker\Delta = \operatorname{ord}\widetilde{U}_{4n+1}(B\Gamma)/\operatorname{ord}\widetilde{U}_{4(n-1)+1}(B\Gamma) = 2^k$ by 2.1. From $G_n \subset \ker\Delta$ and $\operatorname{ord} G_n = \operatorname{ord}\ker\Delta$ we see that the sequence $0 \to G_n \to \widetilde{U}_{4n+1}(B\Gamma) \xrightarrow{\Delta} \widetilde{U}_{4(n-1)+1}(B\Gamma) \to 0$ is exact. A similar proof shows that the sequence $0 \to G'_n \to \widetilde{U}_{4n+3}(B\Gamma) \xrightarrow{\Delta} \widetilde{U}_{4(n-1)+3}(B\Gamma) \to 0$ is exact. □

THEOREM 2.8. *We have* $\operatorname{ord} u_{4n+1} = \operatorname{ord} v_{4n+1} = 2^{n+1}$.

Proof. If $n = 0$ the assertion is clear. Suppose $\operatorname{ord} u_{4i+1} = 2^{i+1}$, $0 \le i \le n-1$. Then $\Delta(2^n u_{4n+1}) = 2^n u_{4(n-1)+1} = 0$ and since the sequence $0 \to G_n \to U_{4n+1}(B\Gamma) \xrightarrow{\Delta} U_{4(n-1)+1}(B\Gamma) \to 0$ is exact, (see 2.7), there are $a \in U_{4n-2}(pt)$, $b \in U_{4n}(pt)$, $c \in U_{4n}(pt)$ such that $2^n u_{4n+1} = aw_3 + bu_1 + cv_1$. It follows that $-bu_1 + 2^n \cdot u_{4n+1} = 0$ and $2^{n+1} \cdot u_{4n+1} = 0$; hence $\operatorname{ord} u_{4n+1} \le 2^{n+1}$. By Theorem 2.4 there are $M(Z), N(Z)$ in Ω_* such that: $2^n Z - bZ^{n+1} = M(Z)(2+J(Z)) + N(Z)$, $\nu(N) > 4(n+1)$. If $M(Z) = h_1 Z + h_2 Z^2 + \cdots$, then we have:

$$\begin{aligned} 2^n Z - bZ^{n+1} &= (2 + \mu_1 Z + \mu_2 Z^2 + \cdots)(h_1 Z + h_2 Z^2 + \cdots) \\ &\quad + e_{n+2} Z^{n+2} + e_{n+3} Z^{n+3} + \cdots, \qquad \mu_1 \notin 2U_*(pt). \end{aligned}$$

A straightforward calculation shows that $2^{n-j} | h_j$ and $2^{n-j+1} \nmid h_j$, $1 \le j \le n$. We have: $-b = 2h_{n+1} + \mu_1 h_n + \mu_2 h_{n-1} + \cdots + \mu_n h_1$; as $2 | h_j$, $1 \le j \le n-1$, $2 \nmid h_n$, $2 \nmid \mu_1$ we have $2 \nmid b$. As a consequence we get $2^n u_{4n+1} \ne 0$ and $\operatorname{ord} u_{4n+1} = 2^{n+1}$. Similarly $\operatorname{ord} v_{4n+1} = 2^{n+1}$. □

III. $\widetilde{U}^*(B\Gamma_k)$, $k \ge 4$: generators, orders and relations. We have seen in **[6]**, Section III, that there are elements $D_k \in \widetilde{U}^4(B\Gamma_k)$, $B_k \in \widetilde{U}^2(B\Gamma_k)$, $C_k \in \widetilde{U}^2(B\Gamma_k)$ defined as Euler classes of irreducible unitary representations η_1, ξ_2, ξ_3 of Γ_k. Moreover in the same article

(Sec. III) we have determined three homogeneous formal power series $T_k(Z) \in \Omega_4$, $J_k(Z) \in \Omega_0$, $G_k(Z) \in \Omega_2$ such that $B_k(2 + J(D_k)) + G_k(D_k) = C_k(2 + J(D_k)) + G_k(D_k) = 0$ and there is $G'_k(Z) \in \Omega_2$ satisfying $G_k(Z) = (2 + J(Z))G'_k(Z)$. Then with $B'_k = B_k + G'_k(D_k)$, $C'_k = C_k + G'_k(D_k)$ and μ being the edge homomorphism: $U^2(B\Gamma_k) \to H^2(B\Gamma_k)$ we see that $\mu(B'_k) = \mu(B_k)$ and $\mu(C'_k) = \mu(C_k)$ are generators of the group $H^2(B\Gamma_k)$; $\mu(D_k)$ is obviously a generator of $H^4(B\Gamma_k)$. Moreover $B'_k(2 + J(D_k)) = C'_k(2 + J(D_k)) = 0$.

Now let $w'_{4n+3} \in \widetilde{U}_{4n+3}(B\Gamma_k)$ be $[S^{4n+3}/\Gamma_k, q']$, q' being the inclusion $S^{4n+3}/\Gamma_k \subset B\Gamma_k$, $u'_{4n+1} = B'_k \cap w'_{4n+3} \in \widetilde{U}_{4n+1}$, $v'_{4n+1} = C'_k \cap w'_{4n+1} \in \widetilde{U}_{4n+1}(B\Gamma_k)$. Then we have the following theorems whose proofs are identical respectively to Theorem 2.2 and Theorem 2.4 and therefore will be omitted.

THEOREM 3.1. *The set $\{u'_{4n+1}, v'_{4n+1}, w'_{4n+3}\}_{n\geq 0}$ is a system of generators for the $U(pt)$-module $\widetilde{U}_*(B\Gamma_k)$.* □

Now let W', V'_1, V'_2 be the $U_*(pt)$-submodules of $\widetilde{U}_*(B\Gamma_k)$ generated respectively by $\{w'_{4n+3}\}_{n\geq 0}$, $\{u'_{4n+1}\}_{n\geq 0}$, $\{v'_{4n+1}\}_{n\geq 0}$.

THEOREM 3.2. (a) $\widetilde{U}_*(B\Gamma_k) = W' \oplus V'_1 \oplus V'_2$.

(b) *In $\widetilde{U}_{2p+1}(B\Gamma_k)$ we have $0 = a_0 w'_3 + a_1 w'_7 + \cdots + a_n w'_{4n+3} = b_0 u'_1 + \cdots + b_m u'_{4m+1}$ iff there are homogeneous polynomials $M(Z), M_1(Z)$ and homogeneous formal power series $N(Z), N_1(Z)$ of Ω_* satisfying: $b_m Z + b_{m-1}Z^2 + \cdots + b_0 Z^{m+1} = M(Z)(2 + J(Z)) + N(Z)$, $a_n Z + a_{n-1}Z^2 + \cdots + a_0 Z^{n+1} = M_1(Z)T_k(Z) + N_1(Z)$, $\nu(N) > 4(m+1)$, $\nu(N_1) > 4(n+1)$. Moreover $b_0 u'_1 + \cdots + b_m u'_{4m+1} = 0$ iff $b_0 v'_1 + \cdots + b_m v'_{4m+1} = 0$.* □

There is a Smith homomorphism Δ: $\widetilde{U}_*(B\Gamma_k) \to \widetilde{U}_*(B\Gamma_k)$ of degree -4 such that

$$\begin{aligned}
\Delta(w'_{4n+3}) &= D_k \cap w'_{4n+3} = w'_{4(n-1)+3},\\
\Delta(u'_{4n+1}) &= D_k \cap u'_{4n+1} = D_k \cap (B'_k \cap w'_{4n+3}) = B'_k \cap (D_k \cap w'_{4n+3})\\
&= B'_k \cap w'_{4(n-1)+3} = u'_{4(n-1)+1}, \Delta(v'_{4n+1}) = v'_{4(n-1)+1}.
\end{aligned}$$

If

$$\begin{aligned}
F_n &= U_{4n}(pt)w'_3 + U_{4n+2}(pt)u'_1 + U_{4n+2}(pt)v'_1,\\
F'_n &= U_{4n-2}(pt)w'_3 + U_{4n}(pt)u'_1 + U_{4n}(pt)v'_1
\end{aligned}$$

then we have:

Lemma 3.3. *The following sequences are exact:*

$$0 \to F_n \to U_{4n+3}(B\Gamma_k) \xrightarrow{\Delta} U_{4(n-1)+3}(B\Gamma_k) \to 0,$$
$$0 \to F'_n \to U_{4n+1}(B\Gamma_k) \xrightarrow{\Delta} U_{4(n-1)+1}(B\Gamma_k) \to 0.$$

Proof. The proof is similar to that of Lemma 2.7. □

It remains to calculate the orders of the generators.

Theorem 3.4. *We have*: $\operatorname{ord} w'_{4n+3} = 2^{2n+k}$, $n \geq 0$.

Proof. We have $0 = T_k(D_k) = 2^k D_k + H(D_k)D_k^2$ and then $0 = (2^k D_k + H(D_k)D_k^2) \cap w_7 = 2^k w_3$ because: $D_k^2 \cap w_7 \in \widetilde{U}_{-1}(B\Gamma_k) = 0$. Now if μ' is the edge homomorphism: $U_3(B\Gamma_k) \to H_3(B\Gamma_k) = \mathbb{Z}_2k$ then we have $\mu'(w'_3) = 1 \in \mathbb{Z}_2k$ and consequently $2^{k-1}w_3 \neq 0$. Then $\operatorname{ord} w_3 = 2^k$.

Suppose that $\operatorname{ord} w'_{4i+3} = 2^{2i+k}$, $0 \leq i \leq n-1$. Then

$$\begin{aligned} 0 = T_k(D_k) \cap w'_{4n+7} &= 2^k w'_{4n+3} + 2^{k-2}\lambda'_2 w'_{4(n-1)+3} \\ &+ \cdots + 2^{k-i}\lambda'_i w'_{4(n-1)+3} + \cdots + 2\lambda'_{k-1} w'_{4(n-k+2)+3} \\ &+ \lambda'_k w'_{4(n-k+1)+3} + \cdots + \lambda'_m w'_{4(n-m+1)+3} + \cdots, \end{aligned}$$

the number of non-zero elements in this sum being finite. If $3 \leq i \leq k-1$ we have $2^{2n-1+k-i} w'_{4(n-i+1)+3} = 0$ because $\operatorname{ord} w'_{4(n-i+1)+3} = 2^{2(n-i+1)+k}$ and $2(n-i+1)+k \leq 2n-1+k-i$ since $i \geq 3$. If $m \geq k$ (≥ 3) we have $2^{2n-1} w'_{4(n-m+1)+3} = 0$ because $\operatorname{ord} w'_{4(n-m+1)+3} = 2^{2(n-m+1)+k}$ and $2(n-m+1)+k \leq 2n-1$ since $k \leq m \leq 2m-3$. It follows that $2^{2n-1+k} w'_{4n+3} + 2^{2n-3+k}\lambda'_2 w'_{4n-1} = 0$. Now $2^{2n-3+k} w'_{4n-1} \neq 0$ because $\operatorname{ord} w'_{4n-1} = 2^{2n-2+k}$; since $\lambda'_2 \notin 2U^{-4}(pt)$ we have $2^{2n-3+k}\lambda'_2 w'_{4n-1} \neq 0$ (see 2.5). Hence

$$2^{2n-1+k} w'_{4n+3} \neq 0 \quad \text{and} \quad 2^{2n+k} w'_{4n+3} = -2^{2(n-1)+k}\lambda'_2 w'_{4n-1} = 0.$$

We have proved that $\operatorname{ord} w_{4n+3} = 2^{2n+k}$. □

Theorem 3.5. *We have*: $\operatorname{ord} u'_{4n+1} = \operatorname{ord} v'_{4n+1} = 2^{n+1}$, $n \geq 0$, *which are therefore independent of* k.

Proof. The proof of 3.5 is based on Theorem 3.2 and Lemma 3.3 and is exactly the same as the one of Theorem 2.8. □

References

[1] J. F. Adams, *Stable Homotopy and Generalized Homology*, University of Chicago Mathematics Lecture Notes, 1971.

[2] J. M. Boardman, *Stable Homotopy Theory*, Mimeographed notes, Chapter V, VI, Warwick, 1966.

[3] P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Springer, 1964.

[4] ——, *Periodic maps which preserve a complex structure*, Bull. Amer. Math. Soc., **70** (1964), 574–579.

[5] ——, *Torsion in* SU-*bordism*, Memoirs Amer. Math. Soc., **60** (1966).

[6] A. Mesnaoui, *Unitary cobordism of classifying spaces of quaternion groups*, to be published.

[7] D. Pitt, *On the complex bordism of the generalized quaternion groups*, J. London Math. Soc., **16** (1977), 142–148.

[8] R. M. Switzer, *Algebraic Topology, Homotopy and Homology*, Vol. 2.2, Springer-Verlag, 1975.

[9] R. H. Szczarba, *On tangent bundles of fibre spaces and quotient spaces*, Amer. J. Math., **86** (1964), 685–697.

[10] G. Wilson, *K-theory invariants for unitary G-bordism*, Quart. J. Math., Oxford, **24** (1973), 499–526.

Received October 5, 1986 and, in revised form, August 15, 1988.

Université Mohammed V
B. P. 1014
Rabat, Morocco