PACIFIC JOURNAL OF MATHEMATICS
Vol. 142, No. 2, 1990

OUTER CONJUGACY OF SHIFTS ON THE HYPERFINITE II_1-FACTOR

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For a shift σ on the hyperfinite II_1 factor R, we define the derived shift σ_∞ to be the restriction of σ to the von Neumann algebra generated by the $(\sigma^k(R))' \cap R$. Outer conjugacy of shifts implies conjugacy of derived shifts. In the case of n-shifts with n prime, we calculate σ_∞ explicitly. Combining this with the known classification of n-shifts up to conjugacy, we obtain useful outer-conjugacy invariants for n-shifts.

Following Powers [**5**], we define a shift σ on a von Neumann algebra M to be a unit-preserving $*$-endomorphism of M such that $\bigcap_{k=1}^{\infty} \sigma^k(M) = \mathbb{C}$, the complex numbers. We define the derived shift σ_∞ to be the restriction of σ to the von Neumann algebra M_∞ generated by all the $(\sigma^k(M))' \cap M$. When two shifts on a factor of type II_1 are outer conjugate, their derived shifts are conjugate (Theorem 1.2, below). This gives us a useful outer-conjugacy invariant. In particular, for shifts σ such that $\sigma_\infty = \sigma$, this shows that outer-conjugacy implies conjugacy (when specialized to binary shifts, this is the affirmative answer to a conjecture of Enomoto and Watatani [**3**]).

In §2, we compute σ_∞ explicitly when σ is an n-shift on the hyperfinite II_1 factor R and n is prime. 2-shifts, called binary shifts in [**5**], were introduced by R. Powers in [**5**]. n-shifts have been studied in [**1**], [**2**] and [**7**]. In the notation of [**1**], every n-shift can be associated with a doubly-infinite sequence $(a(k))_{k \in Z}$ in Z_n which is odd and fails to be periodic $\bmod p$ for all primes p dividing n. Furthermore, every such sequence occurs. In case n is square-free, two shifts with sequences $(a_1(k))$ and $(a_2(k))$ are conjugate if and only if there exists an m in Z_n such that $a_2(k) = m^2(a_1(k))$ for all k. Thus, up to multiplication by a square, the sequence associated with σ_∞ is an outer conjugacy invariant for σ.

The computation of σ_∞ breaks down into three cases. First, if $(a(k))$ fails to be ultimately periodic then $R_\infty = \mathbb{C}$; in this case σ_∞ is trivial and contains no information. Secondly, at the opposite extreme, if $a(k) = 0$ for all but finitely many k then $R_\infty = R$ and $\sigma_\infty = \sigma$; in this case outer conjugacy is equivalent to conjugacy. Finally, the most interesting case occurs when $(a(k))$ is ultimately periodic but doesn't end in 0's: here R_∞ is a factor not equal to $\mathbb{C}$ or R and σ_∞ is an n-shift; we are able (Theorem 2.1) to calculate explicitly the sequence associated with σ_∞ from $(a(k))$.

PROBLEM. If σ_1 and σ_2 are n-shifts with $R_\infty \neq \mathbb{C}$, does conjugacy of the derived shifts $(\sigma_1)_\infty$ and $(\sigma_2)_\infty$ imply outer conjugacy of σ_1 and σ_2? Equivalently, if σ is an n-shift with $R_\infty \neq \mathbb{C}$, are σ and σ_∞ outer conjugate?

In attempting to answer this problem, we present in §3 a method for producing many shifts outer conjugate to a given shift. This yields many interesting examples. But even in simple specific cases, given that $(\sigma_1)_\infty = (\sigma_2)_\infty$ it is still not clear whether σ_1 and σ_2 are outer conjugate.

Acknowledgment. The second named author wishes to thank Ed Granirer for support through NSERC.

1. Definition and properties of σ_∞. As in [5], a shift σ on a von Neumann algebra M is defined to be a unital $*$-endomorphism of M such that $\bigcap_{k=1}^\infty \sigma^k(M) = \mathbb{C}$. Two shifts σ_1 and σ_2, on M_1 and M_2 respectively, are said to be conjugate when there exists a $*$-isomorphism ϕ of M_2 onto M_1 such that $\sigma_1 \circ \phi = \phi \circ \sigma_2$, and outer conjugate when there exists a unitary u in M_1 such that $(adu)\circ\sigma_1$ and σ_2 are conjugate.

Let σ be a shift on M. Define

$$M_k = (\sigma^k(M))' \cap M \qquad \text{for } k = 0, 1, 2, \dots .$$

Evidently M_0 is the center of M and $M_0 \subset M_1 \subset M_2 \subset \cdots$. Let M_∞ be the von Neumann subalgebra of M generated by the M_k and let σ_∞ be the restriction of σ to M_∞. We call σ_∞ the derived shift of σ.

LEMMA 1.1. *σ_∞ is a shift on M_∞.*

Proof. First note that $\sigma_\infty(M_\infty) \subset M_\infty$, since $x \in M_k$ implies that for all $y \in M$,

$$\sigma(x)\sigma^{k+1}(y) = \sigma(x\sigma^k(y)) = \sigma(\sigma^k(y)x) = \sigma^{k+1}(y)\sigma(x),$$

which shows that $\sigma(x) \in M_{k+1} \subset M_\infty$.

Then σ_∞ is a shift because $\bigcap_{k=1}^\infty \sigma_\infty^k(M_\infty) \subset \bigcap_{k=1}^\infty \sigma^k(M) = \mathbb{C}$.

THEOREM 1.2. *Let σ_1 and σ_2 be shifts on the type II_1-factors M_1 and M_2 respectively. If σ_1 and σ_2 are outer conjugate then their derived shifts $(\sigma_1)_\infty$ and $(\sigma_2)_\infty$ are conjugate.*

Proof. Evidently if σ_1 and σ_2 are conjugate then so are $(\sigma_1)_\infty$ and $(\sigma_2)_\infty$. Hence given that σ_1 and σ_2 are outer conjugate we may assume without loss of generality that $M_1 = M_2 = M$ and that $\sigma_2 = (\operatorname{Ad} w)\circ\sigma_1$ for some unitary w in M. Set $w_1 = w$ and for $k = 2, 3, \dots$ set $w_k = w\sigma_1(w)\sigma_1^2(w)\cdots\sigma_1^{k-1}(w)$. Then we can see that:

$$(1.1) \qquad (\operatorname{Ad} w_k)\circ\sigma_1^k = \sigma_2^k \qquad \text{for } k = 1, 2, \dots .$$

For (1.1) holds for $k = 1$, and, for all $y \in M$,

$$\begin{aligned}[][(\operatorname{Ad} w_k)\circ\sigma_1^k]y &= (\operatorname{Ad} w_{k-1})\circ\sigma_1^{k-1}(w)\sigma_1^k(y)(\sigma_1^{k-1}(w))^* \\ &= (\operatorname{Ad} w_{k-1})\circ\sigma_1^{k-1}(w\sigma_1(y)w^*) = [(\operatorname{Ad} w_{k-1})\circ\sigma_1^{k-1}][\sigma_2(y)].\end{aligned}$$

Thus (1.1) follows by induction.

From (1.1), $\operatorname{Ad} w_k$ maps $\sigma_1^k(M)$ isomorphically onto $\sigma_2^k(M)$; therefore $\operatorname{Ad} w_k$ maps $M_k^{(1)} = (\sigma_1^k(M))' \cap M$ isomorphically onto $M_k^{(2)} = (\sigma_2^k(M))' \cap M$. For all $x \in M_k^{(1)}$,

$$(\operatorname{Ad} w_{k+1})(x) = (\operatorname{Ad} w_k)(\sigma_1^k(w)x(\sigma_1^k(w)^*)) = (\operatorname{Ad} w_k)(x).$$

Hence the isomorphisms $\operatorname{Ad} w_k$ are compatible with the inclusions $M_k^{(1)} \subset M_{k+1}^{(1)}$ and $M_k^{(2)} \subset M_{k+1}^{(2)}$; the following diagram is commutative:

$$\begin{array}{ccccccc} \cdots & \to & M_k^{(1)} & \to & M_{k+1}^{(1)} & \to & \cdots \\ & & {\scriptstyle \operatorname{Ad} w_k}\downarrow & & \downarrow{\scriptstyle \operatorname{Ad} w_{k+1}} & & \\ \cdots & \to & M_k^{(2)} & \to & M_{k+1}^{(2)} & \to & \cdots \end{array}$$

Thus there exists a unique $*$-isomorphism ϕ from the C^*-algebra generated by the $M_k^{(1)}$ onto the C^*-algebra generated by the $M_k^{(2)}$ such that

$$\phi(x) = (\operatorname{Ad} w_k)(x) \qquad \text{for all } x \in M_k^{(1)}.$$

Because $\operatorname{Ad} w_k$ preserves the trace τ on M, so does ϕ. Hence ϕ extends to an isomorphism $\overline{\phi}$ of von Neumann algebras from $(M_1)_\infty$ onto $(M_2)_\infty$.

Finally we check that $\overline{\phi}\circ(\sigma_1)_\infty = (\sigma_2)_\infty\circ\overline{\phi}$. For $x \in M_k^{(1)}$:

$$\begin{aligned}\overline{\phi}\circ(\sigma_1)_\infty(x) &= \phi(\sigma_1(x)) = (\operatorname{Ad} w_{k+1})(\sigma_1(x)) \\ &= (\operatorname{ad} w)(\sigma_1(w_k x w_k^*)) = \sigma_2(w_k x w_k^*) = ((\sigma_2)_\infty\circ\phi)(x).\end{aligned}$$

COROLLARY 1.3. *Suppose that σ_1 and σ_2 are shifts on the type II_1-factors M_1 and M_2 respectively. Suppose that $(M_1)_\infty = M_1$ and*

$(M_2)_\infty = M_2$. Then σ_1 and σ_2 are outer conjugate if and only if they are conjugate.

The following are examples of shifts σ such that $M_\infty = M$ so that $\sigma_\infty = \sigma$ and Corollary 1.3 applies.

EXAMPLE 1. Let σ be an n-shift with determining sequence $(a(k))_{k\in Z}$ such that $a(k) = 0$ for all but finitely many k (see §2 for details). Corollary 1.3 applied in this case demonstrates a conjecture of **[3]**.

EXAMPLE 2. Let σ be the canonical shift of the hyperfinite II_1-factor R realized as the von Neumann algebra of the GNS-representation associated with the unique tracial state on a UHF-algebra of type n^∞.

EXAMPLE 3. Let R be realized as the von Neumann algebra generated by a sequence of projections $p_1, p_2, \dots$ satisfying the Jones relations

(i) $p_ip_jp_i = \tau p_i$ for $|i-j| = 1$.
(ii) $p_ip_j = p_jp_i$ for $|i-j| \geq 2$.
(iii) There is a trace on R for which the conditional expectation E_n onto the $*$-algebra generated by $p_1, \dots, p_n$ and 1 satisfies: $E_n(p_{n+1}) = \tau$.

Let σ be the shift $\sigma(p_i) = p_{i+1}$ (see **[4]** and **[1**, §5**]**).

The common feature of these examples is the existence of $a \in R$ such that the $a_k = \sigma^k(a)$ generate R and that each a_j commutes with all a_k for all $k \geq k_0(j)$. Then $a_j \in R_{k_0(j)} \subset R_\infty$, so $R_\infty = R$ and $\sigma_\infty = \sigma$. We have shown:

LEMMA 1.4. *Suppose that σ is a shift on M and that there exists an a in M such that:*

(i) *$a, \sigma(a), \sigma^2(a), \dots$ generate M, and*
(ii) *there is a k_0 such that a commutes with $\sigma^k(a)$ for all $k \geq k_0$.*

Then $M_\infty = M$ and $\sigma_\infty = \sigma$.

LEMMA 1.5. $(M_\infty)_\infty = M_\infty$, $(\sigma_\infty)_\infty = \sigma_\infty$.

Proof. Let $S_k = (\sigma^k(R_\infty))' \cap R_\infty$. Then

$$S_k \supset (\sigma^k(R))' \cap R_\infty = ((\sigma^k(R))' \cap R) \cap R_\infty = R_k \cap R_\infty = R_k.$$

Thus $(R_\infty)_\infty$, the W^*-algebra generated by the S_k, contains R_∞. Since the opposite inclusion is evident, $(R_\infty)_\infty = R_\infty$.

LEMMA 1.6. *Suppose that σ is a group shift, $\sigma = \sigma(G, s, \omega)$ in the notation of* **[1]**, *where s is a shift on the abelian group G, and ω is* *an s-invariant cocycle on G. Define* $\rho(g \wedge h) = \omega(g,h)\overline{\omega(h,g)}$ *for all* $h, g \in G$. *Let, for* $k = 0, 1, 2, \ldots,$

$$D_k = \{g \in G \mid \rho(g \wedge s^k(G)) = 1\}$$

and let $D_\infty = \bigcup_{k=0}^{\infty} D_k$. *Let* $\tilde{s}$ *and* $\tilde{\omega}$ *be the restrictions of* s *and* ω *to* D_∞. *Then* σ_∞ *is the group shift* $\sigma(D_\infty, \tilde{s}, \tilde{\omega})$.

Proof. Use Proposition 1.2 of **[1]**.

COROLLARY 1.7. *There exist shifts on the hyperfinite* II_1*-factor* R *which fail to be outer conjugate to any group shift.*

Proof. By Lemma 1.6 and Theorem 1.2, it suffices to display a shift σ on R which is not a group shift and for which $\sigma_\infty = \sigma$. In Example 3 above, take $\tau = 1/p$ where p is a prime > 4. Then $\sigma_\infty = \sigma$ and σ is not conjugate to a group shift by Proposition 5.4 of **[1]**.

2. n-shifts on the hyperfinite factor: calculation of σ_∞. Fix an integer $n \geq 2$. For the main results of this section n will be assumed prime. Fix $\gamma = \exp(2\pi i/n)$.

An n-shift σ on the hyperfinite factor R may be characterized (see **[1]**, [7], **[2]**) by the existence of a unitary u in R such that:

(i) $u^n = 1$, $u^m \notin \mathbb{C}$ for $m = 1, 2, \ldots, n-1$,
(ii) R is generated by the $\sigma^k(u)$ for $k = 0, 1, 2, \ldots$, and
(iii) u and $\sigma^k(u)$ commute up to scalars:

$$u(\sigma^k(u))u^*(\sigma^k(u))^* \in \mathbb{C} \quad \text{for } k = 1, 2, \ldots.$$

We write:

$$u_k = \sigma^k(u), \qquad u_j u_k u_j^* u_k^* = \gamma^{a(k-j)} \quad \text{for all } j, k = 0, 1, \ldots$$

where $a(k) \in Z_n$. Then we call $(a(k))_{k \in Z}$ a *determining sequence* for σ. The sequence $(a(k))$ is odd and fails to be periodic mod p for every prime p dividing n; furthermore all such sequences occur as the determining sequence of an n-shift σ on R (see **[1]**). When n is square-free, two sequences $(a_1(k))$ and $(a_2(k))$ determine conjugate shifts if and only if there is an $m \in Z_n$ such that $a_2(k) = m^2(a_1(k))$ for all k (see **[1]**).

Here we are concerned with the calculation of σ_∞ and R_∞. σ is a group shift $\sigma(G, s, \rho)$ with $G = \bigoplus_{k=0}^{\infty}(Z_n)^{(k)}$, s the canonical shift $s: e_k \to e_{k+1}$ on G, and $\rho(e_j \wedge e_k) = \gamma^{a(k-j)}$ for $j, k = 0, 1, 2, \ldots$. From Lemma 1.6 we know that σ_∞ is a group shift, namely $\sigma(D_\infty, \tilde{s}, \tilde{\rho})$ where

$\tilde{s}$ and $\tilde{\rho}$ are the restrictions of s and ρ to D_∞ and $D_\infty = \bigcup_{k=0}^{\infty} D_k$. As in Lemma 1.6,

$$D_k = \{g \in G | \rho(g \wedge s^k(G)) = 1\}.$$

σ_∞ is not always an m-shift (see Example 7 at the end of §2). If, however, n is a prime, then σ_∞ is an n-shift. Theorem 2.1 summarizes the calculation of σ_∞ in this case.

THEOREM 2.1. *Let n be a prime and let σ be an n-shift on the hyperfinite* II_1*-factor R with determining sequence $(a(k))$. Let σ_∞ on R_∞ be the derived shift of σ.*

Part A. (i) $R_\infty = R$ if and only if $a(k) = 0$ for all but finitely many k.

(ii) $R_\infty \neq \mathbb{C}$ if and only if $(a(k))$ is ultimately periodic; i.e. there exist $T > 0$ and K such that $a(k+T) = a(k)$ for all $k \geq K$.

(iii) In all cases R_∞ is a factor. If $R_\infty \neq \mathbb{C}$ then σ_∞ is an n-shift and R_∞ is isomorphic to R.

Part B. Suppose now that $(a(k))$ is ultimately periodic so that $R_\infty \neq \mathbb{C}$. Let q_0 be the smallest integer such that $R_{q_0} \neq \mathbb{C}$. Define the length of a nonzero v in G to be L when $v = \sum_{j=0}^{L} v_j e_j$ with $v_L \neq 0$. Then we have:

(iv) Let $v \neq 0$ be in D_{q_0}. Then v spans D_{q_0} and $v, s(v), s^2(v), \ldots, s^k(v)$ is a basis for D_{q_0+k}. Hence D_∞ is isomorphic to $G = \bigoplus_{k=0}^{\infty}(Z_n)^{(k)}$ by the mapping $s^k(v) \to e_k$.

(v) g has minimal length in $D_\infty - \{0\}$ if and only if g spans D_{q_0}.

Part C. Let v be a vector of minimal length L in $D_\infty - \{0\}$. Suppose that $a(k)$ commences its ultimate periodicity at k_0 so that

$$a(k+T) = a(k) \quad \text{for all } k \geq k_0 \quad \text{and} \quad a(k_0 - 1 + T) \neq a(k_0 - 1).$$

Then

(vi) $q_0 = k_0 + L$.

(vii) k_0 is the smallest integer such that $\tilde{v} \perp A^k$ for all $k \geq k_0$, where $\tilde{v} = [v_L, v_{L-1}, \ldots, v_0]$ and $A^k = [a(k), a(k+1), \ldots, a(k+L)]$ are in $(Z_n)^{L+1}$ with the usual inner product.

(viii) L is the rank of the $T \times T$ matrix A with jth row $A_j = [a(k_0 + j - 1), a(k_0 + j), \ldots, a(k_0 + j + T - 2)]$.

(ix) σ_∞ has determining sequence $(b(k))$ given by $\gamma^{b(k)} = \rho(v \wedge s^k v)$. Then $b(q_0 - 1) \neq 0$ and $b(k) = 0$ for all $k \geq q_0$.

(x) The Jones index $[R\colon R_\infty]$ is n^L.

Proof. (i) $R_\infty = R$ if and only if $D_\infty = G$ if and only if $e_0 \in D_\infty$. That happens if and only if, for some m, $\rho(e_0 \wedge e_k) = 1$ for all $k \geq m$, i.e. $a(k) = 0$ for $k \geq m$.

(ii) Suppose that $a(k+T) = a(k)$ for all $k \geq k_0$. Then $g = e_0 - e_T$ is in $D_{k_0} \subset D_\infty$ and $R_\infty \neq \mathbb{C}$.

Conversely, suppose that $R_\infty \neq \mathbb{C}$. Then $D_{k_0} \neq 0$ for some k_0. Taking $g = \sum g_j e_j \neq 0$ in D_{k_0}, we get (Lemma 3.2 of **[1]**)

$$\sum_{j=0}^{\infty} g_j a(k-j) = 0 \quad \text{for all } k \geq k_0.$$

From here, as in the proof of Lemma 3.4 of **[1]**, we easily see that $a(k)$ is ultimately periodic.

(iii) See the proof of (ix).

(iv) LEMMA. *If* $g = \sum_{j=0}^{\infty} g_j e_j$ *is in* D_{q_0+k} *and if* $g_0 = g_1 = \cdots = g_k = 0$ *then* $g = 0$.

Proof of the Lemma. Assume that $g_0 = g_1 = \cdots = g_k = 0$ and $g \in D_{q_0+k}$. Then $g = s^{k+1} g'$ for some $g' \in G$, so $\rho(g' \wedge e_j) = \rho(g \wedge e_{j+k+1}) = 0$ for all j with $j+k+1 \geq q_0 + k$ or for all j with $j \geq q_0 - 1$. Hence g' is in $D_{q_0-1} = 0$ so $g' = 0$ and $g = 0$.

Proof of (iv). Suppose $v, w \in D_{q_0}$ with $v \neq 0$. Then $v_0 \neq 0$ and there exists $\lambda \in Z_n$ such that $(w - \lambda v)_0 = 0$. Then $w = \lambda v$ by the lemma. We have shown that v spans D_{q_0}.

Evidently $v, s(v), \dots, s^k(v)$ are linearly independent (they are in row echelon form) in D_{q_0+k}. For $w \in D_{q_0+k}$ we can successively find $\lambda_0, \lambda_1, \dots, \lambda_k$ such that $w' = w - \sum_{j=-0}^{k} \lambda_j s^j v$ has $w'_0 = w'_1 = \cdots = w'_k = 0$. Then the lemma shows that $w' = 0$, and we have shown that $v, sv, \dots, s^k v$ span D_{q_0+k}.

(v) By (iv), every non-zero g in D_∞ can be written in the form

$$g = \sum_{j=0}^{k} \lambda_j s^j v \quad \text{with } \lambda_k \neq 0.$$

Evidently the length of g is equal to $k + L$ where L is the length of v. Hence g is of minimal length in $D_\infty - \{0\}$ if and only if $g = \lambda v$ for $\lambda \neq 0$.

(vi) Write $v = \sum_{k=0}^{L} v_k e_k$ with $v_0, v_L \neq 0$. Then because v is in D_{q_0},

$$\sum_{j=0}^{L} v_j a(k-j) = 0 \quad \text{for all } k \geq q_0.$$

As in the proof of Lemma 3.4 of [**1**], that implies periodicity of $a(k)$ commencing at $q_0 - L$. Hence $k_0 \le q_0 - L$ or $k_0 + L \le q_0$.

To prove the opposite inequality use $a(k+T) = a(k)$ for all $k \ge k_0$. Combining that with $\sum_{j=0}^{L} v_j a(k-j) = 0$ for k large enough we obtain $\sum_{j=0}^{L} v_j a(k-j) = 0$ for all k such that $k - L \ge k_0$ or $k \ge k_0 + L$. That shows v is in D_{k_0+L} and therefore that $k_0 + L \ge q_0$.

(vii) q_0 is the smallest integer such that, for all $k \ge q_0$, $\rho(v \wedge e_k) = 1$. This is equivalent to

$$0 = \sum_{j=0}^{L} v_j a(k-j) = \sum_{j=0}^{L} \tilde{v}_j a(k - L + j) = (\tilde{v}|A^{k-L}).$$

Hence q_0 is the smallest integer such that $\tilde{v} \perp A^{k-L}$ for all $k \ge q_0$, and $k_0 = q_0 - L$ is the smallest integer such that $\tilde{v} \perp A^k$ for all $k \ge k_0$.

(viii) From $a(k+T) = a(k)$ for all $k \ge k_0$ it follows that $e_0 - e_T$ is in D_∞ so $L \le T$. If $r = \operatorname{rank} A < T$ choose $T - r$ linearly independent vectors $\tilde{v}(1), \tilde{v}(2), \ldots, \tilde{v}(T-r)$ in $(Z_n)^T$ perpendicular to $A_1, A_2, \ldots, A_T$. Taking a suitable linear combination of the $\tilde{v}(k)$ we can find a vector $\tilde{g}$ of the form $[g_r, g_{r-1}, \ldots, g_1, g_0, 0, \ldots, 0]$. Then $g = \sum_{k=0}^{r} g_k e_k$ is in D_∞ so $L \le r$. In all cases, then, we have proved $L \le r$. If $L = T$ then $L = r = T$, so to complete the proof we need only show that $r \le L$ provided $L < T$.

Suppose then that $L < T$. let $\tilde{v} = [v_L, v_{L-1}, \ldots, v_0, 0, \ldots, 0]$ in $(Z_n)^T$ where v has minimal length in D_∞. Then $\tilde{v}, s\tilde{v}, \ldots, s^{T-(L+1)}\tilde{v}$ are $T - L$ linearly independent vectors perpendicular to $A_1, A_2, \ldots, A_T$. Hence $r = \operatorname{rank} A \le T - (T - L) = L$.

(ix) D_∞ is isomorphic to G by $s^k v \to e_k$. Under this isomorphism the restriction of s to D_∞ corresponds to s and the restriction of ρ to D_∞ corresponds to $\tilde{\rho}(e_0 \wedge e_k) = \rho(v \wedge s^k v)$. Hence σ_∞ has defining sequence $(b(k))$ given by:

$$\gamma^{b(k)} = \rho(v \wedge s^k v).$$

Because $v \in D_{q_0}$ and $D_{q_0-1} = 0$, $\rho(v \wedge e_k) = 1$ for all $k \ge q_0$ and $\rho(v \wedge e_{q_0-1}) \ne 1$. That implies $\rho(v \wedge s^k v) = 1$ for all $k \ge q_0$ and $\rho(v \wedge s^{k-1} v) \ne 1$, where we use the fact that $v_0 \ne 0$. Thus $b(k) = 0$ for $k \ge q_0$ and $b(q_0 - 1) \ne 0$.

Then $(b(k))$ is not periodic; therefore R_∞ is a factor and is in fact isomorphic to R by [**1**]. This also proves (iii).

(x) The span of $e_0, e_1, \ldots, e_{L-1}$ is a complement for D_∞ in G. Hence G/G_∞ is isomorphic to $(Z_n)^L$, and, by Proposition 1.4 of [**1**], $[R{:}\,R_\infty] = n^L$.

EXAMPLES. In each case we specify σ by giving the determining sequence $(a(k))_{k\in Z}$: we write $a = a(0), a(1), a(2) \dots$. Similarly we specify σ_∞ by giving its determining sequence $(b(k))$. n can be taken to be an arbitrary prime with the noted exceptions: it is understood that integers are to be reduced $\bmod n$. The first repeating period is underlined.

1. $a = 0, \underline{1}, 1, 1, 1, \dots$.
   $k_0 = 1$, $L = T = 1$, $q_0 = 2$, $v = e_0 - e_1$,
   $b = 0, 1, \underline{0}, 0, \dots$
2. $a = 0, 0, \underline{1, 2}, 1, 2, \dots, n \neq 2, 3$.
   $k_0 = 2$, $T = 2$, $A = \left[\begin{smallmatrix}1 & 2\\ 2 & 1\end{smallmatrix}\right]$ has rank 2,
   $L = r = 2$, $q_0 = 4$.
   Then $v = e_0 - e_2$, $b(k) = 2a(k) - [a(k+2) + a(k-2)]$.
   $b = 0, -2, 1, 2, \underline{0}, 0, \dots$
3. $a = 0, 0, \underline{1, 2}, 1, 2, \dots$ with $n = 3$.
   As in Example 2, $k_0 = 2$ and $T = 2$ but now A has rank 1, so $L = r = 1$ and $q_0 = 3$. $v = e_0 - 2e_1$,
   $b(k) = 2a(k) + a(k-1) + a(k+1)$,
   $b = 0, 1, 1, \underline{0}, 0, \dots$
4. $a = 0, 0, \underline{1, -1}, 1, -1, \dots$
   $k_0 = 2$, $v = e_0 + e_1$, $q_0 = 3$,
   $b(k) = 2a(k) + (a(k+1) + a(k-1))$
   $b = 0, 1, 1, \underline{0}, 0, \dots$
5. $a = 0, 0, 1, 2, 3, 4, \dots$
   $T = n$, $k_0 = 1$, $v = e_0 - 2e_1 + e_2$ is of minimal length in D_∞ because
   $\left[\begin{smallmatrix}1 & 2\\ 2 & 3\end{smallmatrix}\right]$ has rank 2.
   $L = 2$, $q_0 = 3$,
   $b(k) = 6a(k) - 4[a(k+1) + a(k-1)] + [a(k+2) + a(k-2)]$
   $b = 0, -2, 1, \underline{0}, 0, \dots$
6. $a_1 = \underline{0, 0, 1}, 0, 0, 1, \dots$
   $a_2 = \underline{0, 1, 0}, 0, 1, 0, \dots$ for $n \neq 2$
   $a_3 = \underline{0, 2, 2}, 0, 2, 2, \dots$ for $n \neq 2$
   all have $L = T = 3$, $k_0 = 0$, $q_0 = 3$, $v = e_0 - e_3$.
   $b = 0, 1, 1, \underline{0}, 0, \dots$

In the calculation of b_3 we use the fact that multiplying a determining sequence by a square does not change its conjugacy class (see [**1**]).

7. $a = 0, 3, 0, 0, \ldots, 0, 6, 18, \ldots$ for $n \neq 3$, N arbitrary ≥ 3 where $a(0) = 0$, $a(1) = 3$, $a(k) = 0$ for $2 \leq k \leq N-1$,
and for $k \geq N$:

$$a(k) = 2 \sum_{i=k-N}^{k-1} a(i). \tag{2.1}$$

Then (2.1) holds for all $k \geq 2$ but not for $k = 1$ since $2\sum_{i=1-N}^{0} a(i) = 2a(-1) = -6$ and $n \neq 3$. Hence $a(k)$ is not periodic, but is ultimately periodic commencing with $k_0 = -N+2$. A minimal v in D_∞ is given by $v = e_0 - 2\sum_{i=1}^{N} e_i$.

Therefore $L = N$ and $q_0 = 2$. A direct calculation of $b(1)$ gives $9 = 3^2$ so

$$b = 0, 1, \underline{0}, 0, 0, \ldots.$$

8. A 4-shift σ on R such that σ_∞ is not an m-shift for any m:

$$a = 0, 1, \underline{2}, 2, \ldots, \qquad n = 4.$$

Since $(a(k))$ fails to be periodic mod 2 the factor condition is satisfied and σ is a shift on R by **[1]**. In $G = \bigoplus_{k=0}^{\infty}(Z_4)^{(k)}$ take $v_0 = 2e_0$, $v_k = e_{k-1} + e_k$ for $k \geq 1$. Then $s(v_0) = v_0 + 2v_1$, $s(v_k) = v_{k+1}$ for $k \geq 1$. We see easily (as in the proof of Theorem 2.1) that $D_2 = Z_2 v_0$, $D_3 = Z_2 v_0 \oplus Z_4 v_1$ and finally that

$$D_\infty = Z_2 v_0 \oplus Z_4 v_1 \oplus Z_4 v_2 \oplus \ldots.$$

Hence σ_∞ is the group shift $\sigma(D_\infty, \tilde{s}, \tilde{\rho})$ where $\tilde{s}$ and $\tilde{\rho}$ are the restrictions to D_∞ of s and ρ on G. If σ_∞ were an m-shift, there would exist a $g \in D_\infty$ such that $g, s(g), s^2(g), \ldots$ generate D_∞ (see Proposition 5.2 of **[1]**). It is easy to check that this is impossible. It is also easy to check that $\tilde{\rho}$ is non-degenerate on D_∞ so that R_∞ is a factor.

3. Outer conjugacies. Given an n-shift σ with determining sequence $(a(k))$ we give one method for calculating determining sequences of n-shifts outer conjugate to σ. Although this method produces some interesting examples we are unable to exploit it to the extent of showing when σ and σ_∞ are outer conjugate in general.

A basic lemma from operator theory follows.

LEMMA 3.1. *Suppose that n is an integer ≥ 2 and that u is a unitary operator with $u^n = 1$. Then there exists a unitary y in the $*$-algebra generated by u with the following properties:*

1. *$y^n = 1$ in case n is odd; $y^{2n} = 1$ in case n is even.*

2. *Let* $\gamma = \exp(2\pi i/n)$. *For all unitaries* v *such that* $uvu^*v^* = \gamma^a$ *where* $a \in Z_n$,

$$yvy^* = u^a v \quad \textit{for } n \textit{ odd},$$
$$yvy^*(u^a v)^* \in \mathbb{C} \quad \textit{for } n \textit{ even}.$$

Proof. Suppose first that n is odd. Let $T_n = \{\lambda \in \mathbb{C} | \lambda^n = 1\}$. It suffices to produce a function $f: T_n \to T_n$ such that

$$f(\gamma z) = z f(z) \quad \text{for all } z \in T_n. \tag{3.1}$$

For given such a function, let $y = f(u)$. Then y is unitary and $y^n = 1$. If $uvu^*v^* = \gamma^a$ then $vuv^* = \gamma^{-a}u$ so $vf(u)v^* = f(\gamma^{-a}u) = F(u)$ where $F(z) = f(\gamma^{-a}z) = \bar{z}^a f(z)$ by (3.1). Then $F(u) = (u^*)^a f(u)$ so $vyv^* = u^{-a}y$ or $yvy^* = u^a v$.

To show that a function f satisfying (3.1) exists, let

$$f(\gamma^s) = \gamma^{[s(s-1)/2]} \quad \text{for } s = 0, 1, \ldots, n-1. \tag{3.2}$$

We confirm that (3.2) holds for $s = n$ also, since $(n-1)/2$ is an integer, and then easily check that f satisfies (3.1).

Suppose now that n is even. (Then of course a function f satisfying (3.1) cannot exist.) Let $\delta = \exp(\pi i/n)$ and define $f(\gamma^s) = \delta^s \gamma^{[s(s-1)/2]}$ for $s = 0, 1, \ldots, n-1$. Then $f(\gamma z) = \delta z f(z)$ for all $z \in T_n$ and, as in the case when n is odd, $y = f(u)$ has the required properties.

Corollary 3.2. *Suppose that* σ *is an* n*-shift on* M, $\sigma = \sigma(G, s, \rho)$ *where* $G = \bigoplus_{k=0}^{\infty} (Z_n)^{(k)}$. *Let* $g \to u_g$ *be the canonical twisted representation of* G *in* M, *and define a bilinear map* [,] *from* $G \times G$ *to* Z_n *by*:

$$\gamma^{[g,h]} = \rho(g \wedge h) = u_g u_h u_g^* u_h^* \quad \textit{for } g, h \in G.$$

Fix $g \in G$ *and define* $\phi_g: G \to G$ *by*: $\phi_g(h) = h + [g, h]g$ *for all* $h \in G$. *Then there exists a unitary* y_g *in* M *such that*

$$y_g u_h y_g^* = \lambda(g, h) u_{\phi_g(h)} \quad \textit{for all } h \in G$$

where $\lambda(g, h) \in \mathbb{C}$.

Proposition 3.3. *Suppose that* n *is a prime and that the* n*-shift* σ *on the hyperfinite factor* R *has determining sequence* $(a(k))$. *Let* $G = \bigoplus_{k=0}^{\infty} (Z_n)^{(k)}$, *let* s *be the shift* $e_k \to e_{k+1}$ *on* G, *let* ρ *on* G *be defined by* $(a(k))$, *and let* [,] *and* ϕ_g *be defined as in Corollary* 3.2, *so that*

$$[e_i, e_j] = a(j - i) \quad \textit{for all } i, j = 0, 1, 2, \ldots.$$

Suppose that $g(1), g(2), \ldots, g(m)$ *are in* G *and let* ϕ *be* $\phi_{g(1)} \circ \phi_{g(2)} \circ \phi_{g(3)} \circ \cdots \circ \phi_{g(m)}$. *Suppose that* $v(0)$ *in* G *is such that* G *is generated by* $v(0), v(1), v(2), \ldots$ *where* $v(k) = \phi(s(v(k-1)))$. *Then* $b(k) = [v(0), v(k)]$ *defines a determining sequence* $(b(k))$ *of an n-shift* σ' *on* R *which is outer conjugate to* σ.

Proof. We may assume that $\sigma = \sigma(G, s, \rho)$ and that $R = W^*(G, \rho)$. Let $y = y_{g(1)} y_{g(2)} \cdots y_{g(n)}$ where $y_{g(k)}$ is given by Corollary 3.2. Then $y u_h y^* = \lambda(h) u_{\phi(h)}$ for all $h \in G$, where $\lambda(h) \in \mathbb{C}$. Hence

$$[(\operatorname{Ad} y) \circ \sigma](u_{v(k)}) = \lambda_k u_{v(k+1)}$$

for $\lambda_k \in \mathbb{C}$. Now let $\sigma' = (\operatorname{Ad} y) \circ \sigma$ and let $w_0 = u_{v(0)}$. Then

1. $w_0^n = 1$ and $w_0^k \neq 1$ for $k = 1, \ldots, n-1$;
2. the $w_k = (\sigma')^k w_0$ generate R;
3. $w_0 w_k w_0^* w_k^* = \gamma^{[v(0), v(k)]}$.

Therefore (Proposition 4.1 of [**1**]), σ' is an n-shift on R with determining sequence $b(k) = [v(0), v(k)]$.

Examples. 1. Take σ_0 given by the sequence $0, 1, \underline{0}, 0, \cdots$ (i.e. $a(0) = 0$, $a(1) = 1$ $a(2) = 0 \ldots$). Then the shifts given by each of the following sequences are outer conjugate to σ_0, and hence, for each, the derived shift is σ_0 and $q_0 = 2$.

(a) $0, \underline{1}, 1, 1, \ldots$
(b) $\overline{\underline{0, 2}}, 0, 2, 0, \ldots$, for $n \neq 2$,
(c) $0, 1, a, a^2, \ldots$,
(d) $0, \lambda + 1, \lambda^2 - 1, \lambda^3 + 1, \ldots$, for $\lambda \neq -1, n \neq \lambda + 1$,
(e) $0, 1 - \lambda\mu, (1 - \lambda\mu)(\lambda^2 - \mu^2)/\lambda - \mu, \ldots, (1 - \lambda\mu)(\lambda^n - \mu^n)/\lambda - \mu, \ldots$, for $\lambda \neq \mu, \lambda\mu \neq 1$.

The $g(i)$'s in Proposition 3.3 which demonstrate the above outer conjugacies are

(a) $g_1 = e_0$,
(b) $g_1 = -e_1$, $g_2 = e_0$,
(c) $g_1 = (1 + a)e_0$, $g_2 = -e_1$,
(d) $\mu = -1$ in (e),
(e) $g_1 = \mu e_1$, $g_2 = \lambda e_0$.

In each case we can take $v(0) = e_0$.

Remarks. Given a shift σ of forms (c), (d) or (e) for example, the calculation of σ_∞ or q_0 by the methods of §2 might be very difficult even for one prime n. There are, however, shifts which have derived shift σ_0 which are not obviously outer conjugate to σ (see Example 7 of §2).

2. Take σ_0 given by $b = 0, 0, 1, \underline{0}, 0, \ldots$. Then the shifts given by the following defining sequences are outer conjugate to σ_0:

(a) $0, \underline{0, 1}, 0, 1, \ldots,$

(b) $\underline{0, 0, 2, 0}, 0, 0, 2, 0, \ldots,$ for $n \neq 2$ (note $k_0 = -1$),

(c) $0, 0, 1, 0, \lambda, 0, \lambda^2, 0, \ldots$.

The $g(i)$'s in Proposition 3.3 which demonstrate the above outer conjugacies are as follows: (a) $g(0) = e_0$; (b) $g(0) = -e_2$, $g(1) = e_0$; (c) $g(0) = \lambda e_0$.

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Received March 21, 1988.

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