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RADON-NIKODYM PROBLEM FOR THE VARIATION OF A VECTOR MEASURE

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We consider the problem of representing the variation $|m|$ of a vector measure m as an integral in the Dinculeanu sense with respect to M.

Throughout this paper (S, Σ) denotes a measurable space. If X is a Banach space, we write X^* for the dual space and K_X for the closed unit ball of X. We use brackets $\langle\ ,\ \rangle$ for the pairing between a Banach space and its dual. Let $m: \Sigma \to X$ be a vector measure with finite variation $|m|$. Recall that a strongly measurable function $f: S \to X^*$ is said to be integrable in Dinculeanu's sense if there exists a sequence $\{f_n\}_{n\geq 1}$ of simple functions converging $|m|$-a.e. to f such that

$$\lim_{n,p\to\infty} \int \|f_n - f_p\| \, d|m| = 0,$$

i.e., the function $\|f\|$ is $|m|$-integrable. Further, D-$\int_A f\,dm$ denotes the Dinculeanu integral of the function f with respect to m over the set A.

It was proved in [2] that for every $\varepsilon > 0$ there exists an X^*-valued strongly measurable function f defined on the set S such that $\|f\| \leq 1 + \varepsilon |m|$-a.e. and $|m|(A) = $ D-$\int_A f\,dm$ for each $A \in \Sigma$. We are interested in the following question: For which Banach spaces may we obtain the preceding equality when we insist that $\|f\| = 1$ a.e. $|m|$?

We begin our investigation by introducing the following property of Banach spaces. The Banach space X has property (DV) if for every equivalent norm on x, for every measurable space (S, Σ) for every equivalent norm on X and every vector measure $m: \Sigma \to X$ with finite variation $|m|$ there exists a strongly measurable function $f: S \to X^*$ with $\|f\| = 1$ $|m|$-a.e. such that $|m|(A) = $ D-$\int_A f\,dm$ for each $A \in \Sigma$.

THEOREM 1. *If both X and X^* have the Radon-Nikodym Property, then X has property* (DV).

Proof. Let (S, Σ) be a measurable space and $m: \Sigma \to X$ be a measure with finite variation $|m|$. Since X has RNP, there exists a strongly measurable function $f: S \to X$ such that $m(A) = \text{B-}\int_A f\,dm$ for each $A \in \Sigma$. ($\text{B-}\int_A f\,dm$ denotes the Bochner integral of f with respect to m over the set A.) For every $x \in X$ let

$$G(x) = \{x^* \in K_{X^*} : \|x^*\| = 1 \text{ and } \langle x, x^*\rangle = \|x\|\}.$$

Then G is a set-valued mapping, and $G(x)$ is non-empty and w^*-compact for every $x \in X$. We now see that G is upper semi-continuous if X is endowed with the norm topology and K_{X^*} is endowed with the w^*-topology. Indeed, let H be a w^*-closed subset of K_{X^*}. It suffices to show that

$$\{x \in X : G(x) \cap H \neq \varnothing\}$$

is norm closed in X. Let $\|x_n - x\| \to 0$, and suppose that $G(x_n) \cap H \neq \varnothing$, i.e., for every n there exists $x_n^* \in H$ such that $\|x_n^*\| = 1$ and $\|x_n\| = \langle x_n, x_n^*\rangle$. Let x^* be any w^*-cluster point of $\{x_n^*\}$. It is not difficult to see that for every $\varepsilon > 0$ we have $|\ \|x\| - \langle x, x\rangle| < \varepsilon$; i.e. the set is norm closed. Following [**7**, Theorem 8], we see that the set-valued mapping G has a selector which is of the first Baire class when X^* is equipped with the norm topology. Then using [**1**, Lemma 4.11.13] we see that the function $h: S \to X^*$ defined by $h = g \circ f$ is strongly measurable. (The preceding lemma and the fact that f is strongly measurable ensures that h has essentially separable range; the strong measurability of f and the fact that g belongs to the first Baire class ensures that $h^{-1}(u)$ is an element of the $|m|$-completion of Σ for every set u which is open in the norm topology on x^*.) But for every $A \in \Sigma$ we have

$$|m|(A) = \int_A \|f\|\,d|m|.$$

Therefore following [**4**, Theorem 3.4.II], we have

$$|m|(A) = \int_A \|f\|\,d|m| = \int_A \langle f(s), h(s)\rangle\,d|m|(s)$$
$$\mathbb{S} = \text{D-}\int_A h\,df|m| = \text{D-}\int_A h\,dm.$$

PROPOSITION 2. *If X has property* (DV), *then every subspace Y of X has property* (DV).

Proof. Let $m: \Sigma \to Y$ be a vector measure with $|m| < \infty$. Since X has property (DV), there exists a strongly measurable function $f: S \to X^*$ with $\|f(x)\| = 1$ $|m|$-a.e. such that $|m|(A) = \text{D-}\int_A f\,dm$ for each $A \in \Sigma$. Define $g: S \to Y^*$ by $g(s) = f(s)|_{Y^*}$ (the restriction of $f(s)$ to Y). Of course g is strongly measurable and $\|g(s)\| \le \|f(s)\| = 1$. For every $A \in \Sigma$ we have $\text{D-}\int_A g\,dm = \text{D-}\int_A f\,dm$ since m takes its values in Y. But

$$|m|(A) = \text{D-}\int_A f\,dm = \text{D-}\int_A g\,dm \le \int_A \|g\|\,d|m| \le |m|(A);$$

therefore $\|g(s)\| = 1$ $|m|$-a.e.

PROPOSITION 3. *Banach spaces l_1 and c_0 do not have property* (DV).

Proof. Let $(I, \mathscr{B})$ be the unit interval with the Borel σ-algebra.

(1) For $A \in \mathscr{B}$ define m by $m(A) = (\int_A (1/2^n) r_n(t)\,dt)_{n-1}^{\infty}$, where r_n denotes the nth Rademacher function. Then m is a vector measure with values in l_1 such that $|m| = \lambda$, where λ is Lebesgue measure. (It is enough to verify this last equality on intervals of the form $[1/2^i, 1/2^{i-1})$.) Suppose there exists a strongly measurable function $f: I \to l_\infty$, $f(t) = (f_n(t))$, such that $\|f(t)\| = 1$ λ-a.e. and $|m|(A) = \text{D-}\int_A f\,dm$ for each A. Because of the definition of m, we have

$$|m|(A) = \int_A \sum_{n=1}^{\infty} f_n(t)(1/2^n) r_n(t)\,dt.$$

In particular, for $A = [0, 1]$ we have $\sum_{n=1}^{\infty} f_n(t)(1/2^n) r_n(t) = 1$ λ-a.e. Further, it is easy to see that $(f_n(t)) = (r_n(t))$ is the unique element of l_∞ which satisfies the preceding equality. But the function $t \to (r_n(t))$ from I to l_∞ is not weakly measurable **[9]**.

(2) For $A \in \mathscr{B}$ define m by $m(A) = (\int_A (n/n+1) r_n(t)\,dt)_{n=1}^{\infty}$. It is easy to verify that m is a vector measure with values in c_0 and $|m| = \lambda$. (The last statement follows from the equality $\sup_n (n/n+1)$ $r_n(t) = 1$.) Assume there exists a strongly measurable function $f: I \to l_1$, $f(t) = (f_n(t))$ with $\|f(t)\| = \sum_{n=1}^{\infty} |f_n(t)| = 1$ λ-a.e. such that $|m|(A) = \text{D-}\int_A f\,dm$ for every $A \in \mathscr{B}$. Then for $A = [0, 1]$ we have

$$1 = \int_0^1 \sum_{n=1}^{\infty} f_n(t)(n/n+1) r_n(t)\,dt,$$

i.e., $\sum_{n=1}^{\infty} f_n(t)(n/n+1)r_n(t) = 1$ λ-a.e. But this is impossible since for every n we have

$$f_n(t)(n/n+1)r_n(t) \leq |f_n(t)(n/n+1)r_n(t)| < |f_n(t)|.$$

REMARK 1. Propositions 2 and 3 show that none of the assumptions in Theorem 1 can be omitted. Namely, l_1 has RNP, c_0 does not have RNP, and c_0 does not have (DV). Similarly, l_1 has RNP, l_∞ does not have RNP, and l_1 does not have (DV).

REMARK 2. Since c_0 does not have property (DV) and l_1 has RNP, we note that (1) and (2) of the theorem in **[3]** are, in fact, not equivalent. The difficulty with the proof of this equivalence occurs when the author concludes that the w^*-cluster point of a sequence of strongly measurable functions is w^*-measurable. Indeed, it is well known that every pointwise cluster point of the sequence of Rademacher functions is not Lebesgue measurable. We note that there is also a difficulty with the proof that $(3) \Rightarrow (1)$ in **[3]**. The author makes strong use of this Lemma 1 in this proof, and in the proof of Lemma 1 he concludes that if X^* is not separable, then $\bigcap \ker\{x_j^*\} \neq \{\theta\}$ when the intersection is taken over a countable set of indices. However, if $X = l_1$, then X^* is not separable, but it does have a countable total subset. In fact, we note that this formulation of Lemma 1 is incorrect. To see this, let X be separable and let B be a countable subset of smooth points of the unit sphere which is dense in the unit sphere (Mazur's theorem provides us with the set B). If there exist nets $\{x_\alpha\}_{\alpha<\Omega} \subset B$ and $\{x_\alpha^*\}_{\alpha<\Omega} \subset S(X^*)$, with $\langle x_\alpha, x_\alpha^*\rangle = 1$ and $\|x_\alpha - x_\beta\| > 0$ as required in Lemma 1 of **[3]**, then we contradict the smoothness of x_α for some α. Further, Theorem 5.6 of **[8]** shows that Lemma 2 is also incorrect as stated.

We are able to deduce a weaker version of Debieve's conjecture, however. Using the fact that X^* has the weak RNP whenever l_1 does not embed in X **[6]**—and the results of this paper—we obtain the following result.

COROLLARY. *If X has property* (DV), *then X^* has the weak* RNP.

Unfortunately, we are not able to decide if X^* must have RNP whenever X has property (DV).

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