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CLASSICAL LINK INVARIANTS AND THE BURAU REPRESENTATION

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The object of this paper is to show how to use the Burau representation of the Artin braid group to calculate some invariants of an oriented link in $\mathbb{S}^3$. More precisely, we obtain

(a) generators and relations for the Alexander module, and

(b) a unimodular $(-t)$-Hermitian form on the torsion submodule of the Alexander module (see below for a precise statement).

Scaling our form by $(1 - t^{-1})$ yields a Hermitian form which, for knots, is probably the Blanchfield form. If so, it would then follow from Trotter that the S-equivalence class of the Seifert form of a knot can be computed from the Burau representation. Even if this form is the Blanchfield form for knots, the situation for links is less clear because $(1 - t^{-1})$ need not be invertible in the endomorphism ring of the Alexander module.

Introduction. To state the results precisely, let B_n be the n-string braid group, let $R = \mathbb{Z}[t, t^{-1}]$, and let V_n be a free R-module of rank n affording the unreduced[1] Burau representation. For $\gamma \in B_n$, let $\hat{\gamma}$ be the link in $\mathbb{S}^3$ obtained by identifying the ends of a geometrical realization of γ, and set $W(\gamma) = (1 - \gamma)(V_n)$.

THEOREM 1. *$V_n/W(\gamma)$ depends only on $\hat{\gamma}$. In fact, it is the Alexander module of the disjoint union of $\hat{\gamma}$ with the unknot. Let $U_n \subseteq V_n$ afford the reduced Burau representation. Then $W(\gamma) \subseteq U_n$ and $U_n/W(\gamma)$ is the Alexander module of $\hat{\gamma}$.*

The fact that the Burau representation is intimately connected with the Alexander module is well known (cf. [**1**], p. 122) but the exact details may not have appeared previously.

To state Theorem 2, we let $\mathbb{Q}(t)$ be the field of rational functions and let * be the automorphism of R defined by $t^* = t^{-1}$. If M and N are R-modules, a $(-t)$-Hermitian form on M with values in N is an R-module map $f\colon M \otimes_R M \to N$ such that $f(x \otimes y) = -tf(y \otimes x)^*$. Such a map induces a natural map $M \to \mathrm{Hom}_R(M, N)$. When this map is an isomorphism, f is sometimes called a "perfect pairing".

[1] It is essential to use the unreduced Burau representation here.

In the following, M is the R-torsion submodule of the Alexander module, and N is $\mathbb{Q}(t)/R$.

THEOREM 2. *Let $\overline{A} = \overline{A}(\gamma)$ be the R-torsion submodule of $U_n/W(\gamma)$. Then there is a $(-t)$-Hermitian form defined on $\overline{A}(\gamma)$ with values in $\mathbb{Q}(t)/R$ depending only on $\hat{\gamma}$. When the Alexander polynomial is non-zero, the form is a perfect pairing.*

The paper is organized as follows. In §2 we recall the definition of the Burau representation and define an invariant sesqui-linear form which is essentially due to Squier [3]. In §3 we prove Theorem 1. In §4 we prove Theorem 2 by defining, for any braid γ, a $(-t)$-Hermitian form on $\overline{A}(\gamma)$ which we then show is invariant up to isomorphism under the Markov moves ([1], p. 51). We defer the proof that the form is unimodular (i.e. a perfect pairing) when the Alexander polynomial is non-zero to §5, in which we describe an algorithm for calculating the form and we do the calculations for the figure eight knot. In §6 we study the effect of the orientation-reversing symmetries. In §7 we show how to get a rational-valued form (integral when Δ is monic) by taking the trace. Finally, in §8 we apply the results to the (n, m) torus link. We obtain a presentation for the Alexander module as a direct sum of cyclic submodules, and explicit formulae for the (Blanchfield?) form.

2. The Burau representation. Let B_n be the n-string Artin braid group with standard generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, and let F_{n+1} be the free group on $n+1$ free generators $x_0, x_1, \ldots, x_n$. Let $R = \mathbb{Z}[t, t^{-1}]$. Then B_n acts on F_{n+1} via

$$\sigma_i(x_j) = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } j = i, \\ x_i & \text{if } j = i+1, \\ x_j & \text{otherwise.} \end{cases}$$

Note that x_0 is fixed by B_n. Define $\varepsilon\colon F_{n+1} \to \mathbb{Z}$ via $\varepsilon(x_i) = 1$ for all i, and let $K = \ker \varepsilon$. Then $V_n = K/K'$ is a free R-module with basis $f_i = x_0 x_i^{-1} K'$ $(1 \le i \le n)$, where the action of t is given by $t f_i = x_j f_i x_j^{-1}$ for any j (one verifies that this is well defined modulo K'). An easy calculation then shows that

$$\sigma_i(f_j) = \begin{cases} (1-t) f_i + t f_{i+1} & \text{if } j = i, \\ f_i & \text{if } j = i+1, \\ f_j & \text{otherwise.} \end{cases}$$

V_n affords the (unreduced) Burau representation of B_n. The vector $u_n = \sum_{i=1}^n t^{i-1} f_i \in V_n$ is fixed by B_n and so is the augmentation map $\varepsilon_0(f_i) = 1$ $(1 \le i \le n)$ (not to be confused with its cousin defined on the free group). Let $U_n = \ker(\varepsilon_0)$. Then U_n has the basis $e_i = f_i - f_{i+1}$ $(1 \le i < n)$ and affords the so-called "reduced" Burau representation. An important point which may not have been thoroughly appreciated heretofore is that U_n is not a summand of V_n as a B_n-module; in fact $\langle u_n \rangle \oplus U_n$ has index $\varepsilon_0(u_n) = \sum_{i=1}^n t^{i-1}$ in V_n.

The usual geometric interpretation is to let B_n act via the mapping class group on the $(n+1)$-punctured disk $\mathbb{D}_{n+1}$ with $F_{n+1} = \pi_1(\mathbb{D}_{n+1})$. K is then the fundamental group of an infinite cyclic cover C of $\mathbb{D}_{n+1}$ which can be embedded in $\mathbb{R}^3$ as the "parking garage": an infinite vertical stack of $(2n+2)$-gons with $n+1$ ramps going between successive levels as in Figure 1. Consequently, there is an integer pairing on $H_1(C) = K/K'$, where $(x, y)_0$ is the linking number of the push-off of x with y. We define

$$(x, y) = \sum_i t^i (t^i x, y)_0.$$

Then

$$(f_i, f_j) = \begin{cases} 1+t & \text{if } i = j, \\ t & \text{if } i < j, \\ 1 & \text{if } i > j, \end{cases} \qquad (e_i, e_j) = \begin{cases} 1+t & \text{if} i = j, \\ -1 & \text{if } i = j-1, \\ -t & \text{if } i = j+1, \\ 0 & \text{if } |i-j| > 1. \end{cases}$$

If * is the automorphism of $\mathbb{Z}[t, t^{-1}]$ defined by $t^* = t^{-1}$, one checks that $(x, \alpha y + z) = \alpha(x, y) + (x, z)$, and $(x, y) = t(y, x)^*$. An easy calculation shows that this form is invariant under the action of B_n. Up to a scale factor and the change of variable $s^2 = t$, the restriction of this form to U_n was discovered by Squier [3].

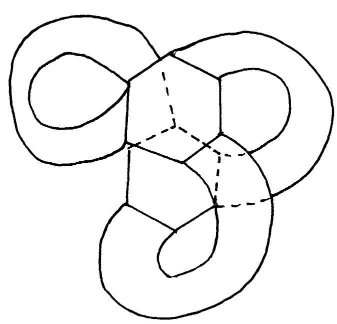


FIGURE 1

3. The Alexander module. Let $\gamma \in B_n$, let g be the union of the geometrical realization of γ with an additional straight string, and let $\hat{g}$ be the corresponding link. $\hat{g}$ is the disjoint union of the link defined by γ with the unknot. The reason for introducing the unknot will be clarified shortly.

Geometrically, it is advantageous to imagine the passage from g to $\hat{g}$ as occurring in two steps. First, we embed the braid g in a solid torus T and identify the ends in such a way that a parametrization of the resulting path(s) has positive longitudinal derivative at all points. Then we attach a second solid torus along the boundary of T to obtain $\mathbb{S}^3$ in the usual way. Let $F = F_{n+1}$. Since the complement $T - g$ is a twisted product of the circle with the $(n+1)$-punctured disk, it's not hard to see that the fundamental group of $T - g$ is the semi-direct product $G = \langle\gamma\rangle F$. When we attach the other torus, the effect on π_1 is to set $\gamma = 1$, and thus $\pi_1(\mathbb{S}^3 - \hat{g}) = G/[\gamma, G]\langle\gamma\rangle \cong F/[\gamma, F]$, where $[\gamma, F] = \langle \gamma x \gamma^{-1} x^{-1} \mid x \in F\rangle$ (see **[1]**, p. 46 for details).

Let $\varphi\colon F \to F/[\gamma, F]$ be the natural map, and let $K = \ker(\varepsilon) \subseteq F$. K defines the "parking garage" whose homology affords the unreduced Burau representation of B_n. Since $\varepsilon(\gamma x \gamma^{-1}) = \varepsilon(x)$ it follows that $[\gamma, F] \subseteq K$. Thus, ε factors through φ and defines a map $\hat{\varepsilon}$: $\pi_1(\mathbb{S}^3 - \hat{g}) \to \mathbb{Z}$. $\hat{\varepsilon}(x)$ is just the linking number of x with $\hat{g}$. Consequently, $\varphi(K)$ defines the infinite cyclic cover of $\mathbb{S}^3 - \hat{g}$ whose homology is the Alexander module. In particular, we see that $K/K'[\gamma, F]$ is isomorphic to the Alexander module of $\hat{g}$.

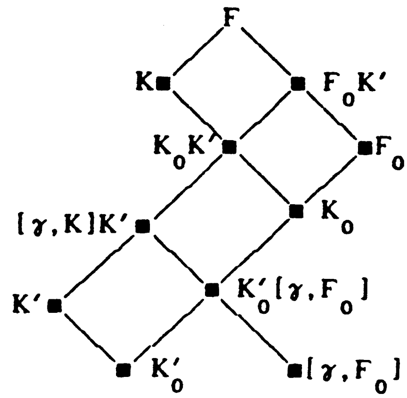
We would like to replace $[\gamma, F]$ by $[\gamma, K]$ because in additive notation the group $K/K'[\gamma, K]$ is just $V_n/(1-\gamma)(V_n)$. Indeed, since γ centralizes F/K it is tempting to conclude that $[\gamma, F] = [\gamma, K]$, but unfortunately this is not in general true. If, however, $F = KX$ where $[\gamma, X] = 1$ then the general identity

$$(*) \qquad [\gamma, kx] = [\gamma, k]k[\gamma, x]k^{-1}$$

implies immediately that $[\gamma, F] = [\gamma, K]$. This is the reason for adding the extra string. Since x_0 is centralized by γ we can take $X = \langle x_0 \rangle$ above and conclude that $V_n/(1-\gamma)V_n$ is the Alexander module of $\hat{g}$.

Finally, set $F_0 = \langle x_1, x_2, \dots, x_n\rangle \subseteq F_{n+1}$ and let $\hat{\gamma}$ be the link defined by γ. Then $\varphi(F_0) = \pi_1(\mathbb{S}^3 - \hat{\gamma})$, and if we put $K_0 = K \cap F_0$, then $K_0/K_0'[\gamma, F_0]$ is the Alexander module of $\hat{\gamma}$. Recall that $f_i = x_0 x_i^{-1} K'$ and that the elements $e_i = f_i - f_{i+1} = x_i x_{i+1}^{-1} K'$ $(1 \le i < n)$ are a $\mathbb{Z}[t, t^{-1}]$-basis for U_n. It follows easily that $K_0K'/K' = U_n$.

In order to keep track of the various relevant subgroups of F, the following lattice diagram may be helpful:



In such a diagram, a downward sloping line from A to B indicates that $A \supseteq B$. A parallelogram with A at the top, B and C at the sides and D at the bottom indicates that $A = BC$ and $D = B \cap C$. The above diagram makes several such assertions:

(a) $F = KF_0$,
(b) $[\gamma, F_0] \subseteq K_0$,
(c) $K_0' = K' \cap K_0$,
(d) $[\gamma, F_0]K' = [\gamma, K]K'$.

All remaining relationships are elementary group-theoretic consequences of these. Assertion (a) follows immediately from $\varepsilon(x_1) = 1$, and (b) follows from the previously noted containment $[\gamma, F] \subseteq K$. As for (c), we have $K_0' \subseteq K' \cap K_0$ and, by the isomorphism theorems, $(K' \cap K_0)/K_0'$ is the kernel of the natural epimorphism $K_0/K_0' \to K_0K'/K'$. But both of these groups are free R-modules of rank $n-1$, so the natural map is an isomorphism.

To prove (d), we already have shown that $[\gamma, K] = [\gamma, F]$, whence $[\gamma, F_0]K' \subseteq [\gamma, K]K'$. Conversely, since K/K' is generated as an R-module by $\{f_1, \dots, f_n\}$, it follows that

$$K = \langle x_0^j(x_0x_i^{-1})x_0^{-j} \mid 1 \le i \le n,\ j \in \mathbb{Z}\rangle K'.$$

Notice that since $[\gamma, F_0]$ is normalized by F_0 and $[\gamma, F_0]K' \trianglelefteq K$, we get $[\gamma, F_0]K' \trianglelefteq F$ by (a). Now repeated application of $(*)$ shows that $[\gamma, K]K'$ is contained in the normal closure in F of

$$\langle [\gamma, x_i^{-1}] \mid 1 \le i \le n\rangle K'$$

which is evidently contained in the normal subgroup $[\gamma, F_0]K'$. It now follows from the isomorphism theorems that

$$K_0/K_0'[\gamma, F_0] \cong K_0K'/[\gamma, K]K',$$

which says precisely that $U_n/(1-\gamma)V_n$ is the Alexander module of $\hat{\gamma}$.

4. The sesqui-linear form. Now fix $\gamma \in B_n$, let $R = \mathbb{Z}[t, t^{-1}]$, $W = W(\gamma) = \operatorname{im}(1-\gamma)$, and let $A = A(\gamma)$ be the complete inverse image of the R-torsion submodule of V_n/W. We want to define a pairing $(\ ,\ )_\gamma \colon A \times W \to R$ which is sesqui-linear, that is, conjugate linear (with respect to the automorphism *) in the first variable, and linear in the second. Set $(a, w)_\gamma = (a, v)$ where v is any element of $(1-\gamma)^{-1}(w)$. Any two choices of v differ by an element u with $\gamma(u) = u$. To see that $(a, w)_\gamma$ is independent of the choice of v, choose $r \in R$ such that $ra \in W$ and let $ra = (1-\gamma)(v_1)$. Then

$$r^*(a, u) = (ra, u) = (v_1 - \gamma(v_1), u) = (v_1, u) - (v_1, \gamma^{-1}(u)) = 0$$

and therefore $(a, u) = 0$. This shows that $(\ ,\ )_\gamma$ is well defined. It is obvious that $(\ ,\ )_\gamma$ is sesqui-linear with values in R.

Let $a \to \overline{a}$ be the natural map $A \to \overline{A} = A/W$, and choose $x, y \in A$. We define an element $\langle \overline{x}, \overline{y} \rangle = \langle \overline{x}, \overline{y} \rangle_\gamma$ of the R-module $\mathbb{Q}(t)/R$ as follows: choose $r \in R$ with $ry \in W$ and put $\langle \overline{x}, \overline{y} \rangle = r^{-1}(x, ry)_\gamma + R$.

We first argue that $\langle \overline{x}, \overline{y} \rangle$ is independent of the choice of r. Suppose $r_1 y \in W$. Then

$$r_1^{-1}(x, r_1 y)_\gamma - r^{-1}(x, ry)_\gamma = \frac{r(x, r_1 y)_\gamma - r_1(x, ry)_\gamma}{rr_1} = 0.$$

We next argue that $\langle \overline{x}, \overline{y} \rangle$ is independent of the representatives x, y. If $y - y_1 = w \in W$, then

$$r^{-1}(x, ry)_\gamma - r^{-1}(x, ry_1)_\gamma = r^{-1}(x, rw)_\gamma = (x, w)_\gamma \in R.$$

On the other hand, if $x - x_1 = (1-\gamma)(v) \in W$ and $ry = (1-\gamma)(v_1)$ then $-r\gamma^{-1}(y) = (1-\gamma^{-1})(v_1)$, and

$$\begin{aligned}(x, ry)_\gamma - (x_1, ry)_\gamma &= (v - \gamma(v), v_1) = (v, v_1) - (v, \gamma^{-1}(v_1)) \\ &= -r(v, \gamma^{-1}(y))\end{aligned}$$

whence $r^{-1}(x, ry)_\gamma - r^{-1}(x_1, ry)_\gamma = -(v, \gamma^{-1}(y)) \in R$.

It is now easily verified that $\langle\ ,\ \rangle_\gamma$ is a sesqui-linear form on $\overline{A}(\gamma)$. To see that it is $(-t)$-Hermitian, choose $x, y \in A(\gamma)$ and $r \in R$ such that

$$rx = u - \gamma(u), \qquad ry = v - \gamma(v)$$

for some $u, v \in V$. By expanding both sides, we easily verify the identity

$$(rx, v) + (u, ry) = (rx, ry).$$

Dividing both sides by rr^* and using $(u, y) = t(y, u)^*$ we have

$$r^{-1}(x, v) + t[r^{-1}(y, u)]^* = (x, y)$$

and thus

$$\langle \overline{x}, \overline{y}\rangle + t\langle \overline{y}, \overline{x}\rangle^* = 0.$$

We will show that the pair $(\overline{A}(\gamma), \langle\ ,\ \rangle_\gamma)$ depends only on the link $\hat{\gamma}$ (up to a form-preserving isomorphism) by showing that it is invariant under the Markov moves ([1], p. 51).

There are two moves. Suppose first that $\xi \in B_n$. Then the identity

$$(1 - \xi\gamma\xi^{-1})(V_n) = \xi(1-\gamma)\xi^{-1}(V_n) = \xi(1-\gamma)(V_n)$$

shows that multiplication by ξ induces an isomorphism $W(\gamma) \cong W(\xi\gamma\xi^{-1})$ and thus isomorphisms $\overline{V}(\gamma) \cong \overline{V}(\xi\gamma\xi^{-1})$ and $\overline{A}(\gamma) \cong \overline{A}(\xi\gamma\xi^{-1})$. We claim that

$$(\xi x, \xi y)_{\xi\gamma\xi^{-1}} = (x, y)_\gamma \qquad \text{for all } x \in A(\gamma),\ y \in W.$$

Namely, let $(1-\gamma)(v) = y$. Then

$$(\xi - \xi\gamma)(v) = \xi y = (1 - \xi\gamma\xi^{-1})(\xi v), \quad \text{so}$$
$$(\xi x, \xi y)_{\xi\gamma\xi^{-1}} = (\xi x, \xi v) = (x, v) = (x, y)_\gamma$$

as required.

The second Markov move is trickier. With the standard embedding $B_n \subseteq B_{n+1}$ we have $B_{n+1} = \langle B_n, \sigma_n\rangle$ and we need to find a form-preserving isomorphism $\overline{A}(\gamma) \to \overline{A}(\sigma_n^{\pm 1}\gamma)$. We have the inclusion $V_n \subseteq V_{n+1}$; in fact $V_{n+1} = V_n \oplus Rf_{n+1} = V_n \oplus Re_n$.

Lemma. $W(\sigma_n^{\pm 1}\gamma) = W(\gamma) \oplus Re_n$.

Proof. We make use of the formal identity

$$(*) \qquad (1 - \sigma_n^{\pm 1}\gamma) = (1 - \sigma_n^{\pm 1}) + (1-\gamma) - (1-\sigma_n^{\pm 1})(1-\gamma).$$

This implies that $W(\sigma_n^{\pm 1}\gamma) \subseteq W(\sigma_n^{\pm 1}) + W(\gamma)$. Note that

$$(1-\sigma_n)(f_i) = \begin{cases} 0 & \text{for } i < n, \\ te_n & \text{for } i = n, \\ -e_n & \text{for } i = n+1 \end{cases}$$

and

$$(1-\sigma_n^{-1})(f_i) = \begin{cases} 0 & \text{for } i < n, \\ -t^{-1}e_n & \text{for } i = n, \\ e_n & \text{for } i = n+1, \end{cases}$$

so $W(\sigma_n^{\pm 1}) = Re_n$. In particular, $W(\sigma_n^{\pm 1}) + W(\gamma) = W(\sigma_n^{\pm 1}) \oplus W(\gamma)$. Moreover, $(1-\sigma_n^{\pm 1}\gamma)(f_{n+1}) = (1-\sigma_n^{\pm 1})(f_{n+1}) = \pm e_n$ because $\gamma(f_{n+1}) = f_{n+1}$, and therefore we have $W(\sigma_n^{\pm 1}) \subseteq W(1-\gamma\sigma_n^{\pm 1})$. Now $(*)$ implies that $W(\gamma) \subseteq W(\sigma_n^{\pm 1}\gamma)$ and the lemma follows. □

Now the inclusion map $V_n \subseteq V_{n+1}$ induces a map $V_n/W(\gamma) \subseteq V_{n+1}/W(\gamma)$, and since $V_{n+1} = V_n \oplus Re_n$ the lemma implies that

$$\frac{V_{n+1}}{W(\sigma_n^{\pm 1}\gamma)} = \frac{V_n \oplus Re_n}{W(\gamma) \oplus Re_n} \cong \frac{V_n}{W(\gamma)}.$$

Explicitly, the map $\varphi\colon \overline{V}(\gamma) \to \overline{V}(\sigma_n^{\pm 1}\gamma)$ given by $\varphi(v + W(\gamma)) = v + W(\sigma_n^{\pm 1}\gamma)$ is an isomorphism. We need to show that φ is form-preserving on the torsion submodule.

We first define linear functionals α, β on V_{n+1} as follows: for $x \in V_{n+1}$ write $x = x_0 + \alpha(x)f_n + \beta(x)f_{n+1}$, where $x_0 \in V_{n-1}$. We next observe that

$$(1-\sigma_n)(x) = (\beta(x) - t\alpha(x))e_n,$$
$$(1-\sigma_n^{-1})(x) = (t^{-1}\beta(x) - \alpha(x))e_n,$$

and that if we define $S(x) = (\beta(x) - t\alpha(x))f_{n+1}$, then

$$(1-\sigma_n)S(x) = (\beta(x) - t\alpha(x))e_n = (1-\sigma_n)(x),$$
$$(1-\sigma_n^{-1})S(x) = (t^{-1}\beta(x) - \alpha(x))e_n = (1-\sigma_n^{-1})(x).$$

Since $\gamma(f_{n+1}) = f_{n+1}$ it follows easily that

$$(1-\sigma_n^{\pm 1}\gamma)S(x) = (1-\sigma_n^{\pm 1})(x) \quad \text{for all } x \in V_{n+1}.$$

Now we get

$$\begin{aligned} 1-\sigma_n^{\pm 1}\gamma &= (1-\sigma_n^{\pm 1})\gamma + (1-\gamma) \\ &= (1-\sigma_n^{\pm 1}\gamma)S\gamma + (1-\gamma) \end{aligned}$$

and thus

$$1-\gamma = (1-\sigma_n^{\pm 1}\gamma)(1-S\gamma).$$

To show that φ is form-preserving on $\overline{A}$, it suffices to show that if $a \in A(\gamma)$ and $w \in W(\gamma)$, then $(a, w)_\gamma = (a, w)_{\sigma_n^{\pm 1}\gamma}$. Choose

$v \in V_{n+1}$ with $(1-\gamma)(v) = w$ and set $u = (1-S\gamma)(v)$. Then $(1-\sigma_n^{\pm 1}\gamma)(u) = w$, so

$$(a, w)_\gamma = (a, v) \quad \text{and} \quad (a, w)_{\sigma_n^{\pm 1}\gamma} = (a, u).$$

We conclude that

$$(a, w)_\gamma - (a, w)_{\sigma_n^{\pm 1}\gamma} = (a, v-u) = (a, S\gamma(v)) = r(a, f_{n+1})$$

for some $r \in R$ because $\mathrm{im}(S) = Rf_{n+1}$. Now choose $r_1 \in R$ with $r_1 a \in W$. Since $\varepsilon_0(\gamma(v)) = \varepsilon_0(v)$ for all $v \in V_{n+1}$, $W(\gamma) \subseteq \ker(\varepsilon_0)$. Thus,

$$r_1 a = \sum_{i=1}^{n-1} \alpha_i f_i \quad \text{where} \sum_{i=1}^{n-1} \alpha_i = 0.$$

But then

$$r_1^*(a, f_{n+1}) = (r_1 a, f_{n+1}) = t\sum_{i=1}^{n-1} \alpha_i = 0$$

and therefore $(a, f_{n+1}) = 0$ as required.

To complete the proof of Theorem 2, we must show that the induced map $\overline{A} \to \mathrm{Hom}_R(\overline{A}, \mathbb{Q}(t)/R)$ is an isomorphism when the Alexander module is torsion. We defer this argument to the next section, where we obtain explicit formulas.

5. Computations. Theorem 1 says that a presentation for the Alexander module of a link $\hat{\gamma}$ can be obtained as follows. Let $w_j = (1-\gamma)(f_j) = \sum_{i=1}^{n-1} w_{ij} e_i$ $(1 \le j \le n)$. Since $\sum t^{j-1} f_j$ is a fixed point for γ we have $\sum t^{j-1} w_j = 0$ and therefore any $n-1$ of the w_j generate $W(\gamma)$. Hence, any $n-1$ columns of the matrix $W = w_{ij}$ are a presentation matrix for the Alexander module of $\hat{\gamma}$.

For example, let $\gamma = (\sigma_1 \sigma_2^{-1})^2$, so $\hat{\gamma} = 4_1$. The matrix of $1-\gamma$ with respect to $\{f_1, f_2, f_3\}$ is

$$\begin{bmatrix} 2t - t^2 & -t^{-1} & t^{-2} - 2t^{-1} + 1 \\ t^2 - t & 1 & -1 \\ -t & t^{-1} - 1 & 2t^{-1} - t^{-2} \end{bmatrix}.$$

Thus we get

$$w_1 = (2t - t^2)e_1 + te_2,$$
$$w_2 = -t^{-1}e_1 + (1 - t^{-1})e_2$$

so $\overline{A}(\gamma)$ is generated by $\{\overline{e}_1, \overline{e}_2\}$ subject to the relations

$$\overline{e}_1 = (t-1)\overline{e}_2, \qquad \overline{e}_2 = (t-2)\overline{e}_1,$$

and we obtain the presentation $\overline{A}(\gamma) = \{\overline{e}_1 \mid (t^2 - 3t + 1)\overline{e}_1 = 0\}$.

Returning to the general situation, let W_0 be a matrix consisting of any $n - 1$ columns of W, and let $\Delta = \det(W_0)$. Evidently, Δ is the Alexander polynomial. Assuming $\Delta \neq 0$, or equivalently that $A(\gamma) = U_n$, $\langle \overline{e}_i, \overline{e}_j \rangle$ can be computed by first solving the equation

$$\Delta e_j = \sum_k u_{kj} w_k$$

for u_{kj}. Thus, $U = u_{kj}$ is the classical adjoint of W_0, and

$$U W_0 = W_0 U = \Delta I.$$

Then

$$\langle \overline{e}_i, \overline{e}_j \rangle = \Delta^{-1} \sum_k u_{kj}(e_i, f_k) = \Delta^{-1}[t u_{ij} - u_{i+1,j}].$$

In the above example, we get

$$(t^2 - 3t + 1)e_1 = (t^{-1} - 1)w_1 + t w_2,$$

$$\langle \overline{e}_1, \overline{e}_1 \rangle = \Delta^{-1}[t(t^{-1} - 1) - t] = \frac{1 - 2t}{t^2 - 3t + 1}.$$

Scaling by $(1 - t^{-1})$ and reducing modulo R, we get

$$(1 - t^{-1})\langle \overline{e}_1, \overline{e}_1 \rangle = \frac{-1}{t - 3 + t^{-1}}.$$

In the general case, we put

$$T = \begin{bmatrix} t & -1 & 0 & \cdots & 0 \\ 0 & t & -1 & 0 & \\ & \cdot & \cdot & & \\ & & & \cdot & \\ & & & t & -1 \\ 0 & 0 & \cdots & 0 & t \end{bmatrix}$$

and $b_{ij} = \langle \overline{e}_i, \overline{e}_j \rangle$. Then $B = \Delta^{-1} T U$. Using this formula, we can now complete the proof of Theorem 2.

For $\overline{x}$, $\overline{y} \in \overline{A}$, define $\varphi_{\overline{x}}(\overline{y}) = \langle \overline{x}, \overline{y} \rangle$. To show that the map $\Phi(\overline{x}) = \varphi_{\overline{x}}$ is an isomorphism, we construct the inverse map as follows. Let $\varphi \in \mathrm{Hom}_R(\overline{A}, \mathbb{Q}(t)/R)$. Since $A = U_n$ is free on $\{e_1, \ldots, e_{n-1}\}$, φ can be lifted to a map $\varphi' \colon A \to \mathbb{Q}(t)$ with $\varphi(W(\gamma)) \subseteq R$. Let $\varphi'(e_i) = y_i$ and let $y = (y_1, \ldots, y_{n-1}) \in \mathbb{Q}(t)^{n-1}$. Then $y W_0 \in R$. Let $x = (x_1, \ldots, x_{n-1})$ be the row vector defined by $x^* = y W_0 T^{-1}$. Then $x^* \in R$ because T is unimodular. Moreover, $x^* T U = y W_0 U = \Delta y$ and thus $x^* B = y$. If we therefore let

$\overline{x} = \sum x_i \overline{e}_i \in \overline{A}$, we see that $\langle \overline{x}, \overline{e}_j \rangle = \varphi(\overline{e}_j)$ for all j. Put $\Psi(\varphi) = \overline{x}$. Then we have shown that $\Phi(\Psi(\varphi)) = \varphi$. Conversely, if we choose any $\overline{x} = \sum x_i \overline{e}_i \in \overline{A}$, set $x = (x_1, \ldots, x_{n-1})$ and let $\varphi = \varphi_{\overline{x}}$, then there is an obvious lift φ' corresponding to the row vector $y = x^* B$. Then $x^* = y W_0 T^{-1}$ and $\Psi(\varphi_{\overline{x}}) = \overline{x}$ as required.

6. Symmetries. Using the standard generators and relations for the braid group, it is easy to see that there is an automorphism $\gamma \to \gamma'$ such that $\sigma_i' = \sigma_i^{-1}$ for all i. Then $\hat{\gamma}'$ is just $\hat{\gamma}$ with all crossings reversed, i.e. the mirror image. Moreover, γ'^{-1} as a word in the σ_i is just γ read backwards, so $\hat{\gamma}'^{-1}$ is the inverse of $\hat{\gamma}$. Then $\hat{\gamma}^{-1}$ is obtained from $\hat{\gamma}$ by reversing both its orientation and the orientation of $\mathbb{S}^3$.

The symmetry $\gamma \to \gamma^{-1}$ is the easiest to analyze, so we will begin there. From the identity $-\gamma^{-1}(1-\gamma) = 1 - \gamma^{-1}$ it follows that multiplication by $-\gamma^{-1}$ induces an isomorphism $U_n/W(\gamma) \cong U_n/W(\gamma^{-1})$. Choose $x, y \in A(\gamma)$, $r \in R$, and $v \in V_n$ with $ry = (1-\gamma)v$. Then $-r\gamma^{-1}y = (1-\gamma^{-1})v$ and since $-\gamma^{-1}$ is unitary we have

$$\begin{aligned}\langle -\gamma^{-1}\overline{x}, -\gamma^{-1}\overline{y}\rangle_{\gamma^{-1}} &= r^{-1}(-\gamma^{-1}x, v)\\ &= -r^{-1}(x, \gamma v) = -r^{-1}(x, v - ry).\end{aligned}$$

It follows that

$$(*) \qquad \langle -\gamma^{-1}\overline{x}, -\gamma^{-1}\overline{y}\rangle_{\gamma^{-1}} = -\langle \overline{x}, \overline{y}\rangle_{\gamma}.$$

To analyze the mirror-image symmetry $\gamma \to \gamma'$, we define a map * on elements $v \in V_n$ (resp. R-linear maps $T\colon V_n \to V_n$) by applying the ring automorphism * to each co-ordinate of v (resp. matrix entry of T) with respect to the basis $\{f_1, f_2, \ldots, f_n\}$. Then $(Tv + w)^* = T^*v^* + w^*$, and $(T_1T_2)^* = T_1^*T_2^*$. For $\gamma \in B_n$ we abuse notation slightly by writing γ^* for the action of the conjugate Burau matrix.

Define $P\colon V_n \to V_n$ via $P(f_i) = f_{n-i-1}$. Then $P^* = P^{-1} = P$. Moreover, from the definition of the Squier form we have $(Pf_i, Pf_j) = (f_j, f_i)$ from which it follows by sesqui-linearity that

$$(Px, Py) = (y^*, x^*) \quad \text{for all } x, y \in V_n.$$

Define $\delta_1 = \sigma_1$, and inductively set $\delta_{i+1} = \sigma_1\sigma_2\cdots\sigma_i\delta_i$. Then δ_n is a half-twist of all $n+1$ strings, and it is easily checked that $\delta_n\sigma_i\delta_n^{-1} = \sigma_{n-i}$ $(1 \le i < n)$. By inspecting matrix entries we verify that

$$\delta P \sigma_i^* P^{-1} \delta^{-1} = \sigma_i^{-1} \qquad (1 \le i < n)$$

from which we obtain the basic identity

$$\delta P \gamma^* P^{-1} \delta^{-1} = \gamma' \quad \text{for all } \gamma \in B_n.$$

Now choose $y, v \in V_n$ and $r \in R$ with $ry = (1-\gamma)v$. Then

$$r^* y^* = (1-\gamma^*)v^* = (\delta P)^{-1}(1-\gamma')\delta P(v^*), \quad \text{and thus}$$
$$r^* \delta P(y^*) = (1-\gamma')\delta P(v^*).$$

It follows that the map $y \mapsto y' = \delta P(y^*)$ defines a (conjugate-linear) isomorphism $U_n/W(\gamma) \to U_n/W(\gamma')$. Moreover, we have

$$\langle x', y' \rangle_{\gamma'} = (\delta P x^*, \delta P v^*)/r^* = (Px^*, Pv^*)/r^* = (v, x)/r^*$$
$$= t(x, v)^*/r^* = t\langle x, y \rangle_\gamma^*.$$

This result together with equation $(*)$ implies that for the inverse symmetry $\gamma \to \gamma'^{-1}$, the map $y \mapsto y'' = -\gamma^{-1}\delta P(y^*)$ defines a conjugate linear isomorphism $U_n/W(\gamma) \to U_n/W(\gamma'^{-1})$ with

$$\langle x'', y'' \rangle_{\gamma'^{-1}} = -t\langle x, y \rangle_\gamma^*.$$

These results have particularly simple consequences in the special case that the Alexander module is cyclic with generator e and annihilator Δ. The form is completely determined by the element $\langle e, e \rangle$, and e' is another generator iff $e' = \alpha e$ for some unit α of $R/\Delta R$. Since the form is $(-t)$-hermitian, we have $\langle e, e \rangle = -t\langle e, e \rangle^*$ which easily implies that non-invertibility cannot be detected in this case. If, however, $\hat{\gamma}$ is amphicheiral, then for some unit α of $R/\Delta R$ we have

$$\alpha\alpha^* \langle e, e \rangle = t\langle e, e \rangle^*.$$

Non-singularity of the form implies that $\langle e, e \rangle$ is a unit, and thus we get the necessary condition $\alpha\alpha^* = -1$ for some unit α.

7. Taking the trace. We first observe that the form $\langle\ ,\ \rangle$ actually takes values in a cyclic submodule of $\mathbb{Q}(t)/R$ isomorphic to R/Rr for some $r \in R$. Namely, recall that R is a noetherian UFD and V_n/W is finitely generated. Thus, $\overline{A}$ has a finite set of generators $\{\overline{a}_1, \dots, \overline{a}_m\}$. Let $r = \operatorname{lcm}\{\operatorname{ann}_R(\overline{a}_i)\}$. Then the form takes values in the R-module Rr^{-1}/R which is isomorphic to R/Rr. Alternatively, we could take r to be the Alexander polynomial of γ.

We can assume that the gcd of the coefficients of r is 1 (this amounts to the assertion that V_n/W is $\mathbb{Z}$-torsion free, which can be easily seen by specializing the Burau representation at $t = 1$). Tensoring with $\mathbb{Q}$ then embeds R/Rr into the finite dimensional algebra $\mathbb{Q}[t, t^{-1}]/(r)$ which admits the canonical linear functional $\operatorname{tr}_{(r)}$, the trace of the regular representation. By applying this functional, we obtain a rational valued form $\langle\ ,\ \rangle_0 = \mathrm{tr}_{(r)}\langle\ ,\ \rangle$ on the Alexander module. This form appears to depend on the choice of denominator r, but in fact it does not. For if we replace r by rs, the coset representing a given value $\langle\overline{x}, \overline{y}\rangle$ of the form is also multiplied by s and thus has zero trace on the submodule $(r)/(rs)$ of $\mathbb{Q}[t, t^{-1}]/(rs)$. Since the isomorphism

$$\frac{\mathbb{Q}[t, t^{-1}]/(rs)}{(r)/(rs)} \cong \mathbb{Q}[t, t^{-1}]/(r)$$

implies that $\mathrm{tr}_{(rs)} = \mathrm{tr}_{(r)} + t'$ where $t'(x)$ is the trace of the restriction of $\mathrm{ad}(x)$ to $(rs)/(r)$, $\langle\ ,\ \rangle_0$ does not depend on r.

For example, in the calculation for the figure eight knot, we got the hermitian

$$(1 - t^{-1})\langle\overline{e}_1, \overline{e}_1\rangle = \frac{-1}{t - 3 + t^{-1}}.$$

since $t - 3 + t^{-1}$ is monic, the Alexander module is finitely generated over $\mathbb{Z}$, in this case it is $\mathbb{Z} \oplus \mathbb{Z}$. If we take the basis $\{e_1, te_1\}$, the action of t is $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 3 \end{smallmatrix}\right]$ and the matrix of the trace form is $\left[\begin{smallmatrix} -2 & -3 \\ -3 & -2 \end{smallmatrix}\right]$.

8. The (n, m) **torus link.** Let $\tau = \sigma_1\sigma_2\cdots\sigma_{n-1}$. Then the obvious braid representative for the (n, m) torus link is $\gamma = \tau^m$. It is easy to see that

$$\begin{aligned} \tau(f_i) &= (1-t)f_1 + tf_{i+1} \qquad (1 \le i < n), \\ \tau(f_n) &= f_1. \end{aligned}$$

Let $e_i = f_i - f_{i+1}$ $(1 \le i < n)$ as above. We will calculate with respect to the basis $\{e_1, e_2, \ldots, e_{n-1}, f_n\}$ of V_n. Define $e_0 = f_n - f_1$ and note that $e_0 = -\sum_{i=1}^{n-1} e_i$. It is easily checked that $\tau(e_i) = te_{i+1}$ for all i with subscripts modulo n. Moreover, $\tau(f_n) = f_1 = f_n - e_0$, so we have

$$\tau^m(e_i) = t^m e_{i+m}, \qquad \tau^m(f_n) = f_n - \sum_{i=0}^{m-1} t^i e_i.$$

The Alexander module A_{nm} for the (n, m) torus link is therefore generated by $\{\overline{e}_0, \overline{e}_1, \ldots, \overline{e}_{n-1}\}$ subject to the relations

(1) $\overline{e}_{i+m} = t^{-m}\overline{e}_i$ $(0 \le i < n)$,

(2) $\sum_{i=0}^{n-1} \overline{e}_i = 0$,

(3) $\sum_{i=0}^{m-1} t^i\overline{e}_i = 0$.

These relations may be unraveled as follows. Let $d = \gcd(n, m)$, $m = kd$, $n = ld$, and $e = \mathrm{lcm}(n, m)$. The relations (1) can be iterated to obtain

$$\overline{e}_{i+rm} = t^{-rm}\overline{e}_i \qquad (0 \le i < n)$$

for any integer r. Note that there are exactly l distinct multiples of m modulo n, namely $\{0, d, 2d, \dots, (l-1)d\}$ and a set of orbit representatives for translation by m is given by $\{0, 1, \dots, d-1\}$. Thus, relations (1) say precisely that A_{nm} is generated by $\{\overline{e}_0, \overline{e}_1, \dots, \overline{e}_{d-1}\}$ and that $(1-t^{lm})\overline{e}_i = 0 \ (0 \le i < d)$. Since $lm = kn = e$, we see that $(1-t^e)$ annihilates A_{nm}. Relations (2) and (3) can be re-written in terms of $\{\overline{e}_0, \overline{e}_1, \dots, \overline{e}_{d-1}\}$ as follows. Let a be the least positive integer such that $am \equiv d \pmod{n}$. Then $\overline{e}_{i+d} = t^{-am}\overline{e}_i$ for all i. Let

$$u = \overline{e}_0 + \overline{e}_1 + \cdots + \overline{e}_{d-1},$$
$$v = \overline{e}_0 + t\overline{e}_1 + \cdots + t^{d-1}\overline{e}_{d-1}.$$

Then relation (2) says that

$$(4) \qquad t^{-(l-1)am}(1 + t^{am} + t^{2am} + \cdots + t^{(l-1)am})u = 0$$

and relation (3) becomes

$$(5) \qquad (1 + t^{d-am} + t^{2(d-am)} + \cdots + t^{(k-1)(d-am)})v = 0.$$

Let $d - am = bn$, and put

$$p_m(t) = \frac{1-t^{lam}}{1-t^{am}} = \frac{1-t^{ae}}{1-t^{am}}, \quad p_n(t) = \frac{1-t^{kbn}}{1-t^{bn}} = \frac{1-t^{be}}{1-t^{bn}}.$$

There are two special cases to consider: $a = 0$, $b = 1$, in which case $d = n$; and $a = 1$, $b = 0$, in which case $d = m$. In the former case, put $p_m(t) = 1$, and in the latter case, put $p_n(t) = 1$. Then is all cases, we can re-write (4) and (5) as

$$p_m(t)u = 0 = p_n(t)v.$$

However, we also have $(1-t^e)u = 0 = (1-t^e)v$, so u (resp. v) is annihilated by the gcd of $p_m(t)$ (resp. $p_n(t)$) and $1-t^e$. Now, $p_m(t)$ is the product of the cyclotomic polynomials $\Phi_r(t)$ as r ranges over all divisors of $ae = aml$ which do not divide am. Since $d = am + bn$, we have $1 = ak + bl$ which implies that if $r|e$ and $r|am$, then $r|m$. We conclude that

$$(6) \qquad \frac{1-t^e}{1-t^m}u = 0 = \frac{1-t^e}{1-t^n}v.$$

It is not difficult now to check that A_{nm} is generated by $\{\overline{e}_0, \overline{e}_1, \dots, \overline{e}_{d-1}\}$ subject to the relations (6) and

$$(7) \qquad (1-t^e)\overline{e}_i = 0 \qquad (0 \leq i < d).$$

To present A_{nm} as a direct sum of cyclic modules, we set

$$v' = \frac{1}{1-t}(u-v) = \sum_{i=1}^{d-1} \frac{1-t^i}{1-t}\overline{e}_i, \quad \text{and}$$

$$u' = \frac{1-t^{bn}}{1-t^d}u + t^{bn}\frac{1-t^{am}}{1-t^d}v = \frac{1-t^{bn}}{1-t^d}u + \left[1 - \frac{1-t^{bn}}{1-t^d}\right]v.$$

Routine calculations then show that $\{u', v', \overline{e}_2, \dots, \overline{e}_{d-1}\}$ is a basis for A_{nm}, provided that $d \geq 2$ and $d \neq n, m$. Moreover, it follows that

$$(8) \qquad \frac{(1-t^e)(1-t^d)}{(1-t^n)(1-t^m)}u' = 0 = \frac{(1-t^e)(1-t)}{(1-t^d)}v',$$

and that relations (7) and (8) imply (6) and (7).

Note that in the knot case ($d = 1$) we have $v = u = u' = \overline{e}_0$. Hence, the Alexander module is cyclic with annihilator

$$(1-t^{nm})(1-t)/(1-t^n)(1-t^m).$$

If either $n \mid m$ or $m \mid n$ then $u' = 0$ and $\{v', \overline{e}_2, \dots, \overline{e}_{d-1}\}$ is a basis for A_{nm}. For example, $A_{2,2k}$ is cyclic with annihilator

$$(1-t^{2k})(1-t)/(1-t^2).$$

Finally, if $n = m$, then $u' = v' = 0$ and $\{\overline{e}_2, \dots, \overline{e}_{d-1}\}$ is a basis.

To evaluate $\langle \overline{e}_i, \overline{e}_j \rangle$ we use relations (1) to write

$$(1-t^e)e_j = (1-\tau^m)(e_j + t^m e_{j+m} + t^{2m} e_{j+2m} + \cdots + t^{(l-1)m} e_{j+(l-1)m}).$$

Hence we obtain

$$(*) \qquad \langle \overline{e}_i, \overline{e}_j \rangle = \frac{1}{1-t^e} \sum_{p=0}^{l-1} t^{pm} (e_i, e_{j+pm}).$$

Recall that

$$(e_i, e_j) = \begin{cases} 1+t & \text{if } i = j, \\ -t & \text{if } i = j-1, \\ -1 & \text{if } i = j+1, \\ 0 & \text{if } |i-j| > 1, \end{cases} \qquad \text{for } 1 \leq i, j < n.$$

In fact, the same formulas extend to $0 \le i,\ j < n$, and using them, it is not difficult to make $(*)$ explicit. For example, in the knot case we get

$$\langle \bar{e}_0, \bar{e}_0 \rangle = \frac{1}{1 - t^{nm}}[1 + t - t^{am} - t^{1-am}] = \frac{(1 - t^{am})(1 - t^{bn})}{1 - t^{nm}}.$$

References

[1] J. Birman, *Braids, Links, and Mapping Class Groups.*, Annals of Mathematics Studies 85, Princeton Univ. Press, 1975.

[2] R. C. Blanchfield, *Intersection theory of manifolds with operators with applications to knot theory*, Ann. of Math., **65** (1957), 340–356.

[3] C. Squier, *The Burau representation is unitary*, Proc. Amer. Math. Soc., **90** (1984), 199–202.

[4] H. F. Trotter, *On S-equivalence of Seifert matrices*, Inventiones Math., **20** (1973), 173–207.

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