PACIFIC JOURNAL OF MATHEMATICS
Vol. 147, No. 1, 1991

RATIONAL FORMAL GROUP LAWS

ROBERT F. COLEMAN AND FRANCIS OISIN MCGUINNESS

In this paper we determine the rational formal groups defined over a field of characteristic zero. This answers a question originally posed by Robert MacPherson.

While one can answer this question using Weil's theorem which asserts that every birational group is birationally isomorphic to an actual algebraic group **[W1]**, below we give an elementary argument using methods similar to those used in **[C]**.

THEOREM. *Every rational formal group law over an algebraically closed field K of characteristic zero is of the form*

$$L^{-1}G(L(x), L(y))$$

where $G(x, y)$ is either $x+y$ or $x+y+xy$ and L is a linear functional transformation over K such that $L(0)=0$.

One deduces easily from this that

COROLLARY. *The rational formal group laws over a field K of characteristic zero are the rational functions*

$$(x+y+cxy)/(1-dxy)$$

where c and d are elements of K. Moreover, this formal group is rationally isomorphic to $x+y$ over K if $c^2-4d=0$ and to $x+y+xy$ over $K(\sqrt{(c^2-4d)})$ otherwise.

Proof of theorem. Recall that now K is algebraically closed. Suppose $F(x, y)$ is a rational formal group.

Let $\omega = dx/F_2(x, 0)$ and $g(x) = F(x, x)$ (the rational function giving multiplication by 2 on F). Then ω and g satisfy the hypothesis of the following proposition:

PROPOSITION. *Suppose $\omega \in K(x)\,dx$ and $g \in K(x)$, $\omega \neq 0$, $\mathrm{ord}_0\,\omega = 0$, $g(0)=0$ and $g^*\omega = 2\omega$. Then $\omega = L^*(dx)$ or $L^*(c\,dx/(x+1))$*

where L is a linear fractional transformation defined over K such that $L(0) = 0$ and $c \in K^*$.

Proof. Let Y denote the set of poles and Z the set of zeros of ω. It follows from the hypothesis that $g^{-1}Y = Y$ and $g^{-1}Z = Z$.

The equation $g^*\omega = 2\omega$ implies that

$$\sum \operatorname{ord}_Q g^*\omega = \sum \operatorname{ord}_Q \omega$$

where the sums run over $Q \in Y = g^{-1}Y$. Suppose $Q \in \mathbb{P}^1(K)$. Then we also have the formula

$$\sum \operatorname{ord}_P g^*\omega = \deg(g) \cdot \operatorname{ord}_Q \omega + (\deg(g) - \# g^{-1}(Q))$$

where the sum runs over $P \in g^{-1}(Q)$. Suppose now Q is a pole of ω. The right-hand side of this formula is less than or equal to $\operatorname{ord}_Q \omega$. Hence the last two formulas imply that

$$\deg(g) \cdot \operatorname{ord}_Q \omega + (\deg(g) - \# g^{-1}(Q)) = \operatorname{ord}_Q \omega$$

for alll $Q \in Y$. This occurs for a given $Q \in Y$ iff $\deg(g) = 1$ or $\operatorname{ord}_Q \omega = -1$ (in which case $\# g^{-1}(Q) = 1$).

Suppose first that $\deg(g) = 1$ and ω has a pole of order greater than one. Since $g^*\omega = 2\omega$, no iterate of g is the identity. As $g(0) = 0$ it follows that there exists exactly one non-zero point fixed by some iterate of g. Since $g^{-1}Y = Y$, $g^{-1}Z = Z$ and $\operatorname{ord}_0 \omega = 0$, we see that ω has only one pole and no zeros. It follows that $\omega = L^*(dx)$ for some linear fractional transformation L which we may assume vanishes at the origin.

Suppose now that ω has only simple poles. If Q is a pole of ω we know that $g^{-1}(Q)$ consists of exactly one point, P say, and we have the formula

$$\operatorname{Res}_P g^*\omega = \deg(g) \operatorname{Res}_Q \omega$$

by a local computation. Since $g^*(\omega) = 2\omega$, this becomes

$$\operatorname{Res}_P \omega = (\deg(g)/2) \operatorname{Res}_Q \omega.$$

Now we know that $g^{-1}Y = Y$. Hence, there exists a Q in Y and a positive integer n such that $\{Q\} = g^{-n}(Q)$. By iterating the previous equation we deduce that

$$(\deg(g)/2)^n \operatorname{Res}_Q \omega = \operatorname{Res}_Q \omega.$$

Hence, as $\operatorname{Res}_Q \omega \neq 0$ and $\deg(g) \in \mathbb{Z}_{>0}$, $\deg(g) = 2$.

The facts that $g^{-1}Y = Y$ and $g^{-1}Z = Z$ imply that the zeros and poles of ω lie among the branch points of $g: \mathbb{P}^1 \to \mathbb{P}^1$. Since g has degree 2 it has only two branch points. Since ω is not equal to zero, has only simple poles and its residues sum to zero it must have exactly two poles and no zeros. Hence $\omega = L^*(c\,dx/(x+1))$ for some linear fractional transformation L and some constant $c \in K^*$. Since $\mathrm{ord}_0\,\omega = 0$, we may assume $L(0) = 0$. This proves the proposition. □

The theorem follows from the proposition noting that $F(x, y) = L^{-1}G(L(x), L(y))$ where $G(x, y) = x + y$ if $\omega = L^*(dx)$ and $G(x, y) = x + y + xy$ if $\omega = L^*(c\,dx/(x+1))$. □

REMARK. The only place in the above argument where the algebraic closedness of K was used in a serious manner was in the last step which required finding a linear fractional transformation which moved one pole of ω to 0 and the other to ∞.

REFERENCES

[C] R. Coleman, *One dimensional algebraic formal groups*, Pacific J. Math., **122** (1986), 35–41.

[W] A. Weil, *On algebraic groups of transformations*, Amer. J. Math., **77** (1955), 355–391.

Received November 6, 1988 and in revised form January 17, 1989.

UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720

FORDHAM UNIVERSITY
BRONX, NY 10458