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ASYMPTOTIC BEHAVIOUR OF SUPERCUSPIDAL CHARACTERS OF p-ADIC GL_3 AND GL_4: THE GENERIC UNRAMIFIED CASE

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This paper describes the singular behaviour of the characters of irreducible supercuspidal representations of π of $G = GL_n(F)$ around 1 in terms of the values at 1 of certain weighted orbital integrals. The weighted orbital integrals are computed when $n = 3$ or 4 and π is generic and unramified.

1. Introduction. Let π be an irreducible supercuspidal representation of $G = GL_n(F)$, where F is a p-adic field of characteristic 0. The character Θ_π of π is a locally constant function on the regular set G_{reg} consisting of all $x \in G$ such that the coefficient of λ^n in the polynomial $\det(\lambda + 1 - \operatorname{Ad} x)$ is nonzero. It is well known that, if $d(\pi)$ is the formal degree of π and $x \in G_{\text{reg}}$ is elliptic and close to the identity, $\Theta_\pi(x) = cd(\pi)$ for some constant c depending only on normalizations of Haar measures. For other $x \in G_{\text{reg}}$ near 1, the value of $\Theta_\pi(x)$ is unknown. Kutzko [**K**] has given a formula for Θ_π when n is prime, but it involves a sum over double cosets in G and cannot easily be evaluated.

The two objects of this paper are as follows. The first is to describe the singular behaviour of the character Θ_π of π around 1 in terms of the values at 1 of certain weighted orbital integrals. To do this, we compare results of Howe and Arthur giving asymptotic expansions for Θ_π. The second is to compute the weighted orbital integrals required to give a formula for Θ_π when $n = 3$ or 4 and π is generic and unramified.

Howe showed that

$$\Theta_\pi(\exp X) = \sum_{\mathcal{O} \in (\mathcal{N}_G)} c_{\mathcal{O}}(\pi)\hat{\mu}_{\mathcal{O}}(X),$$

for $X \in \mathcal{G} = \operatorname{Lie}(G)$ close to zero and such that $\exp X \in G_{\text{reg}}$. $(\mathcal{N}_G)$ denotes nilpotent $\operatorname{Ad} G$-orbits in $\mathcal{G}$, $c_{\mathcal{O}}(\pi)$ is a constant, and $\hat{\mu}_{\mathcal{O}}$ is the Fourier transform of the orbital integral over $\mathcal{O}$. In the case of $GL_n(F)$, the functions $\hat{\mu}_{\mathcal{O}}$ are known. The behaviour of $\Theta_\pi(x)$ as

$x \in G_{\text{reg}}$ approaches 1 is determined by the homogeneity properties of those $\hat{\mu}_{\mathcal{O}}$'s for which $c_{\mathcal{O}}(\pi) \neq 0$. These results are outlined in §2.

In §3 we state results of Arthur **[A3]**, **[A4]** showing that a weighted orbital integral has a germ expansion valid on a neighbourhood of 1, and that Θ_π itself is a multiple of a weighted orbital integral of a sum of matrix coefficients of π.

The equality of Howe's and Arthur's expansions for Θ_π yields one of the main results of this paper—a formula for each constant $c_{\mathcal{O}}(\pi)$ as a multiple of a certain weighted orbital integral evaluated at 1. We derive this formula in §4. It holds for all n and any supercuspidal representation of $\mathrm{GL}_n(F)$.

In §§5 and 7, we consider a generic, unramified, irreducible supercuspidal representation π of $\mathrm{GL}_3(F)$ or $\mathrm{GL}_4(F)$. Such a representation is known to be induced from a representation of some open subgroup of G. The particular sum of matrix coefficients appearing in the weighted orbital integrals is defined in §5 using results of Kutzko which give the character of the inducing representation. §6 contains a description of the normalizations of measures and the evaluation of the weight factor for the weighted orbital integrals. In §7, we obtain explicit expressions for the constants $c_{\mathcal{O}}(\pi)$ as polynomials in the order q of the residue class field of F.

The equality of Arthur's expansion and Harish-Chandra's generalization of Howe's expansion to a reductive p-adic group can be expected to yield information about the character Θ_π of any supercuspidal representation π. However, the functions $\hat{\mu}_{\mathcal{O}}$, which are not known in general, may be difficult to compute, and the germ expansion for weighted orbital integrals is more complicated than that for $\mathrm{GL}_n(F)$.

I would like to thank Paul Sally for helpful discussions and James Arthur for explaining his results about weighted orbital integrals.

2. Fourier transforms and characters of admissible representations. Throughout this section, G will be the F-points of a connected, reductive F-group. Let π be an irreducible admissible representation of G. Θ_π denotes the character of π. We summarize results of Harish-Chandra and Howe relating the values of Θ_π near singular points in G to certain Fourier transforms.

Recall the definition of the Fourier transform on the Lie algebra $\mathcal{G}$ of G. For $f \in C_c^\infty(\mathcal{G})$, the function $\hat{f} \in C_c^\infty(\mathcal{G})$ is given by:

$$\hat{f}(X) = \int_{\mathcal{G}} \psi(B(X, Y)) f(Y)\, dY, \qquad X \in \mathcal{G},$$

where B is a nondegenerate symmetric G-invariant bilinear form on $\mathscr{G}$, ψ is a nontrivial character of F and dY is a Haar measure on the additive group of $\mathscr{G}$. The map $f \mapsto \hat{f}$ is a bijection of $C_c^\infty(\mathscr{G})$. The Fourier transform of a distribution T on $\mathscr{G}$ is defined by $\hat{T}(f) = T(\hat{f})$. Let $\mathscr{G}_{\mathrm{reg}}$ be the set of semisimple elements X in $\mathscr{G}$ such that $\det(\operatorname{ad} X)_{\mathscr{G}/\mathscr{H}} \neq 0$, where $\mathscr{H}$ is a Cartan subalgebra containing X.

THEOREM 2.1 [**HC2**, *Theorem* 3]. *Let T be a G-invariant distribution on $\mathscr{G}$ which is supported on the closure of* $\operatorname{Ad} G(\omega)$ *for some compact set $\omega \subset \mathscr{G}$. Then there exists a locally integrable function ϕ_T on $\mathscr{G}$ such that*

1. $\hat{T}(f) = \int_{\mathscr{G}} \phi_T(X) f(X)\, dX$, $f \in C_c^\infty(\mathscr{G})$.
2. ϕ_T *is locally constant on* $\mathscr{G}_{\mathrm{reg}}$.

Let $X_0 \in \mathscr{G}$ and $\mathscr{O} = \operatorname{Ad} G(X_0)$. If G_{X_0} is the stabilizer of X_0 in G, let dx^* be a G-invariant measure on $G_{X_0}\backslash G$. Then

$$\mu_{\mathscr{O}}(f) = \int_{G_{X_0}\backslash G} f(\operatorname{Ad} x^{-1}(X_0))\, dx^*$$

converges for $f \in C_c^\infty(\mathscr{G})$ and $f \mapsto \mu_{\mathscr{O}}(f)$ is a G-invariant distribution on $\mathscr{G}$.

COROLLARY 2.2 [**HC2**]. *There exists a locally integrable function $\hat{\mu}_{\mathscr{O}} \colon \mathscr{G} \to \mathbf{C}$ which is locally constant on $\mathscr{G}_{\mathrm{reg}}$ and*

$$\hat{\mu}_{\mathscr{O}}(f) = \int_{\mathscr{G}} \hat{\mu}_{\mathscr{O}}(X) f(X)\, dX,$$

for $f \in C_c^\infty(\mathscr{G})$.

Let $(\mathscr{N}_G)$ be the set of nilpotent G-orbits in $\mathscr{G}$. If q is the order of the residue class field of F, $|\cdot|$ denotes the norm on F which satisfies $|\varpi| = q^{-1}$ for any prime element ϖ of F. For $\gamma \in G$, let G_γ be the centralizer of γ in G, and let $\mathscr{G}_\gamma$ be the Lie algebra of G_γ.

PROPOSITION 2.3 [**HC2**]. *For $\mathscr{O} \in (\mathscr{N}_G)$, $X \in \mathscr{G}$ and $t \in F^*$, $\hat{\mu}_{\mathscr{O}}(t^2 X) = |t|^{-\dim \mathscr{O}} \hat{\mu}_{\mathscr{O}}(X)$.*

Proof. For $f \in C_c^\infty(\mathscr{G})$, define $f_t(X) = f(t^{-1}X)$, $X \in \mathscr{G}$. It is well-known that $\mu_{\mathscr{O}}(f_{t^2}) = |t|^{\dim \mathscr{O}} \mu_{\mathscr{O}}(f)$. This, together with $(\widehat{f_t}) = |t|^{\dim \mathscr{G}} (\hat{f})_{t^{-1}}$, proves the proposition.

THEOREM 2.4 [**HC2**, *Theorem* 5]. *Let γ be a semisimple point in G. For any irreducible admissible representation π of G, there exist unique complex numbers $c_{\mathscr{O}}(\pi)$, one for each nilpotent G_γ-orbit $\mathscr{O}$ in $\mathscr{G}_\gamma$, such that*

$$\Theta_\pi(\gamma \exp X) = \sum_{\mathscr{O}} c_{\mathscr{O}}(\pi)\hat{\nu}_{\mathscr{O}}(X),$$

for $X \in \mathscr{G}_\gamma$ sufficiently near 0. Here $\nu_{\mathscr{O}}$ is the G_γ-invariant measure on $\mathscr{O}$, and $\hat{\nu}_{\mathscr{O}}$ is the Fourier transform of $\nu_{\mathscr{O}}$ on $\mathscr{G}_\gamma$.

REMARK. The case $G = \mathrm{GL}_n(F)$ and $\gamma = 1$ is due to Howe [**H**].

The functions $\{\hat{\mu}_{\mathscr{O}} | \mathscr{O} \in (\mathscr{N}_G)\}$ are linearly independent on $V \cap \mathscr{G}_{\mathrm{reg}}$, for any neighbourhood V of 0 in $\mathscr{G}$ [**HC2**, Theorem 4]. Therefore the functions $\{\hat{\mu}_{\mathscr{O}} | c_{\mathscr{O}}(\pi) \neq 0\}$ determine the singular behaviour of Θ_π near 1. Very little is known about the constants $c_{\mathscr{O}}(\pi)$ in general. If π is supercuspidal with formal degree $d(\pi)$, then, if $\{0\}$ denotes the trivial nilpotent orbit, $c_{\{0\}}(\pi) = cd(\pi)$ where $c \neq 0$ depends on the normalization of measures. Howe [**H**] proved that, if π is a supercuspidal representation of $\mathrm{GL}_n(F)$, then $c_{\mathscr{O}}(\pi) = 1$ for the regular (maximal dimension) nilpotent orbit $\mathscr{O}$. Moeglin and Waldspurger [**MW**] have shown a relation between $c_{\mathscr{O}}(\pi)$, for π admissible and some $\mathscr{O}$, and dimensions of certain Whittaker models. As far as the functions $\hat{\mu}_{\mathscr{O}}$ themselves are concerned, there is some information available in [**MW**] for induced nilpotent classes, and for $G = \mathrm{GL}_n(F)$ the $\hat{\mu}_{\mathscr{O}}$'s are known due to Howe (see Lemma 4.1).

3. Weighted orbital integrals and characters of supercuspidal representations. We state several results due to Arthur which will be used in later sections. Theorem 3.4 relates the character Θ_π of a supercuspidal representation π to a weighted orbital integral of a sum of matrix coefficients of π. Theorem 3.5 gives a germ expansion for weighted orbital integrals. A vanishing property for weighted orbital integrals of cusp forms is stated in Proposition 3.9. In Proposition 3.7, we derive a formula for the weighted germ g_M^G corresponding to the trivial unipotent class in a Levi subgroup M.

Our notation follows that of Arthur [**A2**]–[**A4**] except in one respect: the boldface letter $\mathbf{G}$ will be used to denote an algebraic group defined over F, and $G = \mathbf{G}(F)$ will be the F-rational points of $\mathbf{G}$. By a Levi subgroup M of G, we mean $M = \mathbf{M}(F)$, where $\mathbf{P} = \mathbf{MN}$ is a parabolic subgroup of $\mathbf{G}$. If $A_{\mathbf{M}}$ is the split component of $\mathbf{M}$, then $A_M = A_{\mathbf{M}}(F)$. Let $\mathscr{F}(M)$, resp. $\mathscr{L}(M)$, be the collection of parabolic, resp. Levi, subgroups of G which contain

M. Given a parabolic subgroup $P = \mathbf{P}(F)$, M_P and N_P denote its Levi component and unipotent radical, respectively. Let $\mathscr{P}(M) = \{P \in \mathscr{F}(M) \mid M_P = M\}$. The chambers in the real vector space $\underline{a}_M = \mathrm{Hom}(X(\mathbf{M})_F, \mathbf{R})$ parametrize the set $\mathscr{P}(M)$, where $X(\mathbf{M})_F$ is the group of characters of $\mathbf{M}$ which are defined over F.

We now review the notation required in order to define the weights v_M occurring in the weighted orbital integrals. Given M, choose a special maximal compact subgroup K of G which is in good position relative to M. For $P \in \mathscr{P}(M)$ and $x = n_P(x)m_P(x)k(x)$, with $n_P(x) \in N_P$, $m_P(x) \in M_P$, and $k(x) \in K$, set $H_P(x) = H_M(m_P(x))$. Here $H_M\colon M \to \underline{a}_M$ is given by:

$$e^{\langle H_M(m), \chi\rangle} = |\chi(m)|, \qquad m \in M,\ \chi \in X(\mathbf{M})_F.$$

Let $\underline{a}_M^G$ be the kernel of the canonical map from $\underline{a}_M$ onto $\underline{a}_G$. There is a compatible embedding of $\underline{a}_G$ into $\underline{a}_M$ resulting from the embeddings of $X(\mathbf{M})_F$ and $X(\mathbf{G})_F$ into the character groups $X(A_{\mathbf{M}})$ and $X(A_{\mathbf{G}})$ of $A_{\mathbf{M}}$ and $A_{\mathbf{G}}$, respectively. Therefore, $\underline{a}_M = \underline{a}_M^G \oplus \underline{a}_G$. Fix a Weyl-invariant norm $\|\cdot\|$ on $\underline{a}_{M_0}$, where $M_0 \subset M$ is a minimal Levi subgroup. The restriction of $\|\cdot\|$ to each of the subspaces $\underline{a}_M$, $M \in \mathscr{L}(M_0)$, yields a measure on $\underline{a}_M$. We take the quotient measure on $\underline{a}_M^G \simeq \underline{a}_M/\underline{a}_G$.

Let $P \in \mathscr{P}(M)$. The roots of (P, A_M) will be regarded as characters of A_M or as elements of the dual space $\underline{a}_M^*$ of $\underline{a}_M$. Let Δ_P be the set of simple roots of (P, A_M). If $\alpha \in \Delta_P$, the co-root $\alpha^\vee$ is defined as follows. Choose a minimal Levi subgroup $M_0 \subset M$. If β is a reduced root of (G, A_{M_0}), the co-root $\beta^\vee$ is an element of the lattice $\mathrm{Hom}(X(A_{\mathbf{M}_0}), \mathbf{Z})$ in $\underline{a}_{M_0}$. For $P_0 \in \mathscr{P}(M_0)$, with $P_0 \subset P$, there is exactly one root $\beta \in \Delta_{P_0}$ such that $\beta|A_{M_0} = \alpha$. $\alpha^\vee$ is defined to be the projection of $\beta^\vee$ onto $\underline{a}_M^G$. Set $\Delta_P^\vee = \{\alpha^\vee | \alpha \in \Delta_P\}$. The lattice $\mathbf{Z}(\Delta_P^\vee)$ in $\underline{a}_M^G$ generated by $\Delta_P^\vee$ is independent of the choice of $P \in \mathscr{P}(M)$ [**A4**, p. 12]. For $x \in G$, $v_M(x)$ is equal to the volume of the convex hull of the projection of the points $\{-H_P(x) | P \in \mathscr{P}(M)\}$ onto $\underline{a}_M^G$. Set $\theta_P(\lambda) = \mathrm{vol}(\underline{a}_M^G/\mathbf{Z}(\Delta_{\mathbf{P}}^\vee))^{-1} \prod_{\alpha\in\Delta_{\mathbf{P}}} \lambda(\alpha^\vee)$, $\lambda \in i\underline{a}_M^*$. Then, [**A2**, p. 36]

$$v_M(x) = \lim_{\lambda\to 0} \sum_{P\in\mathscr{P}(M)} e^{-\lambda(H_P(x))}\theta_P(\lambda)^{-1}, \qquad \lambda \in i\underline{a}_M^*,$$

and, [**A2**, p. 46]

$$(3.1)\quad v_M(x) = 1/r! \sum_{P\in\mathscr{P}(M)} (-\lambda(H_P(x)))^r \theta_P(\lambda)^{-1},$$

where $r = \dim(A_M/A_G)$.

For $\gamma \in G$, define $D(\gamma) = D_G(\gamma) = \det(1 - \mathrm{Ad}(\sigma))_{\mathfrak{g}/\mathfrak{g}_\sigma}$, where σ is the semisimple part of γ. Let $f \in C_c^\infty(G)$. For a Levi subgroup M, set $A_{M,\mathrm{reg}} = \{a \in A_M \mid \mathbf{G}_a \subset \mathbf{M}^0\}$. The weighted orbital integral is defined for $\gamma \in M$. If $G_\gamma \subset M$, then [**A3**, p. 234]

$$(3.2) \qquad J_M(\gamma, f) = |D(\gamma)|^{1/2} \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) v_M(x)\, dx.$$

More generally, for any $\gamma \in M$ [**A3**, §5],

$$(3.3) \quad J_M(\gamma, f) = \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} r_M^L(\gamma, a) J_L(a\gamma, f), \qquad a \in A_{M,\mathrm{reg}},$$

where $r_M^L(\gamma, a)$, $L \in \mathcal{L}(M)$ is a certain real-valued function. We remark that $f \mapsto J_M(\gamma, f)$ is not an invariant distribution on $C_c^\infty(G)$. If γ_1 and γ_2 are conjugate in M, then $J_M(\gamma_1, f) = J_M(\gamma_2, f)$, so $J_M(\mathcal{O}, f)$ is well-defined for any conjugacy class $\mathcal{O} \subset M$. The restriction of $f \mapsto J_M(\gamma, f)$ to the space of cusp forms is G-invariant.

Let M_{ell} be the set of γ in M which lie in some elliptic Cartan subgroup of M. Recall that an admissible representation π of G is *supercuspidal* if its matrix coefficients are compactly supported modulo A_G.

THEOREM 3.4 [**A4**]. *Let π be a supercuspidal representation of G. Suppose f is a finite sum of matrix coefficients of π. For $\gamma \in M_{\mathrm{ell}} \cap G_{\mathrm{reg}}$, where M is a Levi subgroup,*

$$(-1)^{\dim(A_M/A_G)} \Theta_\pi(f) |D(\gamma)|^{1/2} \Theta_\pi(\gamma) = J_M(\gamma, f).$$

REMARK. 1. Although f is not in $C_c^\infty(G)$, the weighted orbital integrals of f still converge because $\operatorname{supp} f$ is compact modulo A_G.

2. The corresponding result for reductive Lie groups appears in [**A1**].

3. In Theorem 3.4, and, with the exception of the proof of Proposition 3.9, in the remainder of the paper, if $\gamma \in G_{\mathrm{reg}}$, the integral in $J_M(\gamma, f)$ is taken over $A_M \backslash G$ instead of $G_\gamma \backslash G$. The weight factor v_M is invariant under left translation by elements of M, so this is equivalent to multiplying the original definition (3.2) by the measure of $A_M \backslash G_\gamma$.

The measures on $A_G \backslash G$, $A_M \backslash G$, and $\underline{a}_M / \underline{a}_G$ must be normalized correctly in order for Theorem 3.4 to hold. Let $\kappa_M = A_M \cap K$. Given measures on $\underline{a}_M$, $\underline{a}_G$, and $\underline{a}_M / \underline{a}_G$ defined using the restriction of a fixed Weyl-invariant metric on $\underline{a}_{M_0}$, as above, the compatibility requirement for the measures is as follows [**A4**, p. 5]:

$$\mathrm{vol}_{A_M}(\kappa_M) = \mathrm{vol}(\underline{a}_M/H_M(A_M)),$$
$$\mathrm{vol}_{A_G}(\kappa_G) = \mathrm{vol}(\underline{a}_G/H_G(A_G)).$$

The measures on $A_M\backslash G$ and $A_G\backslash G$ are the quotient measures induced by the measures on G, A_M and A_G.

If $\gamma \in G_{\mathrm{reg}} \cap M$, the weighted orbital integral $J_M(\gamma, f)$ has a germ expansion on neighbourhoods of semisimple points in M. The weighted germs are uniquely determined up to orbital integrals on M. Suppose ϕ_1 and ϕ_2 are functions defined on an open subset Σ of σM_σ which contains an M_σ-invariant neighbourhood of the semisimple element σ. ϕ_1 is (M, σ)-equivalent to ϕ_2, $\phi_1(\gamma) \overset{(M,\sigma)}{\sim} \phi_2(\gamma)$, if $\phi_1(\gamma) - \phi_2(\gamma) = J_M^M(\gamma, h)$ for $\gamma \in \Sigma \cap U$, where U is a neighbourhood of σ in M, and $h \in C_c^\infty(M)$. Let $(\sigma\mathscr{U}_{M_\sigma})$ be the finite set of orbits in $\sigma\mathscr{U}_{M_\sigma}$ under conjugation by $M(\sigma) = \mathbf{M}^0(F)_\sigma$. Let $\gamma \in M$. Generalizing the definition of Lusztig and Spaltenstein [**LS**], Arthur [**A3**, p. 255] defines the induced space of orbits $\gamma_M^G = \gamma^G$ in G as the finite union of all $\mathbf{G}^0(F)$-orbits in G which intersect γN_P in an open set for any $P \in \mathscr{P}(M)$.

THEOREM 3.5 [**A3**, *Prop.* 9.1, *Prop.* 10.2]. 1. *There are uniquely determined (M, σ)-equivalence classes of functions $\gamma \mapsto g_M^G(\gamma, \mathscr{O})$, $\gamma \in \sigma M_\sigma \cap G_{\mathrm{reg}}$ parametrized by the classes $\mathscr{O} \in (\sigma\mathscr{U}_{L_\sigma})$ such that, for any $f \in C_c^\infty(G)$,*

$$J_M(\gamma, f) \overset{(M,\sigma)}{\sim} \sum_{L\in\mathscr{L}(M)} \sum_{\mathscr{O}\in(\sigma\mathscr{U}_{L_\sigma})} g_M^L(\gamma, \mathscr{O}) J_L(\mathscr{O}, f),$$

where $J_L(\mathscr{O}, f) \overset{\mathrm{def}}{=} J_L(\sigma u, f)$ for any $\sigma u \in \mathscr{O}$.

2. *Let $t \in F^*$ and $w \in (\mathscr{U}_G)$. Set $d^G(w) = (1/2)(\dim G_w - \mathrm{rank}\, G)$. If $x = \exp(X)$, let $x^t = \exp(tX)$.*

$$g_M^G(\gamma^t, w^t) \overset{(M,1)}{\sim} |t|^{d^G(w)} \sum_{L\in\mathscr{L}(M)} \sum_{u\in(\mathscr{U}_L)} g_M^L(\gamma, u) c_L(u, t)[u^G : w],$$

where the $c_L(u, t)$ are certain real-valued functions and $[u^G : w]$ is 1 if $w \in u^G$, 0 otherwise.

LEMMA 3.6. *Let π be a supercuspidal representation of G and f a matrix coefficient of π. Then $\Theta_\pi(f) = d(\pi)^{-1} f(1)$, where $d(\pi)$ is the formal degree of π.*

Proof. Let $(\, , \,)$ denote a G-invariant inner product on the representation space V of π. Let $e_1, e_2, \ldots$ be an orthonormal basis for V. $f(x) = (v, \pi(x)w)$, some $v, w \in V$. We use the orthogonality relations for matrix coefficients of supercuspidal representations **[HC1**, p. 5] to evaluate

$$\begin{aligned}\Theta_\pi(f) &= \operatorname{tr}\pi(f) = \operatorname{tr}\left(\int_{A_G\backslash G} f(x)\pi(x)\,dx^*\right)\\ &= \sum_i \int_{A_G\backslash G} (v, \pi(x)w)(\pi(x)e_i, e_i)\,dx^*\\ &= \sum_i d(\pi)^{-1}\overline{(v, e_i)}(e_i, w)\\ &= d(\pi)^{-1}(v, w) = d(\pi)^{-1}f(1).\end{aligned}$$

PROPOSITION 3.7. *Assume G is connected. Let $\gamma \in M_{\mathrm{ell}} \cap G_{\mathrm{reg}}$. If l is the F-rank of G and $d(\mathrm{St}_G)$ is the formal degree of the Steinberg representation of G, then*

$$g_M^G(\gamma, 1) \overset{(M,1)}{\sim} (-1)^{(l-\dim A_M)}|D(\gamma)|^{1/2}/d(\mathrm{St}_G).$$

Proof. Let π be a supercuspidal representation of G. Choose a matrix coefficient f of π such that $f(1) \neq 0$. By Lemma 3.6, $\Theta_\pi(f) \neq 0$.

First, let $\gamma \in G_{\mathrm{ell}} \cap G_{\mathrm{reg}}$. From **[R]**, the leading term in the Shalika germ expansion of $J_G(\gamma, f)$ is $(-1)^{(l-\dim A_G)}|D(\gamma)|^{1/2}f(1)/d(\mathrm{St}_G)$. We also have, by Theorem 3.4,

$$J_G(\gamma, f) = \Theta_\pi(f)|D(\gamma)|^{1/2}\Theta_\pi(\gamma).$$

The leading term in Harish-Chandra's asymptotic expansion of $|D(\gamma)|^{1/2}\Theta_\pi(\gamma)$ is $c_{\{0\}}(\pi)|D(\gamma)|^{1/2}$, because $\hat{\mu}_{\{0\}} \equiv 1$. By $\{0\}$, we mean the trivial nilpotent orbit in $\mathscr{G}$. Thus the leading term in $J_G(\gamma, g)$ is also equal to $\Theta_\pi(f)|D(\gamma)|^{1/2}c_{\{0\}}(\pi)$, which means

$$c_{\{0\}}(\pi) = (-1)^{(l-\dim A_G)}f(1)/\Theta_\pi(f)d(\mathrm{St}_G),$$

which, by Lemma 3.6, equals $(-1)^{(l-\dim A_G)}d(\pi)/d(\mathrm{St}_G)$.

Now let $\gamma \in M_{\mathrm{ell}} \cap G_{\mathrm{reg}}$. From Theorem 3.4 and Theorem 3.5(1),

$$\begin{aligned}|D(\gamma)|^{1/2}\Theta_\pi(\gamma) &= (-1)^{\dim(A_M/A_G)}\Theta_\pi(f)^{-1}J_G(\gamma, f)\\ &\overset{(M,1)}{\sim} (-1)^{\dim(A_M/A_G)}\Theta_\pi(f)^{-1}\sum_{L\in\mathscr{L}(M)}\;\sum_{\mathscr{O}\in(\mathscr{U}_L)} g_M^L(\gamma, \mathscr{O})J_L(\mathscr{O}, f).\end{aligned}$$

We will show that $g_M^G(\gamma, 1)$ is the only term occurring in the above expansion having the same homogeneity as $|D(\gamma)|^{1/2}$. Given this, we then have

$$\frac{(-1)^{(l-\dim A_G)}d(\pi)}{d(\mathrm{St}_G)}|D(\gamma)|^{1/2} \overset{(M,1)}{\sim} (-1)^{\dim(A_M/A_G)}\Theta_\pi(f)^{-1}f(1)g_M^G(\gamma, 1),$$

which, using Lemma 3.6, yields the desired expression for $g_M^G(\gamma, 1)$.

Let $L \in \mathscr{L}(M)$ and $u \in (\mathscr{U}_L)$. Since $[u^G : 1] = 1 \Leftrightarrow L = G$ and $u = 1$, and $c_G(1, t) = 1$ (see [A3, §10] for the definition of $c_L(u, t)$), Proposition 3.7(2) reduces to:

$$g_M^G(\gamma^t, 1) \overset{(M,1)}{\sim} |t|^{1/2(\dim G - \operatorname{rank} G)} g_M^G(\gamma, 1).$$

Let $w \in (\mathscr{U}_G)$, $w \neq 1$. The power of $|t|$ in $|t|^{d^G(w)}c_L(u, t)$, $u \in (\mathscr{U}_L)$ such that $[u^G : w] = 1$, is less than $d^G(1)$. Therefore, all other terms in the above weighted germ expansion for $|D(\gamma)|^{1/2}\Theta_\pi(\gamma)$ have smaller homogeneity than $g_M^G(\gamma, 1)$.

LEMMA 3.8 [A3, *Cor.* 6.3]. *Let* $L_1 \in \mathscr{L}(M)$. *Then*

$$J_{L_1}(\gamma^{L_1}, f) = \lim_{a \to 1} \sum_{L \in \mathscr{L}(L_1)} r_{L_1}^L(\gamma, a) J_L(a\gamma, f), \qquad a \in A_{M,\mathrm{reg}}.$$

REMARK. $J_{L_1}(\gamma^{L_1}, f) \overset{\mathrm{def}}{=} \sum_i J_L(\mathscr{O}_i, f)$, where $\gamma^{L_1} = \bigcup_i \mathscr{O}_i$.

Recall that a locally constant function ϕ on G is a *cusp form* if, for all $x \in G$ and all proper parabolic subgroups $P = MN$ of G, $\int_N \phi(xn)\,dn = 0$. The following is a generalization of the well-known fact that orbital integrals of cusp forms vanish at nonelliptic semisimple points in G.

PROPOSITION 3.9. *Let f be a cusp form on G such that* $\operatorname{supp} f$ *is compact modulo A_G. Suppose γ is a semisimple element in a Levi subgroup M and $\gamma \notin M_{\mathrm{ell}}$. Then $J_M(\gamma, f) = 0$.*

Proof. This is due to Arthur. We give a rough outline of the proof. Using results about products of (G, M)-families from §§6 and 10 of [A2], it is possible to show that, for $M_1 \subset M$,

$$v_M(x) = \sum_{\{Q \in \mathscr{F}(M_1),\, Q \neq G\}} a_Q v_{M_1}^Q(x), \qquad x \in G,$$

where $v^Q_{M_1}(x) = \lim_{\lambda\to 0}\sum_{\{P\in\mathscr{P}(M_1)|P\subset Q\}} e^{-\lambda(H_P(x))}\theta^Q_P(\lambda)^{-1}$ and $a_Q \in \mathbf{R}$. Here, $\theta^Q_P(\lambda)$ is defined in the same way as θ_P, but with respect to the set Δ^Q_P of simple roots of $(P\cap M_Q, A_P)$ and the associated set $\{\alpha^\vee|\alpha\in\Delta^Q_P\}$.

Because $\gamma\notin M_{\mathrm{ell}}$, there is a Levi subgroup M_1 properly contained in M with $\gamma\in M_1$. Assume that $M_\gamma = G_\gamma$. Then

$$\begin{aligned} J_M(\gamma, f) &= |D(\gamma)|^{1/2}\int_{M_\gamma\backslash G} f(x^{-1}\gamma x)v_M(x)\,dx \\ &= |D(\gamma)|^{1/2}\sum_{\{Q\in\mathscr{F}(M_1), Q\neq G\}} a_Q\int_{M_\gamma\backslash G} f(x^{-1}\gamma x)v^Q_{M_1}(x)\,dx. \end{aligned}$$

Note that $M_\gamma = M_{1_\gamma}$. By **[A2**, (8.1)], the integral corresponding to Q in the sum above is equal to $J^{M_Q}_{M_1}(\gamma, f_Q)$, where $J^{M_Q}_{M_1}$ is the weighted orbital integral for the Levi subgroup M_1 of M_Q, and $f_Q\colon M_Q\to\mathbf{C}$ is given by $f_Q(m) = \delta_Q(m)^{1/2}\int_{N_Q}\int_K f(k^{-1}mnk)\,dk\,dn$. Since f is a cusp form, $f_Q\equiv 0$ for $Q\neq G$. Therefore, $J_M(\gamma, f) = 0$.

For general γ, and $a\in A_{M,\mathrm{reg}}$ close to 1, the element $a\gamma$ is not elliptic in any $L\in\mathscr{L}(M)$, and $L_{a\gamma} = G_{a\gamma}$. Thus the above argument shows that $J_L(a\gamma, f) = 0$. From (3.3), $J_M(\gamma, f) = 0$.

4. Some results for $G = \mathrm{GL}_n(F)$. Assume π is an irreducible supercuspidal representation of $G = \mathrm{GL}_n(F)$. The main result of this section, Theorem 4.4, expresses the constant $c_{\mathscr{O}}(\pi)$, $\mathscr{O}\in(\mathscr{N}_G)$, as a multiple of a certain weighted orbital integral of a sum of matrix coefficients of π. Because of the one-to-one correspondence between the set $(\mathscr{N}_G)$ of nilpotent G-orbits in $\mathscr{G}$ and the set $(\mathscr{U}_G)$ of unipotent conjugacy classes in G, we can view $c_{\mathscr{O}}(\pi)$ and $\hat{\mu}_{\mathscr{O}}$ as corresponding to $\mathscr{O}\in(\mathscr{U}_G)$. We begin by defining some notation which allows us to state our results in terms of unipotent conjugacy classes. For $\mathscr{O}\in(\mathscr{U}_G)$, let $\mathscr{P}(\mathscr{O}) = \{P = MN \mid \mathscr{O} = 1^G_M\}$. If $P\in\mathscr{P}(\mathscr{O})$, let π_P be the admissible representation of G induced (unitarily) from the character $\delta_P^{-1/2}$ of P, and let Θ_P denote the character of π_P. If $P_1, P_2\in\mathscr{P}(\mathscr{O})$, then P_1 and P_2 are conjugate in G, and π_{P_1} and π_{P_2} are equivalent, so $\Theta_{P_1} = \Theta_{P_2}$. Let $\Theta_{\mathscr{O}}$ denote the common value Θ_P, $P\in\mathscr{P}(\mathscr{O})$.

For a Levi subgroup M of G, set $\mathscr{L}_{\mathscr{O}}(M) = \{L\in\mathscr{L}(M) \mid \mathscr{O} = 1^G_L\}$. If $L_1, L_2\in\mathscr{L}_{\mathscr{O}}(M)$ and K is a special maximal compact subgroup in good position relative to M, then $L_1 = kL_2k^{-1}$ for some $k\in K$ and

[A3, p. 235] $J_{L_2}(1, f) = J_{L_1}(1, f^k)$, where $f^k(x) = f(kxk^{-1})$. Assume f is a cusp form. Then $J_{L_1}(1, f^k) = J_{L_1}(1, f)$, so $J_{L_1}(1, f) = J_{L_2}(1, f)$. We denote the common value by $J_{\mathscr{O}}(1, f)$. Similarly, let $d(\mathrm{St}(\mathscr{O}))$ be the formal degree of the Steinberg representation of any $L \in \mathscr{L}_{\mathscr{O}}(M_0)$, where M_0 is a minimal Levi subgroup. We note that $\mathscr{L}_{\mathscr{O}}(M_0) \neq \emptyset$ for any $\mathscr{O} \in (\mathscr{U}_G)$. Finally, we set $w_{\mathscr{O}} = |N_G(A)/Z_G(A)|$, for A equal to the split component of any $P \in \mathscr{P}(\mathscr{O})$, and $N_G(A)$ (resp. $Z_G(A)$) the normalizer (resp. centralizer) of A in G. Let $K = \mathrm{GL}_n(\mathscr{O}_F)$, where $\mathscr{O}_F$ is the ring of integers in F. K is a special maximal compact subgroup of G. For convenience, we consider only those Levi subgroups M which are in $\mathscr{L}(M_0)$, where M_0 is the subgroup of diagonal matrices in G. For all such M, $G = PK = KP$ if $P \in \mathscr{P}(M)$.

LEMMA 4.1 **[H]**. *Measures can be normalized so that* $\hat{\mu}_{\mathscr{O}}(\log \gamma) = \mathscr{Q}_{\mathscr{O}}(\gamma)$, *for* $\gamma \in G_{\mathrm{reg}}$ *in a sufficiently small neighbourhood of* 1.

REMARK. In §6, we normalize measures on G and its Levi subgroups. We will assume that the measure on the Lie algebra $\mathscr{G}$ has been normalized so that Lemma 4.1 holds.

LEMMA 4.2. *Let* M *be a Levi subgroup of* G. *If* $\gamma \in M_{\mathrm{ell}} \cap K \cap G_{\mathrm{reg}}$ *and* $\mathscr{O} \in (\mathscr{U}_G)$, *then*

$$\hat{\mu}_{\mathscr{O}}(\log \gamma) = |D(\gamma)|^{-1/2} w_{\mathscr{O}} \sum_{L \in \mathscr{L}_{\mathscr{O}}(M)} |D_L(\gamma)|^{1/2}.$$

Proof. Let $P_1 = L_1 N_1 \in \mathscr{P}(\mathscr{O})$ with A_1 the split component of L_1. We have simply rewritten van Dijk's **[D]** formula for the induced character:

$$\Theta_{\mathscr{O}}(\gamma) = \sum_{s \in W(A_1, A_M)} {}^s\delta_{P_1}^{-1/2}(\gamma) \frac{|D_{L_1^s}(\gamma)|^{1/2}}{|D(\gamma)|^{1/2}},$$

where $W(A_1, A_M) = \{s\colon A_1 \to A_M \mid s1 - 1, a^s = a^y, y \in G\}$, and ${}^s\delta_{P_1}^{-1/2}(\gamma) = \delta_{P_1}^{-1/2}(y^{-1}\gamma y)$. $\delta_{P_1}|K \equiv 1$ and y can be taken in K, so ${}^s\delta_{P_1}^{-1/2}(\gamma) = 1$. $W(A_1, A_M) = \emptyset \Leftrightarrow \mathscr{L}_{\mathscr{O}}(M) = \emptyset$. Assume $\mathscr{L}_{\mathscr{O}}(M) \neq \emptyset$ and $L_1 \in \mathscr{L}_{\mathscr{O}}(M)$. Define a map $s \mapsto L$ from $W(A_1, A_M)$ to $\mathscr{L}_{\mathscr{O}}(M)$ by: $L = L_1^s = yL_1y^{-1}$. If $L \in \mathscr{L}_{\mathscr{O}}(M)$, then $L = L_1^y$ for some $y \in K$ and $a \mapsto a^y$ maps A_1 bijectively onto A_L. Since $M \subset L$, $A_L \subset A$. Thus $a \mapsto a^y$ defines an $s \in W(A_1, A_M)$ which maps to L. Suppose $L = L_2^{s_2}$ for some $s_2 \in W(A_1, A_M)$. Then

$A_L = yA_1y^{-1} = y_2A_1y_2^{-1}$, so $y_2^{-1}y \in N_G(A_1)$. Clearly $s = s_2 \Leftrightarrow y_2^{-1}y \in Z_G(A_1)$. Thus $s \mapsto L$ is onto and $w_{\mathscr{O}}$-to-one, which proves the lemma.

LEMMA 4.3. *Let f be a cusp form on G which is compactly supported modulo A_G.*

1. *If $u \in (\mathscr{U}_M)$, and $u \neq 1$, then $J_M(u, f) = 0$.*
2. *$J_M(1, f) = \lim_{a\to 1} J_M(a, f)$, $a \in A_{M,\mathrm{reg}}$.*

Proof. 1. There exists a Levi subgroup $M_1 \subset M$ such that $u = 1_{M_1}^M$. By [A3, Corollary 6.3],

$$J_M^G(u, f) = \lim_{a\to 1} \sum_{L\in\mathscr{L}(M)} r_M^L(1, a)J_L(a, f), \qquad a \in A_{M_1,\mathrm{reg}}.$$

Because $a \in A_{M_1,\mathrm{reg}}$ and $M_1 \neq L$ for each $L \in \mathscr{L}(M)$, a is not elliptic in L. Therefore, by Proposition 3.9, $J_L(a, f) = 0$.

2. For $L \in \mathscr{L}(M), L \neq M$, we have $J_L(a, f) = 0$, since $a \in A_{M,\mathrm{reg}}$ is not elliptic in L. By definition, [A3] $r_M^M(1, a) = 1$. Thus

$$\begin{aligned} J_M(1, f) &= \lim_{a\to 1} \sum_{L\in\mathscr{L}(M)} r_M^L(1, a)J_L(a, f) \\ &= \lim_{a\to 1} J_M(a, f). \end{aligned}$$

Let π be a supercuspidal representation of G. We now express the coefficients $c_{\mathscr{O}}(\pi)$ in the asymptotic expansion about 1 of the character Θ_π in terms of the weighted orbital integrals at 1 of the matrix coefficients of π.

THEOREM 4.4. *Let f be a finite sum of matrix coefficients of the supercuspidal representation π. Assume $f(1) \neq 0$. For $\mathscr{O} \in (\mathscr{U}_G)$,*

$$c_{\mathscr{O}}(\pi) = \frac{(-1)^{n-1}J_{\mathscr{O}}(1, f)d(\pi)}{w_{\mathscr{O}}d(\mathrm{St}(\mathscr{O}))f(1)}.$$

Proof. Let $\gamma \in M_{0,\mathrm{ell}} \cap G_{\mathrm{reg}}$. Recall [HC1] that the matrix coefficients of π are cusp forms. Applying Theorem 3.4, Theorem 3.5(1), and Lemma 4.3(1),

$$\Theta_\pi(\gamma) \overset{(M_0,1)}{\sim} (-1)^{n-1}\Theta_\pi(f)^{-1}|D(\gamma)|^{-1/2} \sum_{L\in\mathscr{L}(M_0)} g_{M_0}^L(\gamma, 1)J_L^G(1, f).$$

Writing the sum over $L \in \mathscr{L}(M_0)$ as a double sum over $\mathscr{O} \in (\mathscr{U}_G)$ and $L \in \mathscr{L}_{\mathscr{O}}(M_0)$ and using Proposition 3.7 to substitute $|D_L(\gamma)|^{1/2}/d(\mathrm{St}_L)$ for $g^L_{M_0}(\gamma, 1)$, we obtain

$$\Theta_\pi(\gamma) \overset{(M_0,1)}{\sim} (-1)^{n-1}\Theta_\pi(f)^{-1}|D(\gamma)|^{-1/2} \sum_{\mathscr{O}\in\mathscr{U}_G} J_{\mathscr{O}}(1, f)/d(\mathrm{St}(\mathscr{O})) \times \left(\sum_{L\in\mathscr{L}_{\mathscr{O}}(M_0)} |D_L(\gamma)|^{1/2}\right).$$

For $\gamma \in M_{0,\mathrm{ell}} \cap G_{\mathrm{reg}}$ close to 1, we also have:

$$\Theta_\pi(\gamma) = \sum_{\mathscr{O}\in(\mathscr{U}_G)} c_{\mathscr{O}}(\pi)\hat{\mu}_{\mathscr{O}}(\log\gamma) = \sum_{\mathscr{O}\in(\mathscr{U}_G)} c_{\mathscr{O}}(\pi) w_{\mathscr{O}} \left(\sum_{L\in\mathscr{L}_{\mathscr{O}}(M_0)} \frac{|D_L(\gamma)|^{1/2}}{|D(\gamma)|^{1/2}}\right). \tag{4.6}$$

The two expressions (4.5) and (4.6) differ by an orbital integral on $M_0 = A_{M_0}$, that is, by $c \mid D(\gamma)|^{-1/2}$, for some constant c. Let $\mathscr{O}_{\mathrm{reg}}$ be the regular unipotent class in G. By Lemma 4.3(2), $J_{M_0}(1, f) = J_{\mathscr{O}_{\mathrm{reg}}}(1, f) = \lim_{a\to 1} J_{M_0}(a, f)$, $a \in A_{M_0,\mathrm{reg}}$. Multiplying (4.5) by $(-1)^{n-1}\Theta_\pi(f)|D(a)|^{1/2}$ and letting $a \to 1$, we get

$$J_{\mathscr{O}_{\mathrm{reg}}}(1, f)/d(\mathrm{St}(\mathscr{O}_{\mathrm{reg}})) + c,$$

which must equal $J_{\mathscr{O}_{\mathrm{reg}}}(1, f)$. Since M_0 is abelian, the Steinberg representation of M_0 is just the trivial representation, so $d(\mathrm{St}(\mathscr{O}_{\mathrm{reg}})) = 1$. Therefore $c = 0$.

The functions $\sum_{L\in\mathscr{L}_{\mathscr{O}}(M_0)} |D_L(\gamma)|^{1/2}/|D(\gamma)|^{1/2}$, $\mathscr{O} \in (\mathscr{U}_G)$, are linearly independent on any neighbourhood of 1 intersected with $A_{M_0,\mathrm{reg}}$. Therefore, the equality of (4.5) and (4.6) implies:

$$c_{\mathscr{O}}(\pi) = \frac{(-1)^{n-1}\Theta_\pi(f)^{-1} J_{\mathscr{O}}(1, f)}{w_{\mathscr{O}} d(\mathrm{St}(\mathscr{O}))}.$$

From Lemma 3.6, $\Theta_\pi(f) = f(1)/d(\pi)$.

REMARK. 1. It follows from the definition of the Steinberg character, that is, the character of St_G (see **[Ca]**), that

$$c_{\mathscr{O}}(\mathrm{St}_G) = (-1)^{n-d(\mathscr{O})}\mathrm{card}\,\mathscr{L}_{\mathscr{O}}(M_0),$$

where $d(\mathscr{O}) = \dim A_M$, $M \in \mathscr{L}_{\mathscr{O}}(M_0)$.

2. If $\pi = \mathrm{Ind}_P^G(\tau \otimes \mathrm{id})$, $P = MN$, τ a supercuspidal representation of M, then, using van Dijk's formula in **[D]** which expresses Θ_π in terms of Θ_τ, it is possible to write $c_{\mathcal{O}}(\pi)$, $\mathcal{O} \in (\mathcal{U}_G)$, terms of the constants $c_{\mathcal{O}'}(\tau)$, $\mathcal{O}' \in (\mathcal{U}_M)$.

3. If π is in the discrete series of G and π is not supercuspidal or a twist of St_G, there is no formula for $c_{\mathcal{O}}(\pi)$, $\mathcal{O} \neq \{1\}$.

5. Characters of inducing representations. To find the constant $c_{\mathcal{O}}(\pi)$ for a supercuspidal representation π of $G = \mathrm{GL}_n(F)$, we must evaluate $J_{\mathcal{O}}(1, f)$ for f equal to a sum of matrix coefficients of π such that $f(1) \neq 0$ (Theorem 4.4). Here, we outline how to produce such a function f. It will be shown in Lemma 6.1 that only the values of f on the unipotent set $\mathcal{U}_G$ are required to compute $J_{\mathcal{O}}(1, f)$. Lemma 5.2 gives a formula for the values of f on $\mathcal{U}_G$ for π generic and unramified.

Carayol **[C]** has constructed an infinite family of irreducible unitary representations of KA_G which are called *very cuspidal.* To each such representation σ is attached a positive integer h, the *level* of σ. Given any (unitary) character χ of F^*, the representation $\pi = \mathrm{Ind}_{KA_G}^G \sigma \otimes \chi \circ \det$ is irreducible and supercuspidal. We will say that any such π is *generic* and *unramified.*

The reason for this terminology is as follows. Let p be the residual characteristic of F. If $(p, n) = 1$, the irreducible supercuspidal representations of G are parametrized by conjugacy classes of admissible characters of extensions of degree n over F. For definitions and a general description, see **[CMS]**. Let θ be such a character. In this setting, those supercuspidal representations which correspond to the case where θ is generic over F and the extension of F is unramified are precisely the generic and unramified representations defined above. We remark that Carayol's construction is valid for arbitrary p, and thus we do not place any restriction on p.

LEMMA 5.1 **[C]**. *Let H be an open subgroup of G. Suppose φ is a matrix coefficient of a representation σ of H. For $x \in G$, define $\tilde{\varphi}(x)$ to be $\varphi(x)$, if $x \in H$, and* 0 *otherwise. Then $\tilde{\varphi}$ is a matrix coefficient of* $\mathrm{Ind}_H^G \sigma$.

Let $\pi = \mathrm{Ind}_{KA_G}^G \sigma \otimes \chi \circ \det$ be generic and unramified. By Lemma 5.1, if χ_σ is the character of σ, then $\tilde{\chi}_\sigma$ is a sum of matrix coefficients of $\mathrm{Ind}_{KA_G}^G \sigma$, and we may take $f = \tilde{\chi}_\sigma \chi \circ \det$ as a finite sum of matrix coefficients of π. Note that $f(1) = \dim \sigma \neq 0$. This particular f

is chosen because $\int_K f(k^{-1}uk)\,dk = f(u)$, for $u \in \mathscr{U}_G$, which will simplify the computation of $J_{\mathscr{O}}(1, f)$ (see Lemma 6.1).

Let ϖ be a prime element in F, and let $\mathscr{P}_F = \varpi\mathscr{O}_F$. If j is a positive integer, define $K_j = \{k \in K \mid k \in I + \mathbf{M}_n(\mathscr{P}_F^j)\}$.

LEMMA 5.2. *If σ is a very cuspidal representation of KA_G having level h, then, for $u \in \mathscr{U}_G \cap K$,*

$$\chi_\sigma(u) = \begin{cases} q^{n(n-1)(h-1)/2}(-1)^{n+s_h(u)} \displaystyle\sum_{j=1}^{s_h(u)-1} (q^j - 1), & \text{if } u \in K_{h-1}, \\ 0, & \text{otherwise.} \end{cases}$$

For $u \in K_{h-1}$, $s_h(u)$ is the number of blocks in the Jordan form of $\varpi^{1-h}(u-1)$ viewed as a matrix over $\mathscr{O}_F/\mathscr{P}_F$.

Proof [**K**, *Lemma* 6.6]. The proof given by Kutzko is for n prime, but in fact uses only the very cuspidal property of σ and therefore is valid for arbitrary n.

6. Weights for $\mathrm{GL}_4(F)$**.** To compute the coefficients $c_{\mathscr{O}}(\pi)$, it is necessary to evaluate $J_{\mathscr{O}}(1, f)$ for f equal to a suitable sum of matrix coefficients of π. Proposition 6.5 gives explicit integral formulas for $J_M(1, f)$ for non-minimal Levi subgroups M of $\mathrm{GL}_4(F)$.

On $G = \mathrm{GL}_n(F)$, we take the Haar measure with respect to which $K = \mathrm{GL}_n(\mathscr{O}_F)$ has measure one. The Haar measure on K is the restriction of this measure to K. If $P = MN$ is a parabolic subgroup with $G = KP$, the measures on M and N are normalized so that the measures of $M \cap K$ and $N \cap K$ equal one. Then we have

$$\int_G \varphi(x)\,dx = \int_K \int_M \int_N \varphi(mnk)\,dk\,dm\,dn, \qquad \varphi \in C_c^\infty(G).$$

LEMMA 6.1. *Let f be a cusp form on G which is compactly supported modulo A_G. Then, if $G = KP$ and $P = MN$,*

$$J_M(1, f) = \lim_{a \to 1} \int_N f_K(u) v_M(n)\,du, \qquad a \in A_{M,\mathrm{reg}}.$$

where $n \in N$ is defined by $u = a^{-1}n^{-1}an$ and

$$f_K(x) = \int_K f(k^{-1}xk)\,dk,$$

for $x \in G$.

Proof. From Lemma 4.3(2) and (3.2),

$$\begin{aligned} J_M(1, f) &= \lim_{a\to 1} J_M(a, f) \\ &= \lim_{a\to 1} |D(a)|^{1/2} \int_{M\backslash G} f(x^{-1}ax) v_M(x)\, dx, \qquad a \in A_{M,\mathrm{reg}}. \end{aligned}$$

The quotient measure on $M\backslash G$ is $dx = dn\, dk$, and **[A2]** $v_M(mnk) = v_M(n)$ for $m \in M$, $n \in N$, and $k \in K$. Therefore,

$$J_M(1, f) = \lim_{a\to 1} |D(a)|^{1/2} \int_N f_K(n^{-1}an) v_M(n)\, dn.$$

Since $n \mapsto a^{-1}n^{-1}an$, $n \in N$, $a \in A_{M,\mathrm{reg}}$, is an invertible polynomial mapping from N to N, we can make the change of variables $u = a^{-1}n^{-1}an$. This introduces the factor $|D(a)|^{-1/2}\delta_P(a)^{1/2}$. f_K is locally constant on G, and therefore is invariant under left and right translation by some open compact subgroup of G. Thus $f_K(au) = f_K(u)$ for all $u \in N$ if a is sufficiently close to the identity. Also, $\delta_P | K \cap P \equiv 1$.

We now describe, for $\mathrm{GL}_n(F)$, the normalizations of measures on $\underline{a}_M$, $\underline{a}_G$, $\underline{a}_M^G$, A_M, A_G and A_M/A_G required by the compatibility conditions of §3. Fix the Weyl-invariant inner product $((x_1, \dots, x_n), (y_1, \dots, y_n)) = \log^{-2} q \sum_{1\le i\le n} x_i y_i$ on $\underline{a}_{M_0}$. The corresponding measure is $\log^{-n} q\, dx_1 \cdots dx_n$, where dx_i denotes the standard Haar measure on $\mathbf{R}$. On $\underline{a}_M$ we take the measure coming from the restriction of the above inner product to $\underline{a}_M$. Suppose M is conjugate to $\prod_{i=1}^r \mathrm{GL}_{n_i}(F)$. The embeddings of $X(\mathbf{M})_F$ and $X(\mathbf{G})_F$ into the character groups $X(A_{\mathbf{M}})$ and $X(A_{\mathbf{G}})$ result in the embedding $x \mapsto (xn_1/n, \dots, xn_r/n)$ of $\underline{a}_G$ into $\underline{a}_M$. It is compatible with the canonical projection $(x_1, \dots, x_r) \mapsto \sum_{1\le i\le r} x_i$ from $\underline{a}_M$ onto $\underline{a}_G$, whose kernel is denoted by $\underline{a}_M^G$. This results in the decomposition $\underline{a}_M = \underline{a}_M^G \oplus \underline{a}_G$.

Let $\kappa_M = A_M \cap K$. The function H_M maps A_M/κ_M bijectively onto a lattice in $\underline{a}_M$. As stated in **[A4**, p. 5], the measure of κ_M in A_M must equal the volume of $\underline{a}_M/H_M(A_M)$. The measures on $A_M\backslash G$, $A_G\backslash G$, and $\underline{a}_M/\underline{a}_G \simeq \underline{a}_M^G$ are the ones induced by those on G, A_M, A_G, $\underline{a}_M$, and $\underline{a}_G$.

The next lemma gives the measures of the κ_M's. We will use these to determine the formal degree $d(\mathrm{St}(\mathscr{O}))$ which appears in the formula for $c_{\mathscr{O}}(\pi)$. Note that, in order to be consistent, the measure of $M_0 \cap K = A_{M_0} \cap K$ must equal one. This determined our choice of inner product on $\underline{a}_{M_0}$.

LEMMA 6.2. *Let M be conjugate to $\prod_{i=1}^{r} \mathrm{GL}_{n_i}(F)$. With the above normalizations, the measure of κ_M is $\sqrt{n_1 \cdots n_r}$.*

Proof. For $m \in M$, $H_M(m) = (\log|\det m_1|, \ldots, \log|\det m_r|)$. Thus $H_M(A_M) = n_1 \log q\mathbf{Z} \times \cdots \times n_r \log q\mathbf{Z}$. The measure on $\underline{a}_M \simeq \mathbf{R}^r$ is $(\log^{-r} q/\sqrt{n_1 \cdots n_r})\, dx_1 \cdots dx_r$. The volume of $\underline{a}_M/H_M(A_M)$ is therefore $\sqrt{n_1 \cdots n_r}$.

In order to evaluate $v_M(x)$, $x \in G$, we need to compute $\mathrm{vol}(\underline{a}_M^G/\mathbf{Z}(\Delta_P^\vee))$ for $P \in \mathscr{P}(M)$. As noted in [A4, p. 12], $\mathbf{Z}(\Delta_P^\vee)$ is independent of the choice of $P \in \mathscr{P}(M)$. Let $\mu_M = \mathrm{vol}(\underline{a}_M^G/\mathbf{Z}(\Delta_P^\vee))$.

LEMMA 6.3. $\mu_M = \sqrt{n/(n_1 \cdots n_r)} \log^{-r+1} q$.

Proof. Let $P = MN \in \mathscr{P}(M)$ be chosen so that N is upper triangular. Then $\Delta_P^\vee = \{\alpha_1, \ldots, \alpha_{r-1}\}$, where α_i has 1 in the ith position and 0 elsewhere. Define variables $y_1, \ldots, y_r$ by

$$y_1\alpha_1 + \cdots + y_r\alpha_{r-1} + y_r(n_1/n, \ldots, n_r/n) = (x_1, \ldots, x_r).$$

Then, since $dy_1 \cdots dy_r = dx_1 \cdots dx_r$, the measure on $\underline{a}_M$ is $(\log^{-r} q/\sqrt{n_1 \cdots n_r})\, dy_1 \cdots dy_r$. The measure on $\underline{a}_G$ is $(\log^{-1} q/\sqrt{n})\, dx$ and $x \in \underline{a}_G$ embeds in $\underline{a}_M$ as $(xn_1/n, \ldots, xn_r/n)$. The quotient measure on $\underline{a}_M^G$ is given by $(\log^{-r+1} q\sqrt{n/(n_1 \cdots n_r)}\, dy_1 \cdots dy_{r-1}$.

Let $u \in \operatorname{supp} f_K$. We want to compute the value of $v_M(n)$, where $u = a^{-1}n^{-1}an$, $a \in A_{M,\mathrm{reg}}$. If $a \in A_M$, then $a = \mathrm{diag}(a_1 I_{n_1}, \ldots, a_r I_{n_r})$, with $a_i \in F^*$, and I_{n_i} the $n_i \times n_i$ identity matrix, $1 \le i \le r$. Let $\mathscr{P}_F$ be the maximal ideal in the ring of integers $\mathscr{O}_F$. For each positive integer d, define $A_{M,d} = \{a \in A_{M,\mathrm{reg}} \mid a_i \in 1 + \mathscr{P}_F^d, |a_i - a_j| = q^{-d}, i \ne j\}$. We will compute $v_M(n)$ for $a \in A_{M,d}$ for large values of d, and to evaluate $J_M(1, f)$, we will let $d \to \infty$. The next lemma gives the values of $v_M(n)$ for certain non-minimal Levi subgroups of $\mathrm{GL}_4(F)$. We take n in the corresponding upper triangular unipotent subgroup. For $x \in F^*$, $\nu(x)$ is defined by $|x| = q^{-\nu(x)}$.

LEMMA 6.4. *Let $u \in N \cap K$, $a \in A_{M,d}$, and n be given by $u = a^{-1}n^{-1}an$.*

1. *Let $M = \mathrm{GL}_3(F) \times \mathrm{GL}_1(F)$. If*

$$u = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is such that $\max\{|x|, |y|, |z|\} \neq 0$, *then*

$$v_M(n) = \tfrac{2}{\sqrt{3}}(d - \min\{\nu(x), \nu(y), \nu(z)\}),$$

for large d.

2. *Let* $M = \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. *If*

$$u = \begin{pmatrix} 1 & 0 & w & x \\ 0 & 1 & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

such that $wz - xy \neq 0$ *then*

$$v_M(n) = 2d - \nu(wz - xy),$$

for large d.

3. *Let* $M = \mathrm{GL}_2(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$. *Let*

$$u = \begin{pmatrix} 1 & 0 & x_1 & y_1 \\ 0 & 1 & x_2 & y_2 \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Define

$$\begin{aligned} A &= \min\{\nu(x_1), \nu(x_2)\}, \\ B &= \min\{\nu(x_1 y_2 - x_2 y_1), \nu(z) + A\}. \end{aligned}$$

If $A \neq 0$, $B \neq 0$, *and* d *is large, then*

$$\begin{aligned} v_M(n) = {} & 3\sqrt{2}d^2 - d(2\sqrt{2}A + 2\sqrt{2}\nu(z) + \sqrt{2}B) \\ & + \frac{1}{\sqrt{2}}B^2 - \sqrt{2}(B - A)^2 + \sqrt{2}B\nu(z). \end{aligned}$$

REMARK. Let $P_0 = A_{M_0} N_0$ be the Borel subgroup of $\mathrm{GL}_n(F)$ such that N_0 is the subgroup of upper triangular unipotent matrices. For $x \in \mathrm{GL}_n(F)$, we use the following fact to find $H_{P_0}(x)$. Suppose $x = nak$, with $n \in N_0$, $a = \mathrm{diag}(a_1, \ldots, a_n) \in A_{M_0}$, and $k \in K$. Then, for $1 \leq i \leq n$, $|a_i \cdots a_n|$ is equal to the maximum of the set of norms of determinants of $(n-i+1) \times (n-i+1)$ matrices which can be formed from the last $n-i+1$ rows of x. For example, $|a_{n-1}a_n| = \max_{1 \leq i \neq j \leq n}\{|x_{n-1,i}x_{n,j} - x_{n,i}x_{n-1,j}|\}$. If $P = MN$, $M \in \mathscr{L}(M)$, $N \subset N_0$, then $H_P(x) = (\log|a_1 \cdots a_{n_1}|, \ldots, \log|a_{n_{r-1}+1} \cdots a_n|)$.

Proof of Lemma 6.4. 1. Let $\overline{P} \in \mathscr{P}(M)$ be the opposite parabolic subgroup. It is not hard to see that $H_{\overline{P}}(n) = -H_P(n^{t^{-1}})$, where t denotes transpose. If $a \in A_{M,d} = \mathrm{diag}(a_1, a_1, a_2, a_2)$ then

$$n = \begin{pmatrix} 1 & 0 & 0 & (1-a_1^{-1}a_2)^{-1}x \\ 0 & 1 & 0 & (1-a_1^{-1}a_2)^{-1}y \\ 0 & 0 & 1 & (1-a_1^{-1}a_2)^{-1}z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the above remark, we obtain

$$\begin{aligned} H_P(n^{t^{-1}}) &= \log\max\{1, q^d|x|, q^d|y|, q^d|z|\}(-1, 1) \\ &= -\log q(d - \min\{\nu(x), \nu(y), \nu(z)\})(1, -1), \qquad d \text{ large.} \end{aligned}$$

By definition, $v_M(n)$ is the volume in $\underline{a}_M^G$ of the convex hull of $H_P(n) = 0$ and $H_{\overline{P}}(n)$, which is, by Lemma 6.3, equal to $\frac{2}{\sqrt{3}}(d - \min\{\nu(x), \nu(y), \nu(z)\})$.

2. We note that, if $a = \mathrm{diag}(a_1, a_1, a_2, a_2) \in A_{M,d}$,

$$n^{t^{-1}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -(1-a_1^{-1}a_2)^{-1}w & -(1-a_1^{-1}a_2)^{-1}y & 0 & 0 \\ -(1-a_1^{-1}a_2)^{-1}x & -(1-a_1^{-1}a_2)^{-1}z & 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} H_{\overline{P}}(n) &= \log\max\{1, q^{2d}|wz - xy|, q^d|w|, q^d|x|, q^d|y|, q^d|z|\}(1, -1) \\ &= \log q(2d - \nu(wz - xy))(1, -1), \qquad d \text{ large.} \end{aligned}$$

To obtain 2, proceed as above for 1.

3. Let $a = \mathrm{diag}(a_1, a_1, a_2, a_3) \in A_{M,d}$. The characters $\alpha = (1, -1, 0)$, $\beta = (1, 0, -1)$ and $\gamma = (0, 1, -1)$ of A_M are viewed as elements of the dual space $\underline{a}_M^*$. Given u as in the statement of the lemma,

$$n = \begin{pmatrix} 1 & 0 & \tilde{x}_1 & \tilde{y}_1 \\ 0 & 1 & \tilde{x}_2 & \tilde{y}_2 \\ 0 & 0 & 1 & \tilde{z} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{aligned} \tilde{x}_i &= (1-a_1^{-1}a_2)^{-1}x_i, \\ \tilde{y}_i &= (1-a_1^{-1}a_3)^{-1}(y_i + a_1a_2^{-1}(1-a_1^{-1}a_2)^{-1}x_iz), \qquad i = 1, 2, \\ \tilde{z} &= (1-a_2^{-1}a_3)^{-1}z. \end{aligned}$$

Define $A = \min\{\nu(x_1, \nu(x_2)\}$ and $B = \min\{\nu(x_1y_2 - x_2y_1), \nu(z) + A\}$. For u in an open dense subset of the unipotent radical, A and B are nonzero. For d sufficiently large, the values $H_P(n)$, $P \in \mathscr{P}(M)$, are given by the table below.

Δ_P	$\log^{-1} q H_P(n)$
$\{\alpha, \gamma\}$	0
$\{-\alpha, -\gamma\}$	$(2d - B)\alpha^\vee + (2d - \nu(z) - A)\gamma^\vee$
$\{\alpha, -\beta\}$	$(-d + \nu(z))\alpha^\vee + (2d - \nu(z) - A)\beta^\vee$
$\{-\alpha, \beta\}$	$(d - A)\alpha^\vee$
$\{-\beta, \gamma\}$	$(2d - B)\beta^\vee + (-d + A)\gamma^\vee$
$\{\beta, -\gamma\}$	$(d - \nu(z))\gamma^\vee$

For the pairs $\{-\alpha, -\gamma\}$, $\{-\alpha, \beta\}$ and $\{\beta, -\gamma\}$, $H_P(n)$ can easily be computed using the remark preceding the lemma. We describe the case $\{\beta, -\gamma\}$. If $P \in \mathscr{P}(M)$ has simple roots $\{\beta, -\gamma\}$, then

$$N_P = \left\{ \begin{pmatrix} 1 & 0 & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c_{43} & 1 \end{pmatrix} \right\}.$$

Note that

$$n = n_P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tilde{z} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $n_P \in N_P$. Also, $\left(\begin{smallmatrix} 1 & \tilde{z} \\ 0 & 1 \end{smallmatrix}\right)$ is the product of $\left(\begin{smallmatrix} 1 & 0 \\ w & 1 \end{smallmatrix}\right)$ and $\mathrm{diag}(\delta_1, \delta_2)$ with a matrix in $\mathrm{GL}_2(\mathscr{O}_F)$, where $|\delta_1| = |\delta_2|^{-1} = |\tilde{z}|$, for large d. Therefore, $H_P(n) = \log(q^d|z|)(0, 1, -1)$.

The values $H_P(n)$ for $\{\alpha, -\beta\}$ and $\{-\beta, \gamma\}$ are determined by the values for the other parabolic subgroups by using the following property (see [**A4**, p. 5]): If $P, P' \in \mathscr{P}(M)$ are adjacent, and τ is the simple root of (P, A_M) in $\Delta_P \cap (-\Delta_{P'})$ which determines the wall shared by the chambers of P and P' in $\underline{a}_M$, then for any $x \in G$, $-H_P(x) + H_{P'}(x)$ is a nonnegative multiple of $\tau^\vee$. That is, $\{-H_P(x) \mid P \in \mathscr{P}(M)\}$ forms a positive orthogonal set for M.

To compute $v_M(n)$ we use formula (3.1):

$$v_M(x) = 1/r! \sum_{\{P \in \mathscr{P}(M)\}} (-\lambda(H_P(x)))^r \theta_P(\lambda)^{-1},$$

$$\lambda \in i\underline{a}_M^*, \quad r = \dim(A_M/A_G),$$

where $\theta_P(\lambda) = \mu_M^{-1}\prod_{\alpha\in\Delta_P}\lambda(\alpha^\vee)$. Setting $\lambda = (it_1, it_2, it_3)$ with t_1, t_2, t_3 distinct real numbers, $\mu_M = \sqrt{2}\log^{-2}q$, and computing $1/2\sum_{\{P\in\mathscr{P}(M)\}}(\lambda(H_P(n)))^2\theta_P(\lambda)^{-1}$, after some algebra, we obtain the desired expression for $v_M(n)$.

PROPOSITION 6.5. *Let f be a cusp form on $\mathrm{GL}_4(F)$ with $\operatorname{supp} f \subset KZ$. Given M, define the variable $u \in N \cap K$ as in Lemma 6.4.*

1. *If $M = \mathrm{GL}_3(F)\times\mathrm{GL}_1(F)$,*

$$J_M(1, f) = -2/\sqrt{3}\int_N f_K(u)\min\{\nu(x), \nu(y), \nu(z)\}\,du.$$

2. *If $M = \mathrm{GL}_2(F)\times\mathrm{GL}_2(F)$,*

$$J_M(1, f) = -\int_N f_K(u)\nu(wz - xy)\,du.$$

3. *If $M = \mathrm{GL}_2(F)\times\mathrm{GL}_1(F)\times\mathrm{GL}_1(F)$, and A and B are as in Lemma 6.4,*

$$J_M(1, f) = \sqrt{2}\int_N f_K(u)(B^2/2 - (B-A)^2 + B\nu(z))\,du.$$

Proof. Let $d \geq 1$ and $a \in A_{M,d}$. For $n \in N$ such that $u = a^{-1}n^{-1}an$, set $\tilde{v}_M(n)$ equal to

$$\begin{aligned}
&(2/\sqrt{3})(d - \min\{\nu(x), \nu(y), \nu(z)\}),\\
&2d - \nu(wz - xy),\\
&3\sqrt{2}d^2 - d(2\sqrt{2}A + 2\sqrt{2}\nu(z) + \sqrt{2}B) + B^2/\sqrt{2}\\
&\qquad\qquad - \sqrt{2}(B-A)^2 + \sqrt{2}B\nu(z),
\end{aligned}$$

in cases 1, 2 and 3, respectively. By Lemma 6.4, for all $u \in N \cap K$, $\lim_{d\to\infty}(v_M(n) - \tilde{v}_M(n)) = 0$. Results of Arthur **[A3]**, imply that $\lim_{d\to\infty}\int_N f_K(u)(v_M(n) - \tilde{v}_M(n))\,du = 0$. Thus

$$\begin{aligned}
J_M(1, f) &= \lim_{d\to\infty}\left(\int_N f_K(u)\tilde{v}_M(n)\,du + \int_N f_K(u)(v_M(n) - \tilde{v}_M(n))\,du\right)\\
&= \lim_{d\to\infty}\int_N f_K(u)\tilde{v}_M(n)\,du.
\end{aligned}$$

Because f, hence f_K, is a cusp form, we have $\int_N f_K(u)\,du = 0$. In the first two cases, $\tilde{v}_M(n)$ is a constant multiple of d plus a term which is independent of d. Thus the lemma follows immediately in these cases.

To prove 3, we first observe that, for large values of d, $v_{M_1}(n)$ is a multiple of $2d - A - \nu(z)$, where $M_1 = \mathrm{GL}_3(F) \times \mathrm{GL}_1(F)$. In the notation used in the proof of the third part of Lemma 6.4,

$$n = \begin{pmatrix} 1 & 0 & \tilde{x}_1 & 0 \\ 0 & 1 & \tilde{x}_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \tilde{y}_1 - \tilde{x}_1\tilde{z} \\ 0 & 1 & 0 & \tilde{y}_2 - \tilde{x}_2\tilde{z} \\ 0 & 0 & 1 & \tilde{z} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$v_{M_1}(n)$ is therefore a multiple of $\log\max\{1, |\tilde{z}|, |\tilde{y}_1 - \tilde{x}_1\tilde{z}|, |\tilde{y}_2 - \tilde{x}_2\tilde{z}|\}$.

$$|\tilde{z}| = q^d|z|,$$
$$|\tilde{y}_i - \tilde{x}_i\tilde{z}| = q^d|y_i - (1 - a_2^{-1}a_3)^{-1}x_iz|, \qquad i = 1, 2.$$

We assume that $x_iz \neq 0$, $i = 1, 2$, and d is large. Then $|\tilde{y}_i - \tilde{x}_i\tilde{z}| = q^{2d}|x_iz|$.

$J_{M_1}(a, f) = \delta_P(a)^{1/2}\int_N f_K(au)v_{M_1}(n)\,du$. This is obtained by the same change of variables used in the proof of Lemma 6.1. $a \in A_{M,d}$ is not elliptic in M_1, so, by Proposition 3.9, $J_{M_1}(a, f) = 0$. By an argument similar to the one above for $J_M(1, f)$, we get:

$$\begin{aligned} \lim_{a\to 1} J_{M_1}(a, f) &= \lim_{d\to\infty}\int_N f_K(u)v_{M_1}(n)\,du \\ &= \lim_{d\to\infty}\int_N f_K(u)(2d - A - \nu(z))\,du \\ &= -\int_N f_K(u)(A + \nu(z))\,du. \end{aligned}$$

Thus $\int_N f_K(u)(A + \nu(z))\,du = 0$.

Similarly, if $M_2 = \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, we can show that $v_{M_2}(n)$ is a multiple of $2d - B$ for large d, so $\int_N f_K(u)B\,du = 0$.

Looking at the formula for $\tilde{v}_M(n)$ given at the beginning of the proof, we see that

$$\int_N f_K(u)\tilde{v}_M(n)\,du = \int_N f_K(u)\sqrt{2}(B^2/2 - (B - A)^2 + B\nu(z))\,du.$$

7. Calculation of $c_{\mathcal{O}}(\pi)$ for $\mathrm{GL}_3(F)$ and $\mathrm{GL}_4(F)$. We now compute the coefficients $c_{\mathcal{O}}(\pi)$ for a generic unramified supercuspidal representation π of $\mathrm{GL}_3(F)$ or $\mathrm{GL}_4(F)$.

Let $M = \prod_{1\leq i\leq r}\mathrm{GL}_{n_i}(F)$. Let St_M be the Steinberg representation of M. If $G = \mathrm{GL}_n(F)$, the formal degree $d(\mathrm{St}_G)$ of St_G is given by

[CMS]:

$$d(\mathrm{St}_G) = 1/n \left(\prod_{k=1}^{n-1} (q^k - 1) \right) \mathrm{vol}_{Z\backslash G}(Z\backslash KZ)^{-1}.$$

Here $Z = A_G$ is the centre of G. We are assuming that $\mathrm{vol}_{Z\backslash G}(Z\backslash KZ) = \mathrm{vol}_G(K)/\mathrm{vol}_Z(K \cap Z)$. With the measures normalized as in §6, we have

$$d(\mathrm{St}_M) = \prod_{i=1}^{r} 1/\sqrt{n_i} \prod_{k=1}^{n_i - 1} (q^k - 1). \tag{7.1}$$

If $\pi = \mathrm{Ind}_{KZ}^G \sigma$, then, by [C, p. 211], the formal degree $d(\pi) = \mathrm{vol}_{Z\backslash G}(Z\backslash KZ)^{-1} \dim \sigma = \sqrt{n} \dim \sigma$.

THEOREM 7.2. *Assume* $G = \mathrm{GL}_4(F)$. *Given any character* χ *of* F^*, *let* $\pi = \mathrm{Ind}_{KZ}^G \sigma \otimes \chi \circ \det$ *be a generic unramified supercuspidal representation of* G, *where* σ *has level* h. *If* M *is a Levi subgroup, let* $\mathcal{O} = 1_M^G$.

1. *If* $M = G$, $c_{\mathcal{O}}(\pi) = -4q^{6(h-1)}$.
2. *If* $M = \mathrm{GL}_3(F) \times \mathrm{GL}_1(F)$, $c_{\mathcal{O}}(\pi) = 4q^{3(h-1)}$.
3. *If* $M = \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, $c_{\mathcal{O}}(\pi) = 2q^{2(h-1)}$.
4. *If* $M = \mathrm{GL}_2(F) \times \mathrm{GL}_1(F) \times \mathrm{GL}_1(F)$, $c_{\mathcal{O}}(\pi) = -4q^{h-1}$.
5. *If* M *is minimal,* $c_{\mathcal{O}}(\pi) = 1$.

Proof. 1 and 5 are due to Howe [H]. Let $\tilde{\chi}_\sigma$ be defined as in §5. The function $f = \tilde{\chi}_\sigma \otimes \chi \circ \det$ is a sum of matrix coefficients of π. Note that $f(u) = \tilde{\chi}_\sigma(u)$ for any unipotent element $u \in G$, so $J_{\mathcal{O}}(1, f)$, hence $c_{\mathcal{O}}(\pi)$, is independent of χ. Since $\dim \sigma = f(1)$, and $n = 4$, $d(\pi) = 2f(1)$. Putting this together with Theorem 4.4, we obtain $c_{\mathcal{O}}(\pi) = -2J_{\mathcal{O}}(1, f)/(w_{\mathcal{O}} d(\mathrm{St}(\mathcal{O})))$. In cases 1–4, $w_{\mathcal{O}} = 1, 1, 2$ and 2, respectively. The values of f on the unipotent set are given in Lemma 5.2. Substitution of these values into each formula for $J_{\mathcal{O}}(1, f)$ given in Proposition 6.5 (note that $f_K = f$), and evaluation of the integral results in:

1. $f(1) = q^{6(h-1)}(q^3 - 1)(q^2 - 1)(q - 1)$,
2. $(-2/\sqrt{3})q^{3(h-1)}(q^2 - 1)(q - 1)$,
3. $-q^{2(h-1)}(q - 1)^2$,
4. $2\sqrt{2}q^{h-1}(q - 1)$.

The calculations are fairly short in cases 2 and 3, and lengthy in case 4. We do not include them here. Using (7.1) to evaluate $d(\mathrm{St}(\mathcal{O}))$ completes the proof.

REMARK. For arbitrary n, and π and f as in the theorem, if $M = \mathrm{GL}_{n-1}(F) \times \mathrm{GL}_1(F)$, it is easy to compute

$$J_M(1, f) = -f(1)(\sqrt{n}q^{-(n-1)h})/((\sqrt{n-1})(1 - q^{-(n-1)})),$$

which results in $c_{\mathcal{O}}(\pi) = (-1)^{n-2}nq^{(n-1)(n-2)(h-1)/2}$ for $\mathcal{O} = 1_M^G$.

PROPOSITION 7.3. *Under the same assumptions as Theorem 7.2, except that* $G = \mathrm{GL}_3(F)$, $c_{\mathcal{O}}(\pi) = 3q^{3(h-1)}$, $-3q^{h-1}$, *and* 1 *for* $M = G$, $\mathrm{GL}_2(F) \times \mathrm{GL}_1(F)$, *and* M_0, *respectively.*

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