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NOTES ON REPRESENTATIONS OF NON-ARCHIMEDEAN SL(n)

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Let F be a non-archimedean local field. In this paper the relation between irreducible representations of $\mathrm{GL}(n, F)$ and $\mathrm{SL}(n, F)$ is studied. Using the results on $\mathrm{GL}(n, F)$ a parametrization of (various classes of) irreducible representations of $\mathrm{SL}(n, F)$ by parameters expressed in terms of cuspidal representations of $\mathrm{GL}(n, F)$ is obtained.

Introduction. Before we give a more detailed description of the content of this paper, a few historical remarks on $\mathrm{SL}(n, F)$ are needed. Gelfand and Naimark gave in **[8]** proof of the irreducibility of unitary principal series representations of $\mathrm{SL}(n, \mathbf{C})$. The same proof gives the irreducibility of unitary principal series for $\mathrm{GL}(n)$ over any local field. Using the fact that the unitary principal series have non-trivial Whittaker models for $\mathrm{GL}(n)$, and the uniqueness of the model proved by Rodier (**[18]**), Howe and Silberger proved in **[10]** that the unitary principal series of $\mathrm{GL}(n, F)$ restricted to $\mathrm{SL}(n, F)$ are multiplicity free. The same idea appears in Labesse and Langlands paper **[14]**. In this way, Howe and Silberger obtained that unitary principal series representations of $\mathrm{SL}(n, F)$ are multiplicity free. Shahidi observed in **[20]** that one can prove, using the same idea of Whittaker models, that any irreducible tempered representation of $\mathrm{GL}(n, F)$ restricted to $\mathrm{SL}(n, F)$ is multiplicity free. In this way one obtains that the parabolically induced representation of $\mathrm{SL}(n, F)$ by irreducible tempered representation is multiplicity free. A general approach to the reducibility and the multiplicities was done by Keys. The structure of the commuting algebras of unitary principal series representations for Chevalley groups was described by him in **[11]** and it turned out the multiplicities are not always one. This was also shown earlier by Knapp and Zuckerman in **[12]**. Gelbart and Knapp gave in **[5]** a description of irreducible constituents of the restriction to $\mathrm{SL}(n, F)$ of the unitary principal series representations of $\mathrm{GL}(n, F)$. Their paper **[6]** is based on two working hypotheses, the second of them is the multiplicity one of the restriction to $\mathrm{SL}(n, F)$ of irreducible representations of $\mathrm{GL}(n, F)$. Bernstein showed in **[1]** that any parabolically induced representation of $\mathrm{GL}(n, F)$ by an irreducible unitary representation is irreducible. In [13] Kutzko and Sally and in [17] Moy and Sally, studying the restriction to $\mathrm{SL}(n, F)$ of cuspidal representations of $\mathrm{GL}(n, F)$ showed in the tame and in the prime case that any cuspidal representation of $\mathrm{SL}(n, F)$ is induced from a compact open subgroup. These papers contain a lot of informations about restrictions of cuspidal representations in these two cases.

Now we give a more detailed description of the content of this paper. In the first paragraph it is shown that the restriction to $\mathrm{SL}(n, F)$ of an irreducible smooth representation of $\mathrm{GL}(n, F)$ is a multiplicity free representation. In particular, it proves "Working Hypothesis 2" of Gelbart and Knapp in [6]. Using the Bernstein result in [1] on the irreducibility of the unitary parabolic induction for $\mathrm{GL}(n, F)$ it is obtained that the parabolically induced representation of $\mathrm{SL}(n, F)$ by an irreducible unitary representation of a Levi subgroup is multiplicity free.

The second paragraph presents some simple general facts about restriction of irreducible representations of a connected reductive group G over F to a connected reductive subgroup G_1 of G which contains the derived group G^{der}. We need those facts in the sequel. Most of them were observed and proved by a few authors, the greatest part by Gelbart and Knapp in [5] and [6]. Here we present proofs because Gelbart and Knapp were dealing with the case of $\operatorname{char} F = 0$. In this case $G/Z(G)G_1$ is a finite group ($Z(G)$ denotes the center of G). This is not always the case in the positive characteristic.

Let $P = MN$ be a parabolic subgroup of $\mathrm{GL}(n, F)$, and $M_1 = \mathrm{SL}(n, F) \cap M$. In particular, one may consider the case of $M = \mathrm{GL}(n, F)$ and $M_1 = \mathrm{SL}(n, F)$. For an irreducible smooth representation π of M, $X_{M_1}(\pi)$ denotes the set of all characters χ of $F^\times$ such that $\pi \cong (\chi \circ \det)\pi$. This is a finite group and it has been introduced by several authors, for example in [5], [14], [17]. Fix a non-trivial unitary character of F. Take a pair consisting of an orbit $\mathscr{O}$ for the action of characters of $F^\times$ on the classes of irreducible representations of M and a from the dual group of $X_{M_1}(\pi)$ where $\pi \in \mathscr{O}$. Considering Whittaker models and the Langlands classification we fix an irreducible subrepresentation $\Delta((\mathscr{O}, a))$ of $\pi|M_1$. In this way a parametrization of all irreducible representations of M_1 is obtained by irreducible representations of $\mathrm{GL}(n, F)$ (Theorem 3.1). One can obtain a parametrization of other classes of irreducible representations of M_1 because $\Lambda((\mathscr{O}, a))$ is square integrable if and only if the orbit $\mathscr{O}$ is square integrable, $\Lambda((\mathscr{O}, a))$ is unitary if and only if the orbit $\mathscr{O}$ is unitary,.... Let us observe that the parameters for the irreducible constituents of unitary principal series of $\mathrm{SL}(n, F)$ introduced in **[5]** are of the same type.

In the last paragraph the parametrization of M_1 is reduced to cuspidal representations of $\mathrm{GL}(n, F)$ and groups $X_{\mathrm{SL}(n,F)}(\rho)$ for cuspidal representations ρ. Further reduction would be a description of the groups $X_{\mathrm{SL}(n,F)}(\rho)$ in terms of a classification of cuspidal representations. A great amount of information and calculations of these groups can be found in the paper **[14]** by Kutzo and Sally, and the paper **[17]** by Moy and Sally. In the tame case these groups appear naturally (see Remark 4.3). In this paragraph we give a necessary and sufficient condition for the irreducibility of parabolically induced representations by irreducible unitary representation (Theorem 4.2).

Note that in the case of $\mathrm{GL}(n, \mathbf{C})$ or $\mathrm{GL}(n, \mathbf{R})$ the question about the multiplicities of the restriction of irreducible unitary representations to $\mathrm{SL}(n)$ is pretty simple. Since $\mathbf{R}^x$ has two characters of finite order and $\mathbf{C}^x$ only one, by (a simple) Lemma 3.2 of **[5]** the multiplicities of the restriction are always one and the length can be at most 2 for $\mathbf{R}$, and 1 for $\mathbf{C}$ (for $\mathbf{C}$ it is evident since $\mathrm{GL}(n, \mathbf{C})$ is a product of $\mathrm{SL}(n, \mathbf{C})$ and its center).

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1. Multiplicities one. 1. We fix a locally compact non-archimedean filed F. By A (resp. A_1) we shall denote the maximal torus in $\mathrm{GL}(n, F)$ (resp. $\mathrm{SL}(n, F)$) of all diagonal matrices. The Borel subgroup of all upper triangular matrices in $\mathrm{GL}(n, F)$ (resp. $\mathrm{SL}(n, F)$) will be denoted by B (resp. B_1). The choice of the Borel subgroup determines in a natural way a set of positive roots and further, the set of simple roots.

Now we have a well known

1.1. LEMMA. *Let (σ, V) be a smooth representation of a Levi factor M of a parabolic subgroup $P = MN$ in $\mathrm{GL}(n, F)$, where N denotes the nilpotent radical of P. Set $M_1 = M \cap \mathrm{SL}(n, F)$. Then $P_1 = M_1 N$ is a parabolic subgroup of $\mathrm{SL}(n, F)$ and $P_1 = M_1 N$ is a Levi decomposition of P_1. The representation $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\sigma)|\,\mathrm{SL}(n, F)$ is isomorphic to $\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\sigma|M_1)$ with an isomorphism given by restriction to $\mathrm{SL}(n, F)$.*

Let π be an irreducible smooth representation of $\mathrm{GL}(n, F)$. Then $\pi|\,\mathrm{SL}(n, F)$ is a finite sum of irreducible representations. This can be obtained from **[21]** (see Lemma 2.1 for a more detailed explanation).

1.2. THEOREM. *For an irreducible smooth representation* (π, V) *of* $\mathrm{GL}(n, F)$, $\pi|\,\mathrm{SL}(n, F)$ *is a multiplicity free representation.*

Proof. We consider Langlands parameters of π. We can choose a parabolic subgroup $P = MN$ of $\mathrm{GL}(n, F)$ containing B, an irreducible tempered representation τ of M and a positive-valued character χ of M satisfying the positiveness condition with respect to roots of Proposition 2.6 in Chapter XI of **[3]**, such that π is a unique irreducible quotient of $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\chi\tau)$. We shall assume that we took a Levi factor M which consists of diagonal block matrices for a suitable partition of $n = n_1 + \cdots + n_k$. Then

$$M \cong \mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_k, F)$$

and we identify M with $\mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_k, F)$. Set $M_1 = M \cap \mathrm{SL}(n, F)$ and $P_1 = M_1 N$.

Note that $\tau = \tau_1 \otimes \cdots \otimes \tau_k$ where τ_i are irreducible tempered representations of $\mathrm{GL}(n_i, F)$. Since τ_i has Whittaker model by **[25]**, in the same way as in **[10]** one obtains that $\tau_i|\,\mathrm{SL}(n_i, F)$ is multiplicity free (this was observed in **[20]**, see also Proposition 2.8). Thus $\tau|\,\mathrm{SL}(n, F) \times \cdots \times \mathrm{SL}(n_k, F)$ is multiplicity free. Since $\mathrm{SL}(n_1, F) \times \cdots \times \mathrm{SL}(n_k, F) \subseteq M_1$, $\tau|M_1$ is multiplicity free.

Note that $\tau|M$ is a direct sum of irreducible representations of M_1 (for a more detailed explanation see Lemma 2.1). Let $\tau = \bigoplus_{i=1}^{p} \tau_i$ be a decomposition into irreducible representations of M_1. Observe that all unipotent radicals in M are contained in M_1 and thus the Jacquet modules for parabolic subgroups of M and M_1 are the same spaces. Applying Theorem 2.8.1 of **[23]** one obtains that $\tau_1, \dots, \tau_p$ are tempered representations of M_1 (central exponents of Jacquet modules of M_1-representations are obtained by restricting central exponents of Jacquet modules of M-representations, see also Proposition 2.7).

The representations τ_i are inequivalent irreducible tempered representations of M_1 and $\chi_1 = \chi|M_1$ satisfies the positiveness condition of Proposition 2.6 in Chapter XI of **[3]**, considered for $\mathrm{SL}(n, F)$. Thus $\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\chi_1\tau_i)$ has a unique irreducible quotient say (π_i', V_i). Since all τ_i are inequivalent, $\pi_1', \dots, \pi_p'$ are all inequivalent. By

Lemma 1.1 we can fix an isomorphism

$$\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\chi\tau)|\,\mathrm{SL}(n,F) \cong \mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}\left(\bigoplus_{i=1}^{p}\chi_1\tau_i\right)$$
$$\cong \bigoplus_{i=1}^{p}\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\chi_1\tau_i).$$

Let $\psi : \bigoplus_{i=1}^{p}\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\chi_1\tau_i) \to W$ be a non-trivial morphism of $\mathrm{SL}(n,F)$-representations where (σ, W) is an irreducible $\mathrm{SL}(n,F)$-representation. Suppose that $\psi|\,\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\chi_1\tau_{i_0}) \neq 0$. Thus $\sigma \cong \pi'_{i_0}$. Since $\pi'_1, \dots, \pi'_p$ are inequivalent, $\psi|\,\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\chi_1\tau_i) = 0$ for $i \neq i_0$. If we have another $\mathrm{SL}(n,F)$-morphism

$$\psi_1 : \bigoplus_{i=1}^{p}\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\psi_1\tau_i) \to W$$

it must be proportional to ψ by the uniqueness of irreducible quotient ([3]):

Consider a decomposition $\pi|\,\mathrm{SL}(n,F) = \bigoplus_{j=1}^{q}\pi_j$ into irreducible representations of $\mathrm{SL}(n,F)$. Consider the natural morphisms of $\mathrm{SL}(n,F)$-representations

$$\varphi_j : \mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\chi\tau) \to \pi = \bigoplus_{i=1}^{q}\pi_i \to \pi_j.$$

Suppose that $\pi_i \cong \pi_j$ for some $i \neq j$. Let Λ be an isomorphism of π_i onto π_j. Then

$$\varphi_j, \Lambda\circ\varphi_i : \mathrm{Ind}_P^{\mathrm{GL}(n,F)}\cdot(\chi\tau) \to \pi_j$$

are non-trivial $\mathrm{SL}(n,F)$-morphisms. Note that $\ker\varphi_j \neq \ker(\Lambda\circ\varphi_i)$. Thus φ_j and $\Lambda\circ\varphi_i$ are not proportional. This contradicts the above observations about $\mathrm{SL}(n,F)$-morphisms ψ. Thus $\pi_j \neq \pi_j$ for $i \neq j$. This proves the theorem.

1.3. Remark. The above observations on $\mathrm{SL}(n,F)$-morphisms ψ imply that $\{\pi_1, \dots, \pi_q\} \subseteq \{\pi'_1, \dots, \pi'_p\}$. It is not difficult to obtain $p = q$ and thus $\{\pi_1, \dots, \pi_q\} = \{\pi'_1, \dots, \pi'_p\}$ (otherwise $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\chi\tau)$ would have two different irreducible quotients). In this way there is a natural bijection between irreducible subrepresentations of $\chi\sigma|M_1$ and irreducible subrepresentations of $\pi|\,\mathrm{SL}(n,F)$.

1.4. THEOREM. *Let P_1 be a parabolic subgroup of* $\mathrm{SL}(n, F)$ *with a Levi decomposition* $P_1 = M_1 N$. *Let* (σ, U) *be an irreducible unitary representation of* M_1. *Then* $\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\sigma)$ *is a multiplicity free representation.*

Proof. Choose a parabolic subgroup P of $\mathrm{SL}(n, F)$ with the Levi decomposition $P = MN$ such that $M_1 = M \cap \mathrm{SL}(n, F)$. Then $P_1 = P \cap \mathrm{SL}(n, F)$. It is not difficult to see that there exists an irreducible unitary representation σ_0 of M such that σ is a subrepresentation of $\sigma_0|M_1$ (for the proof see Propositions 2.2. and 2.7). Now $\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\sigma)$ is a subrepresentation of $\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\sigma_0|M_1)$ which is isomorphic to $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\sigma_0)|\,\mathrm{SL}(n, F)$ by Lemma 1.1. Thus, to prove the theorem it is enough to prove that $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\sigma_0)|\,\mathrm{SL}(n, F)$ is multiplicity one. Since $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\sigma_0)$ is irreducible by Corollary 8.2 of [1], $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\sigma_0)|\,\mathrm{SL}(n, F)$ is multiplicity free by Theorem 1.2.

2. Some general remarks. In this paragraph we collect some general remarks, most of them well-known, about the connection of representations of reductive groups $G_1 \subseteq G$ which are in a position analogous to the position of $\mathrm{SL}(n, F) \subseteq \mathrm{GL}(n, F)$. A great part of this is proved, among other papers, in [5], [6], [18], [20]. For the sake of completeness we shall give proofs for which we do not know a precise reference in considered generality. Usually it was considered the situation when $G_1 Z(G)$ is of finite index in G but this is not necessarily true if $\mathrm{char}\, F \neq 0$. ($Z(G)$ denotes the center of G). Since $G/G_1 Z(G)$ is always compact, the case of infinite $G/G_1 Z(G)$ is a slight modification of the case of finite $G/G_1 Z(G)$.

We shall denote by G the group of rational points of a connected reductive group over a non-archimedean field F, and by G^{der} the group of rational points of its derived subgroup. The center of G is denoted by $Z(G)$. By G_1 it will be denoted rational points of a connected reductive subgroup of G containing G^{der}. The set of all classes of irreducible smooth representations of G will be denoted by $\widetilde{G}$ while the subset of all unitarizable (resp. tempered, square integrable modulo center, cuspidal) classes will be denoted by $\widehat{G}$ (resp. $T^u(G)$, $D^u(G)$, $C(G)$). The subset of $\widetilde{G}$ of essentially square integrable representations (resp. essentially tempered representations) will be denoted by $D(G)$ (resp. $T(G)$). Set $C^u(G) = C(G) \cap \widehat{G}$.

For $(\pi, V) \in \widetilde{G}$ and σ a continuous automorphism of G, π_σ will denote the representation $\pi_\sigma(g) = \pi(\sigma(g))$ which is again in $\widetilde{G}$. Clearly $\pi_{\sigma_1\sigma_2} = (\pi_{\sigma_1})_{\sigma_2}$. Let $x \in G$ and let $\gamma(x)$ be the inner automorphism of G defined by $\gamma(x) : g \to xgx^{-1}$. For $(\tau, V) \in \widetilde{G}_1$ set

$$\tau_x = \tau_{\gamma(x)|G_1}.$$

In this way G acts on $\widetilde{G}_1$. This action factorizes to an action of $G/Z(G)G_1$.

Now we have an easy consequence of **[21]**.

2.1. LEMMA. *For $\pi \in \widetilde{G}$, $\pi|G_1$ is a finite direct sum of irreducible representations of G_1.*

Proof. Let for a moment G, G^{der} and $Z(G)$ will be considered as algebraic groups over an algebraic closure of F. Let $Z(G)_0$ be the connected component of $Z(G)$. Then the multiplication $G^{\mathrm{der}} \times Z(G)_0 \to G$ is an isogeny ([2], 14.2, Proposition). Let us return to the groups of rational points. By **[21]**, $\pi|G^{\mathrm{der}}Z(G)$ is a finite direct sum of irreducible representations of $G^{\mathrm{der}}Z(G)$ and moreover, by the Schur lemma, of G^{der}. Thus $\pi|G_1$ is a finite length representation. This implies that $\pi|G_1$ is completely reducible (see proof of Lemma 3 of **[21]**).

Let $\pi \in \widetilde{G}$. Denote by $\mathscr{O}_{G_1}(\pi)$ the set of all $\tau \in G_1$, which are isomorphic to a subrepresentation of $\pi|G_1$. Clearly, $\mathscr{O}_{G_1}(\pi)$ is a finite set and it is invariant for the action of G (since $\pi \cong \pi_{\gamma(g)}$ for $g \in G$). The action of G on $\mathscr{O}_{G_1}(\pi)$ is transitive (since π is irreducible). Set

$$\pi|G_1 \cong \bigoplus_{\tau\in\mathscr{O}(\pi)} n(\tau)\tau.$$

The linear independence of characters together with the transitivity of the action of G on $\mathscr{O}_{G_1}(\pi)$ implies that all $n(\tau)$ are the same, say $m_{G_1}(\pi)$. Thus

$$\pi|G_1 \cong m_{G_1}(\pi) \bigoplus_{\tau\in\mathscr{O}(\pi)} \tau.$$

The cardinality of $\mathscr{O}_{G_1}(\pi)$ will be denoted by $n_{G_1}(\pi)$.

By ${}^0G$ it is denoted the set of all $g \in G$ such that $|\chi(g)|_F = 1$ for all F-rational characters χ of G. Then ${}^0G/G^{\mathrm{der}}$ is compact, $G/{}^0G$ is a free Z-module of finite rank, say n, and $G/{}^0GZ(G)$ is finite. Thus

$${}^0GZ(G)/{}^0G \cong Z(G)/(Z(G) \cap {}^0G)$$

is a free Z-module of rank n. Therefore

$$Z(G) \to Z(G)/(Z(G) \cap {}^0G)$$

splits. Denote by S the image of a splitting homomorphism. Then S is a closed discrete subgroup of $Z(G)$ which is a free Z-module of rank n, $S \cap {}^0G = \{1\}$, $S(Z(G) \cap {}^0G) = Z(G)$, and ${}^0GZ(G) = {}^0GS$. Note that SG^{der} is a closed subgroup of G and that G/SG^{der} is compact. Also SG^{der} is a direct sum of S and G^{der}. Note that ${}^0GZ(G)/{}^0G = {}^0GS/{}^0G \cong S$ is also of rank n and it is of finite index in $G/{}^0G$. Let k be the rank of ${}^0GG_1/{}^0G$. Then

$$({}^0GS/{}^0G) \cap ({}^0GG_1/{}^0G) \cong ({}^0GS \cap {}^0GG_1)/{}^0G$$

is also of rank k. Let

$$S' = \{s \in S\,;\, s\,{}^0G \subseteq {}^0GS \cap {}^0GG_1\}.$$

Then

$${}^0GS \cap {}^0GG_1 = {}^0GS'.$$

Let S_1 be a maximal subgroup of S among subgroups satisfying $S_1 \cap S' = \{1\}$. Now S_1 is of rank $n-k$ and S_1S' is of finite index in S.

Consider S_1G_1. First note that $S_1 \cap {}^0GG_1 = \{1\}$ (in particular $S_1 \cap G_1 = \{1\}$). This implies $S_1G_1 \cap {}^0GG_1 = G_1$. Since ${}^0G$ is an open subgroup of G and G_1 is closed, it is easy to see that S_1G_1 is a closed subgroup of G. It is a direct product of S_1 and G_1. Note that

$${}^0GS_1G_1/S_1G_1 \cong {}^0G/(S_1G_1 \cap {}^0G)$$

is compact since ${}^0G/G^{\mathrm{der}}$ is compact. Since $G/{}^0GS_1S'$ is finite and ${}^0GS_1S' \subseteq {}^0GS_1G_1$, $G/{}^0GS_1G_1$ is finite. Thus G/S_1G_1 is compact.

2.2. PROPOSITION. *For each $\tau \in \widetilde{G}_1$ there exists $\pi \in \widetilde{G}$ such that τ is isomorphic to a subrepresentation of $\pi|G_1$. If the central character of τ is unitary, then there exists such π with the unitary central character.*

Proof. Let $(\tau, U) \in \widetilde{G}_1$. Extend τ to a representation of S_1G_1 defining that each element of S_1 acts as identity. Let (π_1, V_1) be the representation $\mathrm{Ind}_{S_1G_1}^G(\tau)$. This is an admissible representation. Then $f \to f(1)$, $V_1 \to U$ is a S_1G_1-intertwining whose restriction to any non-zero G-invariant subspace is non-zero (thus it is surjective).

Let V_2 be any non-zero finitely generated G-subrepresentation of V_1. Then we have a surjective S_1G_1-intertwining $\alpha : V_2 \to U$. Since V_2 is finitely generated and admissible, it is of finite length. Therefore, we can choose an irreducible G-subrepresentation V_3 of V_2 with the property $\alpha(V_3) = U$. This completes the proof of existence.

Suppose that the central character ω_τ of τ is unitary. Then for the central character ω_π of π we have $\omega_\pi|S_1 = 1$ by construction. Consider $|\omega_\pi|$. It extends to a character χ of G into $\mathbf{R}_+^x$. First $\chi|G_1 = 1$ since $\chi = 1$ on the center of G_1. Therefore $\chi|S_1G_1 = 1$ and finally $\chi|G = 1$. Thus $|\omega_\pi| = 1$.

2.3. Corollary. *Let* $(\tau, U) \in \widetilde{G}_1$. *Set*

$$G_\tau = \{g \in G : \tau_g \cong \tau\}.$$

Then $Z(G)G_1 \subseteq G_\tau$ *and* G_τ *is an open normal subgroup of* G *of finite index.*

Proof. Choose $(\pi, V) \in \widetilde{G}$ such that there is a G_1-subrepresentation $V_1 \subseteq V$ equivalent to U. Let $v_0 \in V_1$, $v_0 \neq 0$. Denote by K an open subgroup of G fixing v_0. Then $KZ(G)G_1 \subseteq G_\tau$ and $KZ(G)G_1$ is open in G and has finite index.

Similarly as in Lemma 3.2 of [5] we obtain the following:

2.4. Proposition. *Let* (π_1, V_1), $(\pi_2, V_2) \in \widetilde{G}$. *Let* $h_{G_1}(\pi_1, \pi_2)$ *be the number of all characters* χ *of* G/G_1 *such that* $\chi\pi_1 \cong \pi_2$ *as representations of* G. *Then* $h_{G_1}(\pi_1, \pi_2)$ *is finite and equal to the dimension of*

$$\mathrm{Hom}_{G_1}(\pi_1, \pi_2).$$

Proof. First we shall prove the proposition in the case when the restrictions of central characters of π_1 and π_2 to S_1 are the same. Observe that with this assumption

$$\mathrm{Hom}_{G_1}(\pi_1, \pi_2) = \mathrm{Hom}_{S_1G_1}(\pi_1, \pi_2).$$

By Frobenius reciprocity

$$\mathrm{Hom}_{S_1G_1}(\pi_1, \pi_2) \cong \mathrm{Hom}_G(\pi_1, \mathrm{Ind}_{S_1G_1}^G(\pi_2)).$$

Denote by $C^\infty(S_1G_1\backslash G)$ the representation of G by right translations on the space of locally constant functions on G constant on S_1G_1-cosets. We have an isomorphism

$$\alpha : C^\infty(S_1G_1\backslash G) \otimes V_2 \to \mathrm{Ind}_{S_1G_1}^G(\pi_2)$$

given by $f \otimes v \to (x \mapsto f(x)\pi_2(x)v)$. It is obvious that α is a well defined injective intertwining. Let $\varphi \in \mathrm{Ind}_{S_1G_1}^G(\pi_2)$. Let X be an open compact subset such that $S_1G_1X = G$. Choose an open compact subgroup K fixing φ and fixing each element of the finite set $\{\pi_2(x^{-1})\varphi(x); x \in X\}$. Let $g_1, \dots, g_n \in X$ be the representatives for $S_1G_1\backslash G/K$. Define φ_i by $\varphi_i|S_1G_1g_iK = \varphi|S_1G_1gK$ and $\varphi_i(x) = 0$ otherwise. Then $\varphi = \varphi_1 + \dots + \varphi_n$ and $\varphi_i \in \mathrm{Ind}_{S_1G_1}^G(\pi_2)$. Let χ_i be the characteristic function of $S_1G_1g_iK$. Now

$$\chi_i \otimes \pi_2(g_i^{-1})\varphi(g_i) \mapsto \varphi_i$$

and this proves the surjectivity.

Note that $C^\infty(S_1G_1\backslash G)$ is isomorphic to the sum of all characters of G/S_1G_1. Thus the set of all unitary characters χ of G/S_1G_1 such that $\chi\pi_2 \cong \pi_1$ is finite and the number of such χ is the dimension of $\mathrm{Hom}_{S_1G_1}(\pi_1, \pi_2) = \mathrm{Hom}_{G_1}(\pi_1, \pi_2)$. Note that for a character χ of G/G_1 such that $\chi\pi_2 \cong \pi_1$ it must be $\chi|S_1 = 1$ (consider the central character).

Now let π_1 and π_2 be arbitrary. Let ω_{π_i} be the central character of π_i. Consider $\omega_{\pi_i}|S_1$ as a character of

$$S_1{}^0GG_1/{}^0GG_1 \cong S_1.$$

Note that $G/S_1{}^0GG_1$ is finite. It is easy to see that $\omega_{\pi_i}|S_1$ extends to a character of $G/{}^0GG_1$, say χ_i. Then

$$\mathrm{Hom}_{G_1}(\pi_1, \pi_2) = \mathrm{Hom}_{G_1}(\chi_1^{-1}\pi_1, \chi_2^{-1}\pi_2).$$

Now we can apply the first part of the proof and the proposition is proved.

2.5. Corollary. *Let $\pi_1, \pi_2 \in \widetilde{G}$. Then the following statements are equivalent*:

(i) *There exists a character χ of G/G^1 such that $\chi\pi_1 \cong \pi_2$.*
(ii) $\mathcal{O}_{G_1}(\pi_1) \cap \mathcal{O}_{G_1}(\pi_2) \neq \varnothing$.
(iii) $\mathcal{O}_{G_1}(\pi_1) = \mathcal{O}_{G_1}(\pi_2)$.

By the above corollary the orbits of the action of G on $\widetilde{G}_1$ are in the bijection with the orbits of the action of the characters of G/G_1 onto $\widetilde{G}$.

2.6. Remark. Let $\pi \in \widetilde{G}$. We shall denote by $X_{G_1}(\pi)$ the group of all characters χ of G/G_1 such that $\chi\pi \cong \pi$. It is simple to see that a character χ of G which is trivial on G_{π_1}, where π_1 is any irreducible subrepresentation of $\pi|G_1$, is in $X_{G_1}(\pi)$ ([6], Lemma 2.1, (e)). If $\pi|G_1$ is multiplicity free, then the converse is true: if $\chi \in X_{G_1}(\pi)$, then $\chi(G_{\pi_1}) = 1$ ([6], Corollary 2.2). The converse is a consequence of comparison of two finite cardinal numbers.

2.7. Proposition. *Let (π, V) be an irreducible smooth representation of G with unitary central character. Then the following equivalences hold*:

(i) $\pi \in \widehat{G} \Leftrightarrow \mathscr{O}_{G_1}(\pi) \subseteq \widehat{G}_1 \Leftrightarrow \mathscr{O}_{G_1}(\pi) \cap \widehat{G}_1 \neq \varnothing$.
(ii) $\pi \in C^u(G) \Leftrightarrow \mathscr{O}_{G_1}(\pi) \subseteq C^u(G_1) \Leftrightarrow \mathscr{O}_{G_1}(\pi) \cap C^u(G_1) \neq \varnothing$.
(iii) $\pi \in D^u(G) \Leftrightarrow \mathscr{O}_{G_1}(\pi) \subseteq D^u(G_1) \Leftrightarrow \mathscr{O}_{G_1}(\pi) \cap D^u(G_1) \neq \varnothing$.
(iv) $\pi \in T^u(G) \Leftrightarrow \mathscr{O}_{G_1}(\pi) \subseteq T^u(G_1) \Leftrightarrow \mathscr{O}_{G_1}(\pi) \cap T^u(G_1) \neq \varnothing$.

Proof. We shall outline only the proofs of implications which are not completely trivial.

The only such implication in (i) is $\mathscr{O}_{G_1}(\pi) \subseteq \widehat{G}_1 \Rightarrow \pi \in \widehat{G}$. Suppose $\mathscr{O}_{G_1}(\pi) \subseteq \widehat{G}_1$. Now we can choose a G_1-invariant scalar product $(\ ,\ )_1$ on V. Then $\pi|S_1G_1$ is unitary. For $v_1, v_2 \in V$ set

$$(v_1, v_2) = \int_{S_1G_1\backslash G} (\pi(g)v_1, \pi(g)v_2)_1\, dg.$$

This is a G-invariant scalar product on V.

It is easy to obtain directly all implications of (ii).

One obtains implications in (iii) by directly comparing integrals of matrix coefficients (one can also prove (iii) using the criterion for square-integrability in Theorem 2.7.1 of [23]).

Let V_1 be an irreducible tempered G_1-subrepresentation of V. Then V_1 is a subrepresentation of suitable $\mathrm{Ind}_{M_1N}^{G_1}(\delta)$ where M_1N is a parabolic subgroup of G_1 and δ a square-integrable representation of the Levi factor M_1. Now it is easy to see that all elements from the orbit $\mathscr{O}_{G_1}(\pi)$ are subrepresentations of the same type of representation. We can choose M_1N in such a way that there exists a parabolic MN in G and $M_1 = M \cap G_1$, $M_1N = MN \cap G_1$. Then we can choose by Proposition 2.2, an irreducible representation δ_0 of M with the unitary central character such that δ is a subrepresentation of $\delta_0|M_1$. By (iii), δ_0 is square integrable. Then we have a projection of $\mathrm{Ind}_{MN}^{G}(\delta_0)|G$ onto V_1. Thus there exists $\pi' \in T^u(G)$ such that V_1 is a subrepresentation of $\pi'|G_1$. Now Proposition 2.4. implies $\pi' = \chi\pi$ with χ unitary. Thus $\pi \in T^u(G)$. The implication

$\pi \in T^u(G) \Rightarrow \mathscr{O}_{G_1}(\pi) \subseteq T^u(G_1)$ proceeds in the similar way. One can prove also (iv) using the criterion in Theorem 2.8.1 of **[23]**.

One can prove the next proposition in the same way as the Theorem in **[10]**. Nevertheless we shall present the proof because we shall need it in the later discussion.

2.8. Proposition. *Suppose additionally that G is a split group and that $(\pi, V) \in \widetilde{G}$ possesses a Whittaker model. Then $\pi|G_1$ is multiplicity free.*

Proof. Let $B = AN$ be a Borel subgroup of G such that A is a maximal split torus of G and N the nilpotent radical of B. Suppose that π has a Whittaker model with respect to a nondegenerate character ϑ of N. Then there exists a non-trivial linear form φ on V such that $\varphi(\pi(u)v) = \vartheta(n)\varphi(v)$, $n \in N$, $v \in V$.

Let $V = V_1 + \cdots + V_n$ be a decomposition into irreducible G_1-representations. Then $\varphi|V_i \neq 0$ for some i. We may take $i = 1$. The uniqueness of the Whittaker model with respect to ϑ implies $\varphi|V_i = 0$ for $i \geq 2$ (**[18]**). Thus V_i, $i \geq 2$ do not have Whittaker models with respect to ϑ. This implies that V_1 is not isomorphic to V_i for any $i \geq 2$. Therefore, $\pi|G_1$ is multiplicity free.

2.9. Remark. Consider the proof of Proposition 2.8. Take $a \in A$. Denote by ϑ_a a character $\vartheta_a(n) = \vartheta(ana^{-1})$. Now if π_1 has a Whittaker module with respect to ϑ, then $(\pi_1)_a$ has a Whittaker module with respect to ϑ_a. Denote

$$A_{\pi_1} = G_{\pi_1} \cap A.$$

Since $AG_1 = G$,

$$A/A_{\pi_1} \cong G/G_{\pi_1}.$$

Now $a \mapsto (\pi_1)_a$ is a parametrization of $\mathscr{O}_{G_1}(\pi)$ by A/A_{π_1}. Let $a_0 \in A$. For any $a \in a_0 A_{\pi_1}$, $(\pi_1)_{a_0}$ has a Whittaker model with respect to ϑ_a. The proof of the preceding proposition implies that $\pi_1' \in \mathscr{O}_{G_1}(\pi)$ such that $\pi_1' \not\cong (\pi_1)_{a_0}$, cannot have Whittaker model with respect to ϑ_a with $a \in a_0 A_{\pi_1}$. For a finite group X of characters of G set

$$G_X = \{g \in G;\, \chi(g) = 1,\, \forall \chi \in X\},$$
$$A_X = \{a \in A;\, \chi(a) = 1,\, \forall x \in X\}.$$

Since $\pi|G_1$ is multiplicity one, Remark 2.6 implies

$$G_{\pi_1} = G_{X_{G_1}(\pi)}, \qquad A_{\pi_1} = A_{X_{G_1}(\pi)}.$$

Thus for fixed ϑ, $(A/A_{X_{G_1}(\pi)})$ parametrizes $\mathcal{O}_{G_1}(\pi)$ in the following way: for each $aA_{X_{G_1}(\pi)} \in (A/A_{X_{G_1}(\pi)})$ there exists a unique $\sigma \in \mathcal{O}_{G_1}(\pi)$ characterized with the property that σ has a Whittaker model with respect to ϑ_a.

3. Parametrization of representations of SL-groups by GL-parameters. In the rest of this paper we shall consider reductive groups $\mathrm{GL}(n, F)$, $\mathrm{SL}(n, F)$ and Levi factors of their parabolic subgroups. The parabolic subgroup P of $\mathrm{GL}(n, F)$ will always be considered to contain upper triangular matrices, and for a Levi decomposition $P = MN$, M will always be assumed to be diagonal block-matrix (for suitable decomposition $n = n_1 + \cdots + n_k$). Now parabolics in $\mathrm{SL}(n, F)$ will be considered to be of the form

$$\begin{aligned} P_1 &= P \cap \mathrm{SL}(n, F), \\ M_1 &= M \cap \mathrm{SL}(n, F), \\ P_1 &= M_1 N. \end{aligned}$$

For M we know $M \cong \mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_k, F)$ in a natural way and we consider parabolic subgroups of M which are products of the above described parabolics of $\mathrm{GL}(n_i, F)$'s. A similar choice is made for Levi decompositions. The corresponding notions for M_1 we shall assume to be obtained from M by intersecting with M_1. We shall always assume that the maximal torus A in M (and $\mathrm{GL}(n, F)$) consists of diagonal matrices, and the maximal torus A_1 in M_1 to be $A \cap M_1$. We shall always consider identifications

$$\begin{aligned} \det &: M/M_1 \to F^x, \\ \det &: A/A_1 \to F^x. \end{aligned}$$

Using the first identification, we have an action of $(F^x)^\sim$ on $\widetilde{M}$ and $(F^x)^\wedge$ on $\widehat{M}$.

A non-trivial unitary character ψ_0 of F will be fixed. Fixing ψ_0 we have a canonical non-degenerate character ϑ of the unipotent radical of the Borel subgroup of $\mathrm{GL}(n, F)$.

$$\begin{bmatrix} 1 & u_{12} & u_{13} & \cdots & \\ & 1 & u_{23} & & \vdots \\ & & 1 & & \vdots \\ & & & \ddots & u_{n-1,n} \\ & & & & 1 \end{bmatrix} \mapsto \psi_0(u_{12} + u_{23} + \cdots + u_{n-1,n}).$$

For the unipotent radical of the Borel subgroup of a group M (and thus of M_1), we shall consider the nondegenerate character obtained by restricting ϑ, and again denote it by ϑ.

For $R \subseteq \widetilde{M}$ (resp. $\widehat{M}$) invariant for the action of $(F^x)^\sim$ (resp. $(F^x)^\wedge$) we introduce a notation $(R/\sim) \times \widehat{X}_{M_1}$ (resp. $(R/^\wedge) \times \widehat{X}_{M_1}$):

$$(R/\sim) \times \widehat{X}_{M_1} = \bigcup_{(F^x)^\sim \pi \in R/(F^x)^\sim} \{(F^x)^\sim \cdot \pi\} \times (X_{M_1}(\pi))^\wedge$$

$$\left(\text{resp. } (R/^\wedge) \times \widehat{X}_{M_1} = \bigcup_{(F^x)^\wedge \pi \in R/(F^x)^\wedge} \{(F^x)^\wedge \cdot \pi\} \times (X_{M_1}(\pi))^\wedge\right).$$

Here $\{(F^x)^\sim \pi\}$ (resp. $\{(F^x)^\wedge \pi\}$) is considered as a one-element set consisting of one orbit. We shall give a more detailed description of these objects.

First suppose that $\pi' \in (F^x)^\sim \pi$. Then $X_{M_1}(\pi) = X_{M_1}(\pi')$. Thus the above notations are well-defined. Recall that $X_{M_1}(\pi) = \{\chi \in (M/M_1)^\wedge;\ \chi\pi \cong \pi\} = \{\chi \in (A/A_1)^\wedge;\ \chi\pi \cong \pi\}$ (after identification M/M_1 and A/A_1). Let π_1 be an irreducible subrepresentation of $\pi|M_1$. Then $M_{\pi_1} = \{m \in M;\ (\pi_1)_m \cong \pi_1\}$ by Theorem 1.2 and Remark 2.6 equals

$$M_{\pi_1} = \{m \in M;\ \chi(m) = 1,\ \forall \chi \in X_{M_1}(\pi)\},$$

and

$$A_{\pi_1} = M_{\pi_1} \cap A = \{a \in A;\ \chi(a) = 1,\ \forall \chi \in X_{M_1}(\pi)\}.$$

Thus

$$A_{\pi_1} = X_{M_1}(\pi)^\perp$$

in A, and

$$(A/A_{\pi_1})^\wedge \cong X_{M_1}(\pi) \Rightarrow A/A_{\pi_1} \cong (X_{M_1}(\pi))^\wedge$$

canonically.

We have seen that there is a canonical description

$$(R/\sim) \times \widehat{X}_{M_1} = \bigcup_{R/(F^x)^\sim} \{(F^x)^\sim \pi\} \times (A/A_{X_{M_1}(\pi)}),$$

$$(R/^\wedge) \times \widehat{X}_{M_1} = \bigcup_{R/(F^x)^\wedge} \{(F^x)^\wedge \pi\} \times (A/A_{X_{M_1}(\pi)}).$$

Note that $A_1 \subseteq A_{X_{M_1}(\pi)}$ and therefore we can identify using the determinant homomorphism $A/A_{X_{M_1}(\pi)}$ with $F^x/F^x_{X_{M_1}(\pi)}$ where

$$F^x_{X_{M_1}(\pi)} = \det(A_{X_{M_1}(\pi)}).$$

Since $M/M_1 \cong A/A_1 \cong F^\times$ by the determinant homomorphism, we may identify $(M/M_1)^\wedge$, $(A/A_1)^\wedge$ with $(F^\times)^\wedge$ and thus consider $X_{M_1}(\pi) \subseteq (F^\times)^\wedge$. Now

$$F^\times_{X_{M_1}(\pi)} = \{x \in F^\times ; \chi(x) = 1, \forall \chi \in X_{M_1}(\pi)\}.$$

Now we shall give a canonical parametrization of $T(M_1)$.

Take $x \in (T(M)/\sim) \times \widehat{X}_{M_1}$. Then $x = ((F^\times)^\sim\pi, aA_{X_{M_1}(\pi)})$. Now the decomposition $\pi|M_1$ does not depend on π from the orbit $(F^\times)^\sim\pi$. By Remark 2.9. there exists a unique irreducible subrepresentation $\Lambda(x)$ of $\pi|M_1$ possessing a Whittaker model with respect to ϑ_a. Then results of §2 imply

$$\Lambda : (T(M)/\sim) \times \widehat{X}_{M_1} \to T(M_1)$$

is a one-to-one correspondence.

Let $x \in (\widetilde{M}/\sim) \times \widehat{X}_{M_1}$, $x = ((F^\times)^\sim\pi, aA_{X_{M_1}(\pi)})$. Consider the Langlands parameters of π: let $P' = M'N'$ be a parabolic subgroup in M and σ an essentially tempered representation of M' satisfying necessary positiveness condition, such that π is a unique irreducible quotient of $\mathrm{Ind}_{P'}^M(\sigma)$. Set $M_1' = M' \cap M_1$. Note first that $X_{M_1'}(\sigma) \subseteq X_{M_1}(\pi)$. The uniqueness of the Langlands parameters implies that we actually have the equality $X_{M_1'}(\sigma) = X_{M_1}(\pi)$ and thus $A_{X_{M_1}(\pi)} = A_{X_{M_1'}(\sigma)}$, $A/A_{X_{M_1}(\pi)} = A/A_{X_{M_1'}(\sigma)}$.

Now we shall parametrize irreducible subrepresentations of $\pi|M_1$ using tempered representations. We shall use the parametrization obtained in Remark 1.3 (see also the proof of Theorem 1.2). To $x' = ((F^\times)^\sim\sigma, aA_{X_{M_1}(\pi)}) = ((F^\times)^\sim\sigma, aA_{X_{M_1'}(\sigma)})$ we have attached $\Lambda(x') \in T(M_1')$. Recall that $\Lambda(x')$ is an irreducible subrepresentation of $\sigma|M_1'$. Now $\mathrm{Ind}_{P_1'}^{M_1}(\Lambda(x))$ has a unique irreducible quotient which will be denoted by $\Lambda(x)$. Note that $\Lambda(x)$ is an irreducible subrepresentation of $\pi|M_1$. Thus we obtained a mapping

$$\Lambda : (\widetilde{M}/\sim) \times \widehat{X}_{M_1} \to \widetilde{M}_1.$$

Sometimes we shall write $(\pi, aA_{X_{M_1}(\pi)})$ or simply (π, a) instead of $((F^\times)^\sim\pi, aA_{X_{M_1}(\pi)})$ or $((F^\times)^\wedge\pi, aA_{X_{M_1}(\pi)})$. Now §2 implies:

3.1. THEOREM. *The map*

$$\Lambda : (\widetilde{M}/\sim) \times \widehat{X}_{M_1} \to \widetilde{M}_1$$

is a bijection. The restriction

$$\Lambda : (\widehat{M}/\widehat{\ }) \times \widehat{X}_{M_1} \to \widehat{M}_1$$

is a bijection. For $(\pi, \mathbf{a}) \in (\widehat{M}/\widehat{\ }) \times \widehat{X}_{M_1}$ *the following equivalences hold:*

$$\begin{aligned}
\Lambda((\pi, \mathbf{a})) \in C^u(M_1) &\Leftrightarrow \pi \in C^u(M), \\
\Lambda((\pi, \mathbf{a})) \in D^u(M_1) &\Leftrightarrow \pi \in D^u(M), \\
\Lambda((\pi, \mathbf{a})) \in T^u(M_1) &\Leftrightarrow \pi \in T^u(M).
\end{aligned}$$

In particular $(\mathrm{GL}(n, F)^{\sim}/\sim) \times \widehat{X}_{\mathrm{SL}(n,F)}$ *parametrizes* $\mathrm{SL}(n, F)^{\sim}$, $(\mathrm{GL}(n,F)^{\wedge}/^{\wedge}) \times \widehat{X}_{\mathrm{SL}(n,F)}$ *parametrizes* $\mathrm{SL}(n,F)^{\wedge}$, $(D^u(\mathrm{GL}(n,F))/^{\wedge}) \times \widehat{X}_{\mathrm{SL}(n,F)}$ *parametrizes* $D^u(\mathrm{SL}(n, F))$ *etc.*

Now we shall give a description of the unitary induction for $\mathrm{SL}(n, F)$. Recall that for $\pi \in \widehat{M}$, $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\pi)$ is irreducible by **[1]**. It is easy to see that $X_{M_1}(\pi) \subseteq X_{\mathrm{SL}(n,F)}(\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\pi))$ and thus $A_{X_{\mathrm{SL}(n,F)}}(\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\pi)) \subseteq A_{X_{M_1}(\pi)}$.

3.2. Proposition. *Let* $\pi_1 \in \widehat{M}_1$ *and* $\Lambda^{-1}(\pi_1) = (\pi, a_0 A_{X_{M_1}(\pi)})$. *The representation* $\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\pi_1)$ *is multiplicity free, its length is* $\mathrm{card}(X_{\mathrm{SL}(n,F)}(\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\pi))/X_{M_1}(\pi))$ *and the parameters of all irreducible factors are contained in* $\{\pi\} \times \widehat{X}_{\mathrm{SL}(n,F)}(\pi)$.

Proof. One needs only to find the length of $\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\pi_1)$. Set

$$\begin{aligned}
p &= \mathrm{card}\, X_{M_1}(\pi), \\
q &= \mathrm{card}(X_{\mathrm{SL}(n,F)}(\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\pi))/X_{M_1}(\pi)).
\end{aligned}$$

Let $\pi|M_1 = \pi_1 + \cdots + \pi_p$ be the decomposition into irreducible subrepresentations. Then

$$\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\pi)|\,\mathrm{SL}(n, F) \cong \mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\pi|M_1) \cong \bigoplus_{i=1}^{p} \mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\pi_i).$$

Now $A/A_{X_{M_1}(\pi)}$ acts simply transitive on the above decomposition. Thus all $\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\pi_i)$ are of the same length, say r. But the length of $\mathrm{Ind}_P^{\mathrm{GL}(n,F)}(\pi)|\,\mathrm{SL}(n, F)$ is pq and from the other side $pq = pr$. Thus $r = q$.

3.3. COROLLARY. *The representation* $\operatorname{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\pi_1)$ *is irreducible if and only if*

$$X_{\mathrm{SL}(n,F)}(\operatorname{Ind}_P^{\mathrm{GL}(n,F)}(\pi)) \subseteq X_{M_1}(\pi).$$

Note that the irreducibility of $\operatorname{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\Lambda((\pi, a)))$, with $\Lambda((\pi, a))$ unitary, depends only on π.

4. GL-parameters. We continue with the notation of the preceding paragraph.

In the last paragraph we defined a parametrization of $\widetilde{M_1}$ (in particular of $\mathrm{SL}(n, F)^{\sim}$) and some important subclasses, by parameters defined in terms of $\widetilde{M}$ (in particular of $\mathrm{GL}(n, F)^{\sim}$).

In this paragraph we shall describe further $(\widetilde{M}/\sim) \times \widehat{X}_{M_1}$, $(\widehat{M}/\widehat{\ }) \times \widehat{X}_{M_1}$, $(T^u/\widehat{\ }) \times \widehat{X}_{M_1}, \ldots$.

We shall fix an isomorphism of M onto $\mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_k, F)$ and identify these two groups. Now, there are natural bijections given by tensoring representations

$$\widetilde{M} \leftrightarrow \prod_{i=1}^{k} \mathrm{GL}(n_i, F)^{\sim},$$

$$\widehat{M} \leftrightarrow \prod_{i=1}^{k} \mathrm{GL}(n_i, F)^{\wedge},$$

$$C(M) \leftrightarrow \prod_{i=1}^{k} C(\mathrm{GL}(n_i, F)),$$

$$C^u(M) \leftrightarrow \prod_{i=1}^{k} C^u(\mathrm{GL}(n_i, F)),$$

$$D^u(M) \leftrightarrow \prod_{i=1}^{k} D^u(\mathrm{GL}(n_i, F)),$$

$$T^u(M) \leftrightarrow \prod_{i=1}^{k} T^u(\mathrm{GL}(n_i, F)),$$

$$T(M) \leftrightarrow \prod_{i=1}^{k} T(\mathrm{GL}(n_i, F)).$$

We shall identify M/M_1 with F^x and thus $(M/M_1)^{\sim}$ with $(F^x)^{\sim}$. Let

$$\pi = \pi_1 \otimes \cdots \otimes \pi_k \in \widetilde{M}.$$

For $\chi \in (F^{\times})^{\sim}$ we have

$$\chi\pi = (\chi\pi_1) \otimes \cdots \otimes (\chi\pi_k).$$

Thus

$$X_M(\pi) = X_{\mathrm{SL}(n_1, F)}(\pi_1) \cap \cdots \cap X_{\mathrm{SL}(n_k, F)}(\pi_k).$$

Up to now, we made a reduction of the parameters to the $\mathrm{GL}(n, F)$-case. Now we shall continue to describe the parameters in this situation.

For smooth representations τ_i of $\mathrm{GL}(n_i, F)$, $i = 1, 2$, we shall denote by $\tau_1 \times \tau_2$ a smooth representation of $\mathrm{GL}(n_1+n_2, F)$ parabolically induced by $\tau_1 \otimes \tau_2$ from a suitable standard parabolic subgroup (see **[25]**). If we have three representations, then $(\tau_1 \times \tau_2) \times \tau_3$ is naturally isomorphic to $\tau_1 \times (\tau_2 \times \tau_3)$. We denote by ν the character $|\det(\ )|_F$ where $|\ |_F$ is the modulus character of F. Set

$$\mathrm{Irr} = \bigcup_{n=0}^{\infty} \mathrm{GL}(n, F)^{\sim}, \qquad \mathrm{Irr}^u = \bigcup_{n=0}^{\infty} \mathrm{GL}(n, F)^{\wedge},$$

$$D = \bigcup_{n=1}^{\infty} D(\mathrm{GL}(n, F)), \qquad D^u = \bigcup_{n=1}^{\infty} D^u(\mathrm{GL}(n, F)),$$

$$C = \bigcup_{n=1}^{\infty} C(\mathrm{GL}(n, F)), \qquad C^u = \bigcup_{n=1}^{\infty} C^u(\mathrm{GL}(n, F))$$

$$T = \bigcup_{n=1}^{\infty} T(\mathrm{GL}(n, F)), \qquad T^u = \bigcup_{n=1}^{\infty} T^u(\mathrm{GL}(n, F)).$$

For a set Y, $M(Y)$ will denote the set of all finite multisets in Y. They are all finite unordered n-tuples, with any $n \in Z_+$. For $(y_1, \dots, y_n)$, $(y_1, \dots, y_m) \in M(Y)$ put

$$(y_1, \dots, y_n) + (y'_1, \dots, y'_m) = (y_1, \dots, y_n, y'_1, \dots, y'_m).$$

For any $\tau \in T$ there exist a unique $\tau^u \in T^u$ and $e(\tau) \in \mathbf{R}$ such that

$$\tau = \nu^{e(\tau)}\tau^u.$$

Clearly $\tau \in D \Leftrightarrow \tau^u \in D^u$.

For $t = (\tau_1, \dots, \tau_n) \in M(T)$ and $\chi \in (F^{\times})^{\sim}$ we define

$$\chi t = (\chi\tau_1, \dots, \chi\tau_n).$$

In this way one obtains an action of $(F^{\times})^{\sim}$ on $M(T)$. The stabilizer of t will be denoted by $X(t)$.

Let $d = (\delta_1, \dots, \delta_n) \in M(D)$. We can choose a numeration of d such that $e(\delta_1) \geq \cdots \geq e(\delta_n)$. The representation $\delta_1 \times \cdots \times \delta_n$ has a unique irreducible quotient which depends only on d and which will be denoted by $L(d)$. Now $d \mapsto L(d)$ is a Langlands-type parametrization of Irr by $M(D)$ (see for example **[19]**). One has

$$\chi L(d) = L(\chi d)$$

for $\chi \in (F^x)^\sim$. Thus $X_{\mathrm{SL}(n,F)}(L(d)) = X(d)$.

For $\rho \in C$ and $n \in \mathbf{N}$ the representation $\nu^{(\frac{n-1}{2})}\delta \times \nu^{(\frac{n-1}{2})-1}\delta \times \cdots \times \nu^{-(\frac{n-1}{2})}\delta$ has a unique essentially square integrable subquotient which will be denoted by $\delta(\rho, n)$. Now $(\rho, n) \mapsto \delta(\rho, n)$ is a parametrization of D (resp. D^u) by $C \times \mathbf{N}$ (resp. $C^u \times \mathbf{N}$). Similarly as above

$$\chi\delta(\rho, n) = \delta(\chi\rho, n)$$

(see **[25]**).

The mapping

$$(M(D^u)\backslash\{\varnothing\}) \ni (\tau_1, \dots, \tau_n) \mapsto \tau_1 \times \cdots \times \tau_n \in T^u$$

is a parametrization of T^u by $M(D^u)\backslash\{\varnothing\}$ (see **[22]** and **[25]**).

For $\delta \in D^u$ set

$$u(\delta, n) = L((\nu^{\frac{n-1}{2}}\delta, \nu^{\frac{n-1}{2}-1}\delta, \dots, \nu^{-\frac{n-1}{2}}\delta)),$$
$$\pi(u(\delta, n), \alpha) = \nu^\alpha u(\delta, n) \times \nu^{-\alpha} u(\delta, n)$$

where $0 < \alpha < 1/2$. Set

$$B = \{u(\delta, n), \pi(u(\delta, n), \alpha); \delta \in D^u, n \in \mathbf{N}, 0 < \alpha < 1/2\}.$$

Then by **[24]**

$$M(B) \ni (\pi_1, \dots, \pi_n) \mapsto \pi_1 \times \cdots \times \pi_n \in \mathrm{Irr}^u$$

is a parametrization of Irr^u by $M(B)$. Again

$$\chi u(\delta, n) = u(\chi\delta, n), \qquad \chi \in (F^x)^\wedge$$

and

$$\chi\pi(u(\delta, n), \alpha) = \pi(u(\chi\delta, n), \alpha), \qquad \chi \in (F^x)^\wedge.$$

According to formulas

$$\chi L(d) = L(\chi d),$$
$$\chi\delta(\rho, n) = \delta(\chi\rho, n)$$

and previous observations, we have a reduction of parameters $(\widetilde{M}/\sim)\times\widehat{X}_{M_1}$ to computing of $X_{\mathrm{SL}(n,F)}(\rho)$ for $\rho\in C$.

4.2. REMARK. Consider an essentially square integrable representation δ of $\mathrm{GL}(m, F)$. Let $P = MN$ be the minimal parabolic subgroup among those for which the Jacquet module of δ for P is non-trivial (it is the parabolic subgroup from which is induced $\nu^{\frac{n-1}{2}}\rho\times\cdots\times\nu^{-\frac{n-1}{2}}\rho$ if $\delta=\delta(\rho, n)$). This parabolic subgroup is homogeneous and the Jacquet module is cuspidal and irreducible. Take $a\in\widehat{X}_{\mathrm{SL}(n,F)}(\delta)$. Then $\Lambda((\delta, a))$ is an essentially square integrable representation and all such representations are obtained in this way. Set $M_1 = \mathrm{SL}(n, F)\cap M$ and $P_1 = M_1N$. It is easy to see that the Jacquet module of $\Lambda((\delta, a))$ for P_1 is irreducible and cuspidal.

We can express now the irreducibility condition of Corollary 3.3 for unitary parabolic induction more explicitly:

4.2. THEOREM. *Let $P=MN$ be a parabolic subgroup of $\mathrm{GL}(n, F)$, $M_1 = M\cap\mathrm{SL}(n, F)$ and $P_1 = M_1N$. Let $\pi_1 = \Lambda((\pi, a))$ be an irreducible unitary representation of M_1. We may suppose $M\cong\mathrm{GL}(n_1, F)\times\cdots\times\mathrm{GL}(n_k, F)$. Let $\pi=\pi^1\times\cdots\times\pi^k$ and $\pi^i = L(d_i)$, $d_i\in L(D)$. Then $\mathrm{Ind}_{P_1}^{\mathrm{SL}(n,F)}(\pi_1)$ is irreducible if and only if*

$$X(d_1+\cdots+d_k)d_i\subseteq d_i \quad \textit{for each } i=1,\ldots,k,$$

or equivalently $X(d_1+\cdots+d_k)\subseteq X(d_i)$, $i=1,\ldots,k$.

4.3. REMARK. We have reduced the parameters to the computation of $X_{\mathrm{SL}(n,F)}(\rho)$ for ρ cuspidal (equivalently to $X_{\mathrm{SL}(n,F)}(\rho)^\frown$ or $A_{X_{\mathrm{SL}(n,F)}(\rho)}$ or $F^x_{X_{\mathrm{SL}(n,F)}(\rho)}$). The following step would be to express (some of) these groups in terms of a parametrization of C. R. Howe constructed in **[9]** cuspidal representations in the tame case. H. Carayol in **[4]** classified the cuspidal representations in the prime case. A great number of informations on the above groups in these two cases can be found in papers **[13]** by P. Kutzko and P. Sally and **[17]** of A. Moy and P. Sally. Let us illustrate this by an example. Suppose that we are in the tame case. Then the cuspidal representations of $\mathrm{GL}(n, F)$ are parametrized by admissible characters of the multiplicative groups of n-dimensional extensions E of F, modulo conjugacy. In **[17]** A. Moy and P. Sally showed that in two of the three possible cases the answer is particularly nice:

$$F^x_{X_{\mathrm{SL}(n,F)}(\pi)} = N_{E/F}(E^x)$$

where $N_{E/F} : E^x \to F^x$ denotes the norm map (char $F = 0$). For details one should consult **[17]**.

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