PACIFIC JOURNAL OF MATHEMATICS
Vol. 155, No. 1, 1992

QUADRATIC CENTRAL POLYNOMIALS WITH DERIVATION AND INVOLUTION

CHARLES LANSKI

The main result of this paper shows that if R is a prime ring with involution and with derivation D, then if $p(x, y) = c_1xy^D + c_2x^Dy + c_3yx^D + c_4y^Dx$ is central for all (skew-) symmetric elements of R, then R must embed in $M_2(F)$, with two explicit exceptions. As a consequence of the special case when $x = y$, one obtains generalizations of existing results about (skew-) centralizing derivations of the (skew-) symmetric elements.

Introduction. The motivation for this paper lies in an attempt to classify the minimal homogeneous identities with derivation which hold for the (skew-) symmetric elements in an ideal of a prime ring R with involution. As a consequence of **[11]**, there are two specific types of such identities $f(x^d, y^h)$ of degree two if R does not satisfy a polynomial identity and $\operatorname{char} R \neq 2$, and no such identity can be of the form $f(x, y^h)$. The situation when either $\operatorname{char} R = 2$ or R is a PI ring, and about other degree two homogeneous identities not of these forms, remains to be studied. In this paper we investigate identities of the form $p(x, y) = c_1xy^D + c_2x^Dy + c_3yx^D + c_4y^Dx$, and more generally show that $p(x, y)$ cannot be a central polynomial for the (skew-) symmetric elements except in two specific cases, or when R embeds in $M_2(F)$. The results in **[11]** are not applicable here since now the same variable appears both with and without a derivation applied.

Throughout the paper, R will denote a prime ring with center Z, extended centroid C, and Martindale quotient ring Q **[15]**. Henceforth, we shall assume that R has an involution, *, and for any ideal I of R we set $T(I) = \{r + r^* | r \in I\}$, $S(I) = \{r \in I | r^* = r\}$, and $K(I) = \{r - r^* | r \in I\}$. It is easy to to show that * extends to C **[16]**. We say that * is of the first kind if $C = C \cap S$, and is of the second kind otherwise. In general, the latter case is easy to deal with. For D a nonzero derivation of R, it is easy to check that D extends uniquely to a derivation of Q, so restricts to a derivation of the central closure $RC + C$ of R (see **[8]**). We say that D is inner if its extension to Q is the inner derivation $\operatorname{ad}(A)(x) = xA - Ax$, for $A \in Q$, and otherwise call D outer.

Although our main result is for a two variable quadratic expression $p(x, y)$ as above, we actually obtain a more general theorem for central quadratic identities in n variables, with a single derivation applied. Specifically, consider the expression

$$p(x_1, \dots, x_n) = \sum_{i,j=1}^{n} a_{ij}x_i^{D_{ij}}x_j + b_{ij}x_ix_j^{D_{ij}}$$

for a_{ij}, $b_{ij} \in C$, and D_{ij} a derivation of R for each $\{i, j\}$. We call $p(x_1, \dots, x_n)$ a *quadratic central* (*skew-*) *trace identity* for a nonzero ideal I of R, if $p(r_1, \dots, r_n) \in C$ for every $(r_1, \dots, r_n) \in W_1(I) \times \cdots \times W_n(I)$, where $W_i(I) = T(I)$ or $W_i(I) = K(I)$. Any such identity decomposes into a sum of similar ones, each homogeneous in its variables. For example, taking $x_i \neq x_j$ and $x_k = 0$ for $k \neq i, j$ yields the central identity $a_{ij}x_i^Dx_j + b_{ij}x_ix_j^D + a_{ji}x_j^Dx_i + b_{ji}x_jx_i^D$ on $W_i(I) \times W_j(I)$, where $D = D_{ij}$. Similarly setting $x_j = 0$ for all $j \neq i$ gives the central identity $a_{ii}x_i^Dx_i + b_{ii}x_ix_i^D$ on $W_i(I)$. Consequently, it will suffice to consider these two special cases.

The special case when $p(x) = cxx^D + zx^Dx$ is central generalizes the notion of centralizing and skew-centralizing derivations. A derivation D is centralizing when $xx^D - x^Dx \in C$ and is skew-centralizing when $xx^D + x^Dx \in C$. A result of Posner **[17]** shows that if D is centralizing for all $x \in R$, then R is commutative, and by **[14]**, if x is restricted to be (skew-) symmetric, then R satisfies the standard identity S_4 when $\operatorname{char} R \neq 2$. When D is skew-centralizing on R, then again R is commutative **[4]**, and R satisfies S_4 if D is skew-centralizing for all (skew-) symmetric elements **[14]**. An extension in the centralizing case was obtained in **[10]** by restricting x to be a (skew-) symmetric element in an ideal of R. These will all clearly be included in our result here as the special case when $c^2 = z^2$ in $p(x)$, and both x and y are (skew-) symmetric.

The proofs of our results will use the theory of generalized *-differential identities **[8]** to show that D must be inner, and to reduce to matrices. The value of this reduction is to make the required computations reasonably straightforward in nature, although they remain considerable. In addition, we need the noninvolution versions of the theorems from **[12]**.

Preliminaries. Before getting to our main results, it will be useful to make a few preliminary remarks concerning rings with involution. These are implicit in the standard literature, but we state them as lemmas for convenience of reference. Recall that R satisfies the standard identity S_4 exactly when R embeds in $M_2(F)$, for F a field.

LEMMA 1. *Let A be the additive subgroup of R generated by $\{t^2 | t \in T\}$, and let B be the subgroup generated by $\{k^2 | k \in K\}$. If $A \subset Z$ or if $B \subset Z$, then R satisfies S_4. When R is a simple ring, either A or B generates R as a ring unless either* char $R \neq 2$ *and* $\dim_Z R \leq 4$, *or* char $R = 2$ *and* $\dim_Z R \leq 36$.

Proof. If either A or B is central, then R satisfies S_6 by [**1**; Theorem 1, p. 63]. Thus $RC = RZ^{-1} = R(Z \cap S)^{-1}$ is a simple ring satisfying S_6 [**5**; Theorem 2, p. 57] with its A, or B, still central, so we can replace R with RC. Should R not satisfy S_4, then $R = M_3(Z)$ or R is a division algebra with $\dim_Z R = 9$. But in the first case, neither the square of $e_{12} + e_{12}^*$ nor of $e_{12} - e_{12}^*$ is central, and in the second there are no extension fields of degree two over Z. Consequently, R must satisfy S_4.

Suppose that R is a simple ring. If char $R \neq 2$, then unless $\dim_Z R \leq 4$, by using results in [**2**; Chapter 1 and Chapter 2] it follows that $A = T$ since A is a Jordan ideal of T, that $T \subset B$, and that T generates R as a ring. When char $R = 2$, $T = K$, and linearizing shows that $[T, T] \subset A$. Now unless $\dim_Z R \leq 36$, then $[T, T]$ generates R [**13**; Theorem 25, p. 129].

The next lemma will serve to eliminate the case of involutions of the second kind. The result is essentially [**18**; Theorem 7, p. 473], but since it is easy, crucial in what follows, and is required for ideals, we provide an argument.

LEMMA 2. *Assume that * is of the second kind, and let $p(x_1, \ldots, x_n) \in Q * C\{x_i\}$, the free product over C of Q and the free algebra over C, be multilinear and homogeneous. If $p(t_1, \ldots, t_n) = 0$ for all $t_i \in W_i(I)$, where $W_i(I) = T(I)$ or $W_i(I) = K(I)$ for I a nonzero ideal of R, then there is a nonzero ideal J of R so that $p(x_1, \ldots, x_n)$ is an identity on J.*

Proof. Let $c \in C$ satisfy $c^* \neq c$, and let J be a nonzero ideal of R so that $J + cJ + c^*J \subset I$ [**15**]. Thus, for $y \in J$, one may write $(c - c^*)y = (cy + c^*y^*) - c^*(y + y^*)$, and $(c - c^*)y = (cy - c^*y^*) - c^*(y - y^*)$, so it follows that $J \in CT(I) \cap CK(I)$. Since $p(x_1, \ldots, x_n)$ is multilinear and homogeneous, if the substitution of t_i for x_i gives zero, then so does the substitution of $c_i t_i$ for x_i. Consequently, $p(x_1, \ldots, x_n)$ is an identity for J.

As we mentioned, the approach we take will reduce to the situation where R is a matrix ring over a field. Our next lemma will provide a useful computation in this case, but before stating it, we recall some important facts about involutions, and establish some notation. Suppose that $R = M_n(C)$ with involution * of the first kind, so * is the identity on C. It is well known that * on R is either of transpose type or is symplectic. In the first case, * is matrix transpose followed by conjugation by an invertible diagonal matrix. In particular, $(e_{ii})^* = e_{ii}$, and for $i \neq j$, $e_{ij} + ae_{ji} \in T$ and $e_{ij} - ae_{ji} \in K$ for a suitable choice of $a = a(i, j) \in C$. When * is symplectic, then $n = 2m$, and if $B \in M_{2m}(C)$ is written as $B = (B_{ij})$ for $B_{ij} \in M_2(C)$, then $B^* = (H_{ij})$ where $H_{ij} = \operatorname{Adj}(B_{ji})$, the classical adjoint. We may also write $B = \sum B_{ij}E_{ij}$, where E_{ij} is the identity of $M_2(C)$ in the (i, j) position of $M_m(M_2(C))$. The context will make clear whether A_{ij} refers to the (i, j) entry of $A \in M_n(C)$, or to the (i, j) 2×2 block of $A = \sum A_{ij}E_{ij} \in M_{2m}(C)$. In the case of either involution we will often write $t_{ij} = e_{ij} + (e_{ij})^*$ and $v_{ij} = e_{ij} - (e_{ij})^*$.

We come now to our last lemma about involutions.

LEMMA 3. *Let* $R = M_n(C)$ *for* $n > 2$, *have an involution * of the first kind. If* W *is any of* $[T, T]$, $[T, K]$, *or* $[K, K]$, *then* W *generates* R *as a ring, and* $[W, W]$ *also generates* R *unless* char $R = 2$,* *is symplectic, and* $n = 4$.

Proof. As above, set $t_{ij} = e_{ij} + (e_{ij})^*$ and $v_{ij} = e_{ij} - (e_{ij})^*$. If * is of transpose type, then it is easy to see that $[T, K] = \sum Ct_{ij}$ for $i \neq j$, and that the other sets contain $\sum Cv_{ij}$. Thus for any W, $[W, W] = \sum Cv_{ij}$, and so clearly generates R. When * is symplectic, by using $t_{ii} \neq 0$ and $v_{ii} \neq 0$, it is easy to see that for $\{i, j\} \in G = \{\{i, j\} | \{i, j\} \neq \{2k-1, 2k\}\}$, all Ct_{ij} are contained in $[T, K]$, and that the other sets contain all Cv_{ij}. It is straightforward to see that $\{Ct_{ij} | \{i, j\} \in G\}$ and $\{Cv_{ij} | \{i, j\} \in G\}$ each generate R, proving that W does as well. Furthermore, if $\{i, j\}, \{j, k\} \in G$ for i, j, k distinct, then $[v_{ij}, v_{jk}] = [t_{ij}, t_{jk}] = v_{ik}$, and it follows that when $n > 4$, $[W, W]$ generates R. Finally, if $n = 4$ and char $R \neq 2$, then one checks that in all cases $[W, W] = K$, so generates R.

The reduction to matrices. We discuss in some detail how one uses the theory from **[8]** to reduce considerations about identities with derivation to matrix rings. The first main step is to show that the derivation D is inner, and is accomplished by citing [**8**; Theorem 7, p. 783]. For the convenience of the reader, we give a very special statement of the result in the form required here. Consider the $4n$ noncommuting indeterminates $x_1, \ldots, x_n, x_1^*, \ldots, x_n^*, x_1^D, \ldots, (x_n^*)^D$ over C, for D a derivation of R. A nonzero polynomial $h(x_1, \ldots, x_n)$ over C in these $4n$ indeterminates is called a G^*-DI for a nonzero ideal I of R if $h(r_1, \ldots, r_n) = 0$ for all $r_i \in I$. Call $h(x_1, \ldots, x_n)$ multilinear, if it is both homogeneous and multilinear in its subscripts, or equivalently, if each of its monomials contains exactly one of x_i, x_i^*, x_i^D, or $(x_i^*)^D$ for each $i \leq n$. In this case, each monomial M of $h(x_1, \ldots, x_n)$ has an exponent sequence $(E_1, \ldots, E_n)$, where $E_i = D$ if x_i^D, or $(x_i^*)^D$ appears in M and $E_i = 1$ if x_i or x_i^* appears in M. The sum of all monomials of $h(x_1, \ldots, x_n)$ having this same exponent sequence, but with all occurrences of D deleted, is denoted $h_{(E_1, \ldots, E_n)}$; that is, $h_{(E_1, \ldots, E_n)}$ is the sum of all monomials with exponent sequence $(E_1, \ldots, E_n)$, but with any x_i^D replaced with x_i and any $(x_i^*)^D$ replaced with x_i^*. We now state the form of [**8**; Theorem 7, p. 783] which we require.

THEOREM A. *Let D be an outer derivation of R and $h(x_1, \ldots, x_n)$ a nonzero multilinear G^*-DI for an ideal I of R. If $E = (E_1, \ldots, E_n)$ is an exponent sequence for any monomial of $h(x_1, \ldots, x_n)$ containing a maximal number of occurrences of D among all such sequences, then h_E is a nonzero *-polynomial identity for R.*

Applying Theorem A will tell us that D is inner or that R satisfies a certain *-PI. Our next lemma shows that the particular *-PI which will arise, forces R to satisfy S_4.

LEMMA 4. *If $cxy + zyx \in C$ for all $(x, y) \in W_1 \times W_2$, where $c, z \in C$ and each W_i is either $T(R)$ or $K(R)$, then $c = z = 0$ or R satisfies S_4.*

Proof. Assume that $c \neq 0$ or $z \neq 0$, and observe that $f(x, y, w) = [cxy + zyx, w]$ is a *-identity for R, so by Amitsur's result [**1**; Theorem 1, p. 63], R must satisfy S_6. If * is of the second kind, then Lemma 2 implies that $f(x, y, w)$ is an identity for any ideal of R. But R is a prime ring, so the ideal is as well, and is commutative since $f(x, y, w)$ is of degree three. Thus R is commutative, proving the lemma. We may now assume that * is of the first kind, and since

$f(x, y, w)$ is multilinear, it remains an identity on the appropriate subsets of $RC = RZ^{-1}$, and of $RC \otimes F = M_n(F)$ for F a splitting field of RC. As we have seen, R satisfies S_6, so $M_n(F)$ does as well, forcing $n \leq 3$. Consequently, we are finished unless $n = 3$.

We may now assume that $R = M_3(C)$ and * is of the first kind, so * must be of transpose type because R is of odd degree over C. If $W_1 = T$ and $W_2 = K$, take $x = e_{12} + ae_{21}$, $y = e_{23} - be_{32}$, and $w = e_{33}$. Then $f(x, y, w) = 0$ yields $ce_{13} + zabe_{31} = 0$, contradicting $c \neq 0$ or $z \neq 0$. When $W_1 = K$ and $W_2 = T$, interchanging the choices of x and y gives the same contradiction. Hence, we may assume that $W_1 = W_2$. If $c + z \neq 0$, then setting $x = y$ yields $(c+z)x^2 \in C$, so $x^2 \in C$ for all $x \in T$, or all $x \in K$, and applying Lemma 1 contradicts $R = M_3(C)$. Finally, should $c + z = 0$ we have $xy - yx \in C$ for all x, $y \in T$, or all x, $y \in K$, contradicting Lemma 3. Therefore, either $c = z = 0$, or R satisfies S_4, proving the lemma.

We can now see how to use Theorem A and Lemma 4 to make our first major reduction: the derivations appearing in a quadratic central (skew-) trace identity must be inner.

LEMMA 5. *For D a derivation of R and $c_i \in C$, let $p(x, y) = c_1xy^D + c_2x^Dy + c_3yx^D + c_3y^Dx$, and if I is a nonzero ideal of R set $W_i = T(I)$ or $W_i = K(I)$ for each of $i = 1, 2$. Then $p(x, y) \in C$ for all $(x, y) \in W_1 \times W_2$ implies that either $D = d(A)$ for $A \in Q$, R satisfies S_4, or all $c_i = 0$.*

Proof. The expression $f(x, y, x_3) = [p(x, y), x_3]$ is an identity on $W_1 \times W_2 \times R$ and gives rise to a multilinear $G^* - DI$ for I. Specifically, in $f(x, y, x_3)$, if $W_1 = T(I)$ replace x with $x_1 + x_1^*$ and x^D with $x_1^D + (x_1^*)^D$, and if $W_1 = K(I)$ replace x with $x_1 - x_1^*$ and x^D with $x_1^D - (x_1^*)^D$. Make corresponding substitutions for y and y^D using x_2, x_2^*, x_2^D, and $(x_2^*)^D$. It is clear that $f(x_1, x_2, x_3)$ is a multilinear $G^* - DI$, and that each of its monomials contains exactly one indeterminate with exponent D. Assume that D is an outer derivation and apply Theorem A to the exponent sequences $(D, 1, 1)$ and $(1, D, 1)$ to obtain the *-PI's $f_{(D,1,1)}$ and $f_{(1,D,1)}$ for R. One of these *-PI's is not zero if some $c_i \neq 0$. In either case we get a nonzero identity $[cXY + zYX, x_3]$ for $(X, Y, x_3) \in W_1(R) \times W_2(R) \times R$, so $cXY + zYX \in C$ on $W_1 \times W_2$, which forces R to satisfy S_4 by Lemma 4.

In our main results, we apply Lemma 5 to conclude that D is inner, and so our quadratic central identity will be a *-generalized polynomial identity on I. This fact will enable us to assume that $I = R = \mathrm{Soc}(R)$ is a simple ring with $eRe = eC$ for any primitive idempotent $e \in R$. Our last lemma shows that in this situation it suffices to assume that $R = M_n(C)$. The proof was suggested by the referee and replaces arguments originally given in each of our first three theorems to follow.

LEMMA 6. *Let* $q(x) = cxx^D + zx^Dx$ *and* $p(x, y) = c_1xy^D + c_2x^Dy + c_3yx^D + c_4y^Dx$ *be nonzero polynomials over* C. *Assume that whenever* $q(x)$ *or* $p(x, y)$ *is a central (skew-) trace identity on* $M_n(C)$ *with involution of the first kind,* $n > 4$ *and* D *an inner derivation of* $M_n(C)$, *then* $D = 0$. *Let* $R = \mathrm{Soc}(R)$ *have involution of the first kind,* $\dim_C R$ *infinite, and* $eRe = Ce$ *for each primitive idempotent* $e \in R$. *If* $q(x)$ *or* $p(x, y)$ *is a central (skew-) trace identity for* R, *and if* $D = \mathrm{ad}\, A$ *for* $A \in Q$, *then* $D = 0$.

Proof. The assumptions on R imply that any $r \in R$ has finite rank, that $fRf \cong M_n(C)$ for any idempotent $f \in R$ of rank n, and since * is of the first kind, that for any $r_1, \dots, r_k \in R$ there is a symmetric idempotent $e \in R$ of arbitrary large rank satisfying $r_1, \dots, r_k \in eRe$ [**6**; Theorem 4, p. 89]. Also, it is an easy consequence of the definition of Q that for $e \in R$ an idempotent, $eQe = Q(eRe) = eRe$.

It will be clear that our argument is the same for $q(x)$ or for $p(x, y)$, so for simplicity we assume that $q(x)$ is a central identity for $x \in W(R)$, where $W = T$ or $W = K$. Using $D = \mathrm{ad}(A)$, write $q(x) = q(x, A)$. Let $e \in R$ be any symmetric idempotent with rank $e > 4$, so that $eRe = M_n(C)$ and $n > 4$. Now for $x \in W(eRe)$, $q(x, eAe) = eq(x, A)e \in eCe$, so $q_1(x) = cxx^E + cx^Ex$ is central on $W(eRe)$ where $E = \mathrm{ad}(eAe)$. By assumption, $E = 0$, forcing $eAe = z_ee$ for $z_e \in C$. Similarly, if $f \in R$ is a symmetric idempotent with rank $f > 4$, then $fAf = z_ff$ for $z_f \in C$. As we observed above, there is a symmetric idempotent $g \in R$ with e, $f \in gRg$, and again $gAg = z_gg$ for $z_g \in C$. But $z_ge = z_geg = e(z_gg)e = egAge = eAe = z_ee$, so $z_g = z_e$, and similarly $z_g = z_f$ showing that $eAe = z_Ae$ for $z_A \in C$ independent of $e \in R$ with rank $e > 4$. To see that this implies that $A \in C$, let $r \in R$ and choose a symmetric idempotent $e \in R$ with rank $e > 4$ so that r, rA, $Ar \in eRe$. We note that since $D = \mathrm{ad}(A)$ and $R = \mathrm{Soc}(R)$, it is indeed the case that

rA, $Ar \in R$ [**8**; p. 766 and Lemma 7, p. 779]. But now,

$$Ar = eAr = eAer = z_A er = z_A r = rz_A$$
$$= rez_A = reAe = rAe = rA.$$

Therefore, A centralizes R, forcing $A \in C$, and so, $D = 0$.

Main results. We begin this section by considering a quadratic central identity in one variable. It is a *-version of a generalization of Posner's theorem [**17**; Theorem, p. 1097], of [**14**], and of [**10**; Theorem 3, p. 284]. One must expect the exception in the next theorem that R can satisfy S_4, since counterexamples exist even for centralizing derivations on T or K [**10**; p. 283]. Also, our arguments rely on the noninvolution version of our theorems from [**12**]. For convenience, we state next [**12**; Theorem 3] for ideals in the form we need.

THEOREM B. *If I is a nonzero ideal of R, D is a derivation of R, and $p(x, y) = c_1 xy^D + c_2 x^D y + c_3 yx^D + c_4 y^D x \in C$ for all x, $y \in I$, where $c_i \in C$, then either $D = 0$, all $c_i = 0$, or R satisfies S_4.*

THEOREM 1. *Let R be a prime ring, I a nonzero ideal of R, and D a nonzero derivation of R. If c, $z \in C$ so that $p(x) = cxx^D + zx^D x \in C$ for all $x \in T(I)$, for all $x \in K(I)$, then either $c = z = 0$, or R satisfies S_4.*

Proof. Assume throughout that R does not satisfy S_4, and that not both c and z are zero. Since both $T(I)$ and $K(I)$ are additive, we can linearize $p(x)$ to $g(x, y) = c(xy^D + yx^D) + z(x^D y + y^D x)$, and apply Lemma 5 to see that $D = \mathrm{ad}(A)$ for $A \in Q$.

Observe that $f(x, y, w) = [g(x, y), w]$ is now a nonzero multilinear GPI on $T(I)$, or on $K(I)$. If * is of the second kind, then by Lemma 2, $f(x, y, w)$ is an identity for a nonzero ideal J of R, and so $g(x, y) \in C$ for all x, $y \in J$. Applying Theorem B gives the contradiction that R satisfies S_4. Hence, $a = a^*$ for all $a \in C$, so * naturally extends to $RC \otimes F$ for any extension field of C. We use this observation to reduce to matrices over a field.

The existence of $f(x, y, w)$ means that R satisfies a GPI [**8**; Theorem 7, p. 783], so by Martindale's theorem [**15**; Theorem 3, p. 579] $H = \mathrm{Soc}(RC) \neq 0$. For each $h \in H$ there is $a \in C$ with $ah \in I$ [**9**; Theorem 3, p. 245], so if $t = h + h^* \in T(H)$, then $at = ah + ah^* = ah + (ah)^* \in T(I)$. Thus, $T(H) \subset CT(I)$, and similarly, $K(H) \subset CK(I)$. Now $T(H \otimes F) = T(H) \otimes F$, from which we may conclude that $g(x, y) \in F$ for x, $y \in T(H \otimes F)$ (or $K(H \otimes F)$). Because $p(x)$ is a homogeneous quadratic expression in x, it is straightforward to see that $p(x) \in F$ for $x \in T(H \otimes F)$ (or $K(H \otimes F)$). Note that D is naturally a derivation of $H \otimes F$ by [**8**; p. 766 and Lemma 7, p. 779], since $HA + AH \subset H$. Also, if we choose F to be an algebraic closure of C, then for any primitive idempotent $e \in H \otimes F$, $e(H \otimes F)e = eF$. Our argument so far shows that we may reduce to the case that $I = R = \mathrm{Soc}(R)$ is a simple ring with $eRe = eC$ for any primitive idempotent $e \in R$, and $a = a^*$ for all $a \in C$. It follows from Lemma 6 that to finish the proof it suffices to assume that $R = M_n(C)$.

Since * is the identity on C, as we indicated before, * on R is either of transpose type or is symplectic. Assume the former and recall that for $i \neq j$, $e_{ij} + ae_{ji}$ is in either T or K for a suitable choice of $a \in C$. Now $D = \mathrm{ad}(A)$, so we have

$$p(x) = cx^2A + (z - c)xAx - zAx^2. \tag{1}$$

The off-diagonal entries of each evaluation of $p(x)$ must be zero, so for i, j, and k distinct, the (i, k) and (k, i) entries of $p(e_{ij} + ae_{ji})$ are zero. These are cA_{ik} and zA_{ki}, and $c \neq 0$ or $z \neq 0$, so $A_{ik} = 0$ whenever $i \neq k$. If $c = z$, then $c^{-1}p(x) = x^2A - Ax^2$, so letting $x = e_{ij} + ae_{ji} + e_{ik} + be_{ki}$, and computing the (k, j) entry of $c^{-1}p(x)$ yields $A_{jj} = A_{kk}$, since $A_{kj} = 0$. This shows that $A \in C$ and gives the contradiction $D = 0$. If $c \neq z$, then using the fact that A is diagonal shows that $p(e_{ij} + ae_{ji}) = a(c - z)(A_{ii} - A_{jj})(e_{ii} - e_{jj}) \in C$. But $n > 2$, so it follows again that $A_{ii} = A_{jj}$, and $D = 0$ results.

Next, we may assume that * is symplectic. Then $n = 2m$, and each $B \in M_{2m}(C)$ may be written in 2×2 block form. We proceed much as in the case of transpose type, assuming that $p(x) \in C$ for $x \in T$. The case $x \in K$ has an argument which is virtually the same, only requiring some changes from "+" to "−". Begin by setting $x = E_{ii} = e_{2i-12i-1} + e_{2i2i}$, and use $p(x)E_{kk} = E_{kk}p(x) = 0$ for $k > 2$ to see that in block form, A has nonzero entries only in its 2×2 diagonal blocks.

Suppose first that $c = z$, so that $p(x) = x^2A - Ax^2$. The off-diagonal blocks of $p(x)$ are zero, so considering the $(3, 1)$ entry of $p(e_{11} + e_{22} + e_{14} - e_{32})$ shows $A_{21} = 0$, since we know that $A_{31} = 0$, where the subscripts refer to the entries of A and not its 2×2 blocks. We also get $A_{12} = 0$ by looking at the $(3, 2)$ entry of $p(e_{11} + e_{22} + e_{24} + e_{31})$. Interchanging 1 with 3 and 2 with 4 in these computations shows that $A_{43} = A_{34} = 0$, and similar computation when $n > 4$ will prove that A is a diagonal matrix. But now, since $p(x)$ is commutation of x^2 with a diagonal matrix, if x^2 has a nonzero entry in the (i, j) position, then $A_{ii} = A_{jj}$. Using $x = t + t^*$ for $t = e_{11} + e_{13} + e_{14} + e_{23}$ gives $A_{11} = A_{22} = A_{33} = A_{44}$, and if $n > 4$ we can similarly show that A is scalar, so $D = 0$.

We may now assume that $c \neq z$. Note that if $x^2 = 0$, then $p(x) = (z - c)xAx \in C$. For $x = e_{13} + e_{42}$, one obtains $A_{34} = A_{21} = 0$, and for $x = e_{24} + e_{31}$ we get $A_{12} = A_{43} = 0$. Thus, in its first two 2×2 blocks, A is diagonal. Now $x = e_{13} + e_{14} + e_{42} - e_{32}$, yields $A_{33} = A_{44}$, and $p(e_{31}+e_{32}+e_{24}-e_{14}) \in C$ gives $A_{11} = A_{22}$. Next, take $x = e_{11}+e_{22}+e_{13}+e_{42}$ and use $p(x)$ as given in (1). Since $p(x) \in C$, its $(1, 3)$ entry is zero, resulting in $cA_{11} = cA_{33}$, and considering the $(3, 1)$ entry of $p(e_{11}+e_{22}+e_{31}+e_{24})$ shows that $zA_{11} = zA_{33}$. Since not both c and z are zero, we have $A_{11} = A_{33}$, and A is scalar in its first two 2×2 blocks. When $n > 4$, the computations above with all subscripts increased by two, repeated as necessary, will show that A is scalar, finishing the proof with the contradiction that $D = 0$.

Our next result considers the two variable identity $p(x, y) = c_1xy^D + c_2x^Dy + c_3yx^D + c_3y^Dx$. It is a linearized version of Theorem 1, and shows that such a $p(x, y)$ can be central for all x, $y \in T(I)$, or in $K(I)$, only in certain special cases. It turns out that this result is needed to handle the mixed case when one variable is replaced by an element of $T(I)$, and the other by an element of $K(I)$. As expected from Theorem 1, if R satisfies S_4, then $p(x, y)$ can be central. Interestingly, there are two other exceptions in the linear case which we now indicate.

EXAMPLE 1. Let C be a field with char $C = 2$, and let $R = M_4(C)$ have symplectic involution. Then $[[T, T], [T, T]] \subset C$, so for x, $y \in T$ and $D = \mathrm{ad}(A)$ for $A \in [T, T]$, $[x, y]^D = xy^D + x^Dy + yx^D + y^Dx \in C$.

EXAMPLE 2. Let C be a field with char $C = 3$, and let $R = M_3(C)$ with transpose as involution. If $A \in R$ is a symmetric matrix whose trace is zero, then by a tedious but straightfoward computation, $xy^D - yx^D - x^Dy + y^Dx = 0$ for all x, $y \in K$.

THEOREM 2. *Let R be a prime ring, I a nonzero ideal of R, and D a nonzero derivation of R. If $p(x, y) = c_1xy^D + c_2x^Dy + c_3yx^D + c_4y^Dx \in C$ for all x, $y \in T(I)$, or all x, $y \in K(I)$, where $c_i \in C$,* *then if F is the algebraic closure of C, one of the following holds:*

(i) *all* $c_i = 0$;

(ii) R *satisfies* S_4;

(iii) char $R = 2$, R *embeds in* $M_4(F)$ *with symplectic involution, and* $D = \mathrm{ad}(A)$ *for* $A \in [T(RC), T(RC)]$; *or*

(iv) char $R = 3$, $c_1 = -c_2 = -c_3 = c_4$, R *embeds in* $M_3(F)$, $p(x, y) = 0$ *for all* x, $y \in K(M_3(F))$, *and on* $M_3(F)$, $D = \mathrm{ad}(A)$ *for* A *a symmetric matrix with zero trace.*

Proof. Assume throughout that (i) and (ii) fail to hold. Apply Lemma 5 to conclude that $D = \mathrm{ad}(A)$ for $A \in Q$. Next specializing to $x = y$ results in $(c_1 + c_3)xx^D + (c_2 + c_4)x^D x \in C$, so by Theorem 1 we must conclude that $c_1 + c_3 = c_2 + c_4 = 0$. Thus $p(x, y) = c(xy^D - yx^D) + z(x^D y - y^D x)$. Should * be of the second kind, then by Lemma 2 there is a nonzero ideal J of R so that $p(x, y) \in C$ for all x, $y \in J$. It follows directly from Theorem B that all $c_i = 0$ or R satisfies S_4. This shows that we may henceforth assume that $a^* = a$ for all $a \in C$.

Our immediate goal is to handle the special case when $c = z$, and as above there is no loss of generality in assuming that $p(x, y) = xy^D - yx^D + x^D y - y^D x = [x, y]^D$. Let W be either $[T(I), T(I)]$ or $[K(I), K(I)]$ as appropriate, so that $W^D \subset C$ and $[W, W]^D = 0$. Now $[W, W] \subset K$ and is invariant under commutation by K, so if R does not satisfy a polynomial identity then the subring generated by $[W, W]$ contains a nonzero ideal of R by [7; Theorem A, p. 1756] when char $R \neq 2$, and by [**13**, Corollary 32, p. 132] when char $R = 2$. This would result in $D = 0$, so R must satisfy a polynomial identity. But now $RC = RZ^{-1}$ is a simple ring satisfying the same identity [**5**; Theorem 2, p. 57], as is $RC \otimes F = M_n(F)$. Furthermore, since * is of the first kind, $p(x, y) \in F$ holds for all x, $y \in T(M_n(F))$, or in $K(M_n(F))$, so as above $[W, W]^D = 0$, but we may now assume that R is a matrix ring. Applying Lemma 3 gives $D = 0$, unless char $R = 2$, $n = 4$, and * is symplectic. In this latter case, it is straightforward to calculate from $p(x, y) = [x, y]^D \in F$, that $A \in [T(RC), T(RC)]$. Therefore, either $D = 0$ or condition (iii) of the theorem holds, finishing the proof when $c = z$.

We may henceforth assume that $c \neq z$, and since $D = \mathrm{ad}(A)$ we may write

$$(2) \qquad p(x, y) = c[x, y]A + (z - c)(xAy - yAx) - zA[x, y].$$

Also, recall that $C = C \cap S$. Because $p(x, y)$ is a multilinear expression and $a^* = a$ for all $a \in C$, as in the proof of Theorem 1, we can extend our hypothesis to $T(\mathrm{Soc}\, RC \otimes F) = T(H)$, or to $K(H)$, where F is an algebraic closure of C. Thus, we can assume that $I = R = \mathrm{Soc}(R)$ is a simple ring and $eRe = eC$ for e any primitive idempotent in R. Note also that as in Theorem 1, D is a derivation of H, and so of R as just described.

Using Lemma 6, we may assume that $R = M_n(C)$, $n > 2$, and $C = C \cap S$. As in Theorem 1, the involution is either of transpose type or is symplectic. Our calculations at this point will go through several cases depending on the type of involution, the characteristic of R, and whether the theorem holds for T or for K. We begin with the case that the theorem holds for T. If either * is symplectic, or $\mathrm{char}\, R \neq 2$ and * is of transpose type, then the identity matrix $I_n \in T$, and from (2), $p(I_n, y) = (z-c)[A, y]$. Consequently, $[A, T] \subset C$, from which it follows that $[A, [T, T]] = 0$, so Lemma 3 forces $A \in C$ and so $D = 0$. Now $T = K$ when $\mathrm{char}\, R = 2$, so to complete the proof it suffices to assume that the theorem holds for K and that either * is symplectic and $\mathrm{char}\, R \neq 2$, or that * is of transpose type.

Next we consider the case that the theorem holds for K and that $\mathrm{char}\, R \neq 2$, with * symplectic, Write $A = \sum A_{ij}E_{ij}$ in 2×2 block form, set $U = e_{11} - e_{22} \in M_2(C)$, and note that $UE_{ii} \in K$. Now from (2), $p(UE_{ii}, UE_{jj}) = (z-c)(UA_{ij}UE_{ij} - UA_{ji}UE_{ji}) \in C$ implies that $A_{ij} = 0$ for $i \neq j$, and then considering the coefficient of E_{ij} in $p(UE_{ii}, YE_{ij} - Y^*E_{ji})$ forces

$$(3) \quad cUYA_{jj} + (z-c)UA_{ii}Y - zA_{ii}UY = 0 \quad \text{for all } Y \in M_2(C).$$

If $Y = e_{11}$, then the $(1, 2)$ entry of equation (3) is $c(A_{jj})_{12} = 0$, and the $(1, 1)$ entry is $c((A_{jj})_{11} - (A_{ii})_{11}) = 0$. If $Y = e_{22}$, then looking at the $(2, 1)$ and $(2, 2)$ entries show $c(A_{jj})_{21} = 0$, and $c((A_{ii})_{22} - (A_{jj})_{22}) = 0$. One can replace c with z in these four relations by first considering the coefficient of E_{ji} instead of E_{ij} in $p(UE_{ii}, YE_{ij} - Y^*E_{ji})$, and then looking at the $(1, 1)$ and $(2, 1)$ entries when $Y^* = e_{11}$, and the $(2, 2)$ and $(1, 2)$ entries when $Y^* = e_{22}$. Since not both c and z are zero, we may conclude that A_{jj} is a diagonal matrix which is the diagonal of A_{ii}. Hence, interchanging i and j everywhere in the argument just given shows that $A_{ii} = A_{jj}$ is a diagonal matrix. It follows that U commutes with A_{ii}, so (3) reduces to $cU(YA_{ii} - A_{ii}Y) = 0$, and so A_{ii} is scalar if $c \neq 0$. Finally, if $c = 0$, then the coefficient of E_{ji} in $p(UE_{ii}, YE_{ij} - Y^*E_{ji})$ is zero, and so $z(Y^*A_{ii} - A_{ii}Y^*)U = 0$, and again A_{ii} is scalar. Since $A_{ii} = A_{jj}$, we obtain the contradiction $A \in C$, and $D = 0$.

The last case we need consider is when * is of transpose type on R and the theorem holds for K. First assume that $R = M_n(C)$ with $n > 3$, and recall that $v_{ij} = e_{ij} - ae_{ji} \in K$, for a suitable $a \in C$. For distinct subscripts, $p(v_{ij}, v_{uw}) = (z - c)(v_{ij}Av_{uw} - v_{uw}Av_{ij}) \in C$, and it follows that $A_{ju} = 0$, so A is diagonal. The (u, w) entry of $p(v_{iu}, v_{iw})$ yields $cA_{ww} + (z-c)A_{ii} - zA_{uu} = 0$. Replacing i with j in this last computation, and comparing the two equations results in $(z - c)(A_{ii} - A_{jj}) = 0$. Consequently, A is scalar, and again $D = 0$.

Finally, we may assume that $R = M_3(C)$. Let a C-basis of K be $v_{12} = e_{12} - ae_{21}$, $v_{13} = e_{13} - be_{31}$, and $v_{23} = e_{23} - ba^{-1}e_{32}$, note that $[v_{12}, v_{23}] = v_{13}$, and set $p(v_{12}, v_{23}) = Y \in C$. The diagonal entries of Y must be equal, and these are $Y_{11} = cA_{31} + zbA_{13}$, $Y_{22} = (z - c)(bA_{13} - A_{31})$, and $Y_{33} = -cbA_{13} - zA_{31}$, so that $Y_{11} - Y_{33} = (c + z)(A_{31} + bA_{13}) = 0$. If $c + z \neq 0$, then $A_{31} = -bA_{13}$, and using this shows that $Y_{22} = 2Y_{11} = 2Y_{22}$. Hence, $Y_{22} = Y_{11} = 0$ forces $A_{13} = 0$ since $c - z \neq 0$, and now $A_{31} = 0$ follows. Using $Y_{ij} = 0$ for $i \neq j$, one computes $Y_{13} + b^{-1}Y_{31} = (c + z)(A_{33} - A_{11}) = 0$. Thus $A_{11} = A_{33}$ under our assumption that $c + z \neq 0$. Repeating these computations with other subscripts shows that $A \in C$, and so, $D = 0$. Specifically, from $p(v_{12}, v_{13}) \in C$ we get $A_{23} = A_{32} = 0$ and $A_{22} = A_{33}$, and $p(v_{13}, v_{23}) \in C$ yields $A_{12} = A_{21} = 0$.

We may now assume that $c + z = 0$, so in particular $c \neq 0$ and replacing $p(x, y)$ with $c^{-1}p(x, y)$ enables us to take

$$p(x, y) = [x, y]A - 2(xAy - yAx) + A[x, y].$$

As above, for $p(v_{12}, v_{23}) = Y \in C$, $Y_{11} = Y_{22}$ yields $A_{31} = bA_{13}$, and similar computations with subscripts permuted show that $A_{32} = ba^{-1}A_{23}$, and $A_{21} = aA_{12}$. Now $Y_{13} = A_{33} - 2A_{22} + A_{11} = 0$, so if char $R = 3$, A must be a symmetric matrix of trace zero and (iv) holds, proving the theorem. If char $R \neq 3$, then combining the last relation with $p(v_{13}, v_{12})_{32} = b(-A_{33} - A_{22} + 2A_{11}) = 0$, shows that $A_{11} = A_{22}$, and so $Y_{13} = 0$ gives $A_{22} = A_{33}$ as well. To finish the proof, it suffices to show that $A_{ij} = 0$ for $i \neq j$. For example, we just indicated that $A_{32} = ba^{-1}A_{23}$, so together with $Y_{12} = A_{32} + 2ba^{-1}A_{23} = 0$ one obtains $A_{23} = 0$, and then $Y_{12} = A_{32} = 0$. In a similar way, using suitable permutations of subscripts, one sees that $A_{ij} = 0$ and that A is scalar, completing the proof.

Our last major result uses Theorem 2 to complete consideration of degree two identities with one derivation by examining the mixed case when evaluations in $p(x, y)$ are made from both $T(I)$ and $K(I)$. In this case, unlike in Theorem 2, the only possibilities arise when R satisfies S_4, assuming that char $R \neq 2$. When char $R = 2$, $T(I) = K(I)$, and the situation is covered by Theorem 2. If char $R \neq 2$ and $R = M_2(C)$ with symplectic involution, then T is central, so $cx^D y + zyx^D = 0$ for D inner and any $x \in T$ and $y \in K$, or $x^D y - yx^D = 0$ for any derivation D. When R has transpose involution, K is commutative so $cxy^D + zy^D x = 0$ for $x \in T$, $y \in K$ and $D = \mathrm{ad}(A)$ for $A \in K$.

Theorem 3. *Let R be a prime ring with* char $R \neq 2$, *I a nonzero ideal of R, and D a nonzero derivation of R. If $p(x, y) = c_1xy^D + c_2x^Dy + c_3yx^D c_4 y^D x \in C$ for all $x \in T(I)$ and all $y \in K(I)$ (or $x \in K(I)$ and $y \in T(I)$), then either all $c_i = 0$ or R satisfies S_4.*

Proof. The proof follows the outline of Theorem 2. Assume throughout that some $c_i \neq 0$ and that R does not satisfy S_4. Also, we assume without loss of generality that $x \in T(I)$ and $y \in K(I)$, since it will be clear that the other case follows by a completely parallel argument. By Lemma 5, $D = \mathrm{ad}(A)$ for $A \in Q$.

Now $g(x, y, w)$ is an identity to which we may apply Lemma 2 if * is of the second kind. In this case, for some nonzero ideal J of R, $p(x, y) \in C$ for all x, $y \in J$, and Theorem B forces either all $c_i = 0$, or R to satisfy S_4. Therefore, we may henceforth assume that * is of the first kind. As in our previous results, we can extend the hypothesis to hold in the ring $\mathrm{Soc}(RC \otimes F)$, and so assume that $I = R = \mathrm{Soc}(R)$, and that $eRe = Ce$ for any primitive idempotent $e \in R$. It suffices to assume that $R = M_n(C)$ by Lemma 6.

We may now assume that $R = M_n(C)$, for $n > 2$ and since $D = \mathrm{ad}(A)$ for $A \in R$, we re-write $p(x, y)$ to get the expression

$$(4)\quad (c_1xy + c_3yx)A + (c_2 - c_1)xAy + (c_4 - c_3)yAx - A(c_2xy + c_4yx).$$

Since char $R \neq 2$, regardless of the type of *, the identity matrix $I_n \in T$, so we may consider $p(I_n, y) = (c_1 + c_4)[y, A] \in C$, using (4). If $c_1 + c_4 \neq 0$, then $[K, A] \subset C$, so $[[K, K], A] = 0$, and it follows from Lemma 3 that $A \in C$, a contradiction to $D \neq 0$. Hence $c_1 + c_4 = 0$. At this point we want to prove that $c_1 \neq 0$. If $c_1 = 0 = c_4$, then

$$(5)\qquad p(x, y) = c_2x^Dy + c_3yx^D.$$

If in addition, $c_2 = 0$, then $KT^D \subset C$, since some $c_i \neq 0$. But K is additively generated by its elements of rank two and $n > 2$, so in fact $KT^D = 0$. Using Lemma 3 again shows that $T^D = 0$, and then that $D = 0$. The same contradiction results from assuming that $c_3 = 0$, so if $c_1 = 0$, then $c_2c_3 \neq 0$.

Evaluating (5) with $xy = yx = 0$ yields $p(x, y) = c_2xAy - c_3yAx \in C$. In particular, when * is of transpose type and we take $x = e_{ii}$ and $y = e_{jk} - be_{kj}$ for i, j, k distinct, then since the (i, k) entry of $p(x, y)$ is zero, it follows that $A_{ij} = 0$, and so A is diagonal. Use (5) to compute

$$p(e_{ij} + ae_{ji}, e_{jk} - be_{kj})$$
$$= zc_2(e_{ij} - ae_{ji})(e_{jk} - be_{kj}) + zc_3(e_{jk} - be_{kj})(e_{ij} - ae_{ji}) \in C,$$

where $z = A_{jj} - A_{ii}$. This reduces to $zc_2e_{ik} + zc_3abe_{ki} \in C$, so shows that $zc_2 = 0$, which forces $z = 0$, and proves that A is scalar. This contradiction means that when $c_1 = 0$, * must be symplectic.

Set $A = \sum P_{ij}E_{ij}$ with $P_{ij} \in M_2(C)$, as in Theorem 2, let $x = E_{ii}$, $y = UE_{jj}$ for $U = e_{11} - e_{22} \in M_2(C)$, and observe that as in the case above, using (5) gives $c_2P_{ij}UE_{ij} - c_3UP_{ji}E_{ji} \in C$, and so, forces $P_{ij} = 0$ when $i \neq j$. Now computing the (i, j) entry of $p(BE_{ij} + B^*E_{ji}, UE_{ii})$, where $B \in M_2(C)$ is arbitrary, shows that $c_3U(BP_{jj} - P_{ii}B) = 0$. Since $c_3 \neq 0$, and U is invertible, we must have $BP_{jj} = P_{ii}B$, and this forces $P_{ii} = P_{jj}$ to be scalar. Consequently, A is scalar and $D = 0$. We have shown that $c_1 = 0$ is impossible.

To complete the proof we need to consider two more possibilities, the first of which is that $c_2 - c_1 = c_4 - c_3 = 0$. In this case, since $c_1 = -c_4 \neq 0$, $(c_1)^{-1}p(x, y) = [[x, y], A]$. Thus $[[T, K], A] \subset C$, so A commutes with $[[T, K], [T, K]]$, which generates R by Lemma 3. This gives the contradiction $D = 0$, and shows that either $c_2 - c_1 \neq 0$ or $c_4 - c_3 = -(c_1 + c_3) \neq 0$.

Using the expression for $p(x, y)$ in (4), suppose that * is of transpose type, let $x = e_{ii}$ and $y = e_{jk} - be_{kj}$ for i, j, k distinct, and observe that since the (i, k) and (j, i) entries are zero, one has $(c_2 - c_1)A_{ij} = (c_1 + c_3)A_{ki} = 0$. The assumption on the c_i forces either $A_{ij} = 0$ or $A_{ki} = 0$, and appropriate permutations of the subscripts show that A is a diagonal matrix. But now $(e_{ii})^D = 0$, so $p(e_{ii}, y) = c_1e_{ii}y^D + c_4y^De_{ii} \in C$. Choosing $y = e_{ij} - ae_{ji}$ and computing the (i, j) entry yields $c_1(A_{ii} - A_{jj}) = 0$, so A must be scalar and $D = 0$.

Finally, we may assume that $*$ is symplectic, and as above set $A = \sum P_{ij}E_{ij}$ for $P_{ij} \in M_2(C)$. Once again, use (4) with $x = E_{ii}$ and $y = UE_{jj}$ for $i \neq j$, and $U = e_{11} - e_{22} \in M_2(C)$, together with one of $c_2 - c_1 \neq 0$ or $c_4 - c_3 \neq 0$ to see that $P_{ij} = 0$. It is clear that now $(E_{ii})^D = 0$, so $p(E_{ii}, y) = c_1 E_{ii} y^D + c_4 y^D E_{ii}$. Computing the (i, j) 2×2 block when $y = BE_{ij} - B^* E_{ji}$, for $B \in M_2(C)$, shows that $BP_{jj} = P_{ii}B$, so as we have seen before, $P_{ii} = P_{jj}$ is a scalar matrix. Therefore, A is scalar, so $D = 0$, and this contradiction completes the proof of the theorem.

Our introductory comments showed that the investigation of a general quadratic central (skew-) trace identity reduces to those identities considered in our three theorems. We end this paper with a general statement putting those results together.

THEOREM 4. *If* $p(x_1, \dots, x_n) = \sum_{i,j=1}^n a_{ij} x_i^{D_{ij}} x_j + b_{ij} x_i x_j^{D_{ij}}$ *is a nonzero quadratic central (skew-) trace identity for a nonzero ideal* I *of* R, *then either all* $D_{ij} = 0$, R *satisfies* S_4, $\operatorname{char} R = 2$ *and* R *embeds in* $M_4(F)$ *with symplectic involution, or* $\operatorname{char} R = 3$ *and* R *embeds in* $M_3(F)$.

Acknowledgment. The author thanks the referee for the careful reading of the paper and the suggestions for the reorganization of the content. The referee suggested the general notion of a quadratic central (skew-) trace identity and the proof of Lemma 6, which replaces three separate arguments originally given in our first three theorems.

REFERENCES

[1] S. A. Amitsur, *Identities in rings with involution*, Israel J. Math., **7** (1969), 63–68.

[2] I. N. Herstein, *Topics in Ring Theory*, The University of Chicago Press, Chicago, 1969.

[3] ——, *Rings with Involution*, The University of Chicago Press, Chicago, 1976.

[4] Y. Hirano, A. Kaya, and H. Tominaga, *On a theorem of Mayne*, Math. J. Okayama Univ., **25** (1983), 125–132.

[5] N. Jacobson, *PI-Algebras*, Lecture Notes in Math., vol. 441, Springer-Verlag, New York, 1975.

[6] C. Lanski, *Invariant submodules in semi-prime rings with involution*, Comm. Algebra, **6** (1978), 75–96.

[7] ——, *Lie structure in semi-prime rings with involution* II, Comm. Algebra, **6** (1978), 1755–1775.

[8] ——, *Differential identities in prime rings with involution*, Trans. Amer. Math. Soc., **291** (1985), 765–787; Correction to: *Differential identities in prime rings with involution*, Trans. Amer. Math. Soc., **309** (1988), 857–859.

[9] ——, *Minimal differential identities in prime rings*, Israel J. Math., **56** (1986), 231–246.

[10] ——, *Differential identities, Lie ideals, and Posner's theorems*, Pacific J. Math., **134** (1988), 275–297.

[11] ——, *Minimal * differential identities in prime rings*, J. Algebra, **133** (1990), 472–489.

[12] ——, *Lie ideals and central identities with derivation*, Canad. J. Math., (to appear).

[13] C. Lanski and S. Montgomery, *Lie structure of prime rings of characteristic* 2, Pacific J. Math., **42** (1972), 117–136.

[14] P. H. Lee and T. K. Lee, *Derivations centralizing symmetric or skew elements*, Bull. Inst. Math. Acad. Sinica, **14** (1986), 249–256.

[15] W. S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra, **12** (1969), 576–584.

[16] ——, *Prime rings with involution and generalized polynomial identities*, J. Algebra, **22** (1972), 502–516.

[17] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., **8** (1957), 1093–1100.

[18] L. Rowen, *Generalized polynomial identities*, J. Algebra, **34** (1975), 458–480.

Received June 11, 1990 and in revised form June 17, 1991.

UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CA 90089-1113