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A_∞ AND THE GREEN FUNCTION

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Let $G(x)$ be the Green function in a domain $\Omega \subseteq \mathbb{R}^m$ with a fixed pole, and Γ be an $(m-1)$-dimensional hyperplane. We give conditions on Ω and $\Omega \cap \Gamma$ so that $|\nabla G|$ is A_∞ with respect to the $(m-1)$-dimensional measure on $\Omega \cap \Gamma$. Certain properties of the Riemann mapping of a simply-connected domain in $\mathbb{R}^2$ are extended to the Green function of domains in $\mathbb{R}^m$.

In [3], Fernández, Heinonen and Martio have proved the following:

THEOREM A. *Let f be a conformal mapping from a simply-connected planar domain Ω onto the unit disk Δ and L be a line segment in Ω. Then $f(L)$ is a quasiconformal arc. Moreover, if L is a line segment on the boundary of a half plane contained in Ω, then $|f'| \in A_\infty(ds)$ on L with respect to the linear measure ds.*

If L is any line segment in Ω, $|f'|$ need not be in $A_\infty(ds)$ on L. In fact, Heinonen and Näkki [9] have proved the following:

THEOREM B. *Let f be a conformal mapping from a simply-connected domain Ω onto the unit disk Δ and L be a line segment in Ω. Then the following are equivalent*:

(1) $|f'| \in A_\infty(ds)$ *on* L,
(2) $f|L$ *is quasisymmetric,*
(3) *there exists a chord arc domain* $D \subseteq \Omega$ *so that* $L \subseteq \overline{D}$,
(4) *there exists a quasidisk* $D \subseteq \Omega$ *so that* $L \subseteq \overline{D}$.

Let μ and ν be two measures on $\mathbb{R}^m$ $(m \geq 2)$. Recall that μ belongs to the Muckenhoupt class $A_\infty(d\nu)$ if there exist $\alpha, \beta \in (0, 1)$ such that whenever E is a measurable subset of a cube Q,

$$\nu(E)/\nu(Q) < \alpha \quad \text{implies} \quad \mu(E)/\mu(Q) < \beta. \tag{0.1}$$

If μ and ν have the doubling property, then $\mu \in A_\infty(d\nu)$ if and only if $\nu \in A_\infty(d\mu)$ ([2]). We say a function is in $A_\infty(d\nu)$ on L, provided that (0.1) holds with $d\mu = g\, d\nu$ for all cubes $Q \subseteq L$.

$f|L$ is quasisymmetric provided that for all $a, b, x \in L$, $|a-x| \leq |b-x|$ implies $|f(a)-f(x)| \leq c|f(b)-f(x)|$ for some constant $c > 0$.

Let G be the Green function for Ω with pole $f^{-1}(0)$ and $\delta(z)$ be $\operatorname{dist}(z, \partial\Omega)$. From the distortion theorem, it follows that

$$|\nabla G(z)| \cong |f'(z)| \cong \frac{1-|f(z)|}{\delta(z)} \cong \frac{G(z)}{\delta(z)} \tag{0.2}$$

when $f(z)$ is away from 0. Thus it is natural to study the analogue of Theorem B for general domains Ω in $\mathbb{R}^m$ $(m \geq 2)$, that is, to find conditions on Ω and the planar section $L \subseteq \Omega$, so that $|\nabla G| \in A_\infty(d\sigma)$ on L with respect to the $(m-1)$-dimensional measure $d\sigma$. Because $|\nabla G|$ may vanish, we study $G(z)/\delta(z)$ instead.

From now on, Ω denotes a domain in $\mathbb{R}^m$ $(m \geq 2)$, G the Green function on Ω, P a fixed point in Ω and $G(x) = G(P, x)$. Let Γ be an $(m-1)$-dimensional hyperplane in $\mathbb{R}^m$ which does not contain P, and σ be the $(m-1)$-dimensional measure on Γ. If L is a domain in Γ, denote by $\partial' L$ its boundary relative to Γ. We shall prove the following:

THEOREM 1. *Suppose that Ω is a nontangentially accessible (NTA) domain and that $L \subseteq \Omega$ is a uniform domain on the hyperplane Γ. Furthermore, there exists $0 < c < 1$ so that for each $x \in L$, at least one component of $B(x, c\operatorname{dist}(x, \partial' L)) \backslash L$ is contained in Ω. Then $\frac{G(x)}{\delta(x)}|_L$ can be extended to become an $A_\infty(d\sigma)$ function on the entire hyperplane Γ.*

THEOREM 2. *Suppose that Ω is a quasiball and is a* BMO_1 *domain. Then $\frac{G(x)}{\delta(x)}|_{\Gamma\cap\Omega}$ can be extended to become an $A_\infty(d\sigma)$ function on the entire hyperplane Γ.*

The assumption that L is a uniform domain arises naturally in defining A_∞ and in extending $G(x)/\delta(x)$ by the method of reflection. The additional condition on L is needed in view of the following:

EXAMPLE. For each $m \geq 2$, there exists an NTA domain so that $\Omega \cap \{x_m = 0\}$ is an $(m-1)$-dimensional cube, but $\frac{G(x)}{\delta(x)} \notin A_\infty(d\sigma)$ on $\Omega \cap \{x_m = 0\}$.

The additional condition on L is satisfied when $L \subseteq \overline{D}$ for some domain $D \subseteq \Omega$ whose complement $\mathbb{R}^m \backslash D$ has the linearly locally connected property (LLC). Examples of such D are quasidisks in $\mathbb{R}^2$ or domains quasiconformally equivalent to a ball in $\mathbb{R}^m$ $(m \geq 3)$, see [7] and [**8**].

In Theorem 2, no condition is imposed on $\Omega \cap \Gamma$, and it may be any open set. Lipschitz domains which are homeomorphic to a ball satisfy the conditions in Theorem 2. The theorem remains true for all quasidisks in $\mathbb{R}^2$ (Theorem B).

In the core of our proof is the following theorem, which in its most general form is proved by B. Davis [4] by probabilistic methods. Special cases and related results can be found in [5], [13] and [15].

THEOREM C. *Let Ω be a domain in $\mathbb{R}^m$, $m \geq 2$, and $\{D_j\}$ be a sequence of closed sets contained in Ω with $\operatorname{dist}(D_i, D_j) > 0$ whenever $i \neq j$. Set $\Omega_j = \Omega \backslash \bigcup_{k \neq j} D_k$. If $\{D_j\}$ are uniformly separated in the sense:*

$$\inf_j \inf_{z \in D_j} \omega(z, \partial\Omega, \Omega_j) = a > 0, \tag{0.3}$$

then for any $x \in \Omega \backslash \bigcup D_j$,

$$\sum_j \omega(x, D_j, \Omega \backslash D_j) < \frac{1}{a} \omega\left(x, \bigcup D_j, \Omega \backslash \bigcup D_j\right).$$

1. Preliminary Theorems. For a domain Ω and a set S in $\mathbb{R}^m$, denote by $\delta(S)$ the *distance* from S to $\partial\Omega$, $d(S)$ the *diameter* of S and $l(S)$ the *side length* of S if S is a cube. If S is a ball, a cube or a square, denote by cS the ball, the cube, or the square on the same hyperplane, concentric to S, of diameter $cd(S)$. Denote by $B(x, r)$ the ball centered at x of radius r.

Ω is called a nontangentially accessible (NTA) domain [10], if it is bounded and there exist constants $r_0 > 0$, $M > 10$ and $N > 10$ depending on Ω so that the following conditions are satisfied:

(1.1) *Corkscrew condition*: for any $Z \in \partial\Omega$, $0 < r < r_0$, there exist $A = A_r(Z) \in \Omega$ such that $M^{-1}r < |A - Z| < r$ and $\operatorname{dist}(A, \partial\Omega) > M^{-1}r$.

(1.2) $\mathbb{R}^m \backslash \overline{\Omega}$ satisfies the corkscrew condition.

(1.3) *Harnack chain condition*: if X_1 and X_2 are in Ω, $\operatorname{dist}(X_i, \partial\Omega) > \varepsilon > 0$, $i = 1, 2$, and $|X_1 - X_2| \leq 10M\varepsilon$, then there exist balls $B_j = B(Y_j, r_j)$, $1 \leq j \leq n$ with $n \leq N$, so that $Y_1 = X_1$ and $Y_n = X_2$ and that the balls satisfy

$$M^{-1}r_j < \operatorname{dist}(B_j, \partial\Omega) < Mr_j, \qquad 1 < j < n,$$

and

$$B\left(Y_j, \frac{r_j}{2}\right) \cap B\left(Y_{j+1}, \frac{r_{j+1}}{2}\right) \neq \varnothing, \qquad 1 \leq j \leq n-1.$$

Suppose Ω is an NTA domain. For $Z \in \partial\Omega$, denote by $\Delta(Z, r)$ the surface ball $B(Z, r) \cap \partial\Omega$. Let P be a fixed point in Ω. Then the Green function in Ω and the harmonic measure ω on $\partial\Omega$ have the following properties, **[10]**:

(1.4) *Doubling property of* ω : there exists $C > 0$ depending only on Ω and P so that

$$\omega(P, \Delta(Z, 2r), \Omega) \leq C\omega(P, \Delta(Z, r), \Omega)$$

for any surface ball $\Delta(Z, r) \equiv B(Z, r) \cap \partial\Omega$.

(1.5) *Relation between* ω *and* G: suppose that $A \in \Omega$, $Z \in \partial\Omega$ with $c^{-1}\delta(A) \leq |A - Z| \leq c\delta(A)$; then there exists $C > 0$ depending on Ω, P and c only so that

$$C^{-1} \leq \frac{G(P, A)\delta(A)^{m-2}}{\omega(P, \Delta(Z, \delta(A)), \Omega)} \leq C.$$

Let Ω be an NTA domain, Q be a cube in Ω satisfying $\operatorname{dist}(P, Q) \geq \delta(Q) \geq d(Q) \geq \frac{1}{2}\delta(Q)$, and Γ be an $(m-1)$-dimensional hyperplane in $\mathbb{R}^m$ passing through the center of Q. Following the arguments in **[10]**, we may find constants c, $C > 0$ depending on Ω and P, so that

$$\begin{aligned} C^{-1}\omega(P, Q, \Omega\backslash Q) &\leq G(P, x)\delta(x)^{m-2} \\ &\leq C\omega(P, Q, \Omega\backslash Q), \qquad x \in Q, \end{aligned} \tag{1.6}$$

and

$$\omega(x, \partial\Omega\backslash\Gamma, \Omega\backslash(\Gamma\backslash Q)) > c, \qquad x \in \tfrac{1}{2}Q. \tag{1.7}$$

Ω is called a *uniform domain* if it satisfies the interior corkscrew condition (1.1) and the interior Harnack chain condition (1.2) in the definition of NTA domain. It is also called a BMO extension domain because of its characterization in terms of extension properties of $\mathrm{BMO}(\Omega)$ by Jones **[11]**. For properties of uniform domains, see [7]. In $\mathbb{R}^2$, a simply-connected uniform domain is a quasidisk.

A bounded domain $\Omega \subseteq \mathbb{R}^m$ is called a BMO_1 domain if its boundary is given locally in some C^∞ coordinate system as the graph of a function ϕ with $\nabla\phi \in \mathrm{BMO}$. BMO_1 domains are defined and studied by Jerison and Kenig in **[10]**. They are NTA domains and can be regarded as the analogue of chord arc domains in $\mathbb{R}^m$ $(m \geq 3)$; note that the graph of $y = \phi(x)$ is a chord arc curve if $\phi' \in \mathrm{BMO}(\mathbb{R}^1)$. It is proved in **[10]** that

THEOREM D. *If* Ω *is a* BMO_1 *domain, then the harmonic measure* ω *on* $\partial\Omega$ *belongs to* $A_\infty(d\sigma)$.

An extension of Hall's Lemma is proved in **[19]**; it is stated here with constants given more precisely.

THEOREM E. *Let* Ω *be a* BMO_1 *domain and* $C_0 > 1$ *be given. There exist constants* λ, $c > 0$ *depending on* Ω *and* C_0 *only, so that for any point* $A \in \Omega$ *and closed set* $E \subseteq \Omega \cap B(A, C_0\delta(A))$,

$$\omega(A, E, \Omega\backslash E) \geq c(M_{m-1}(E)\delta(A)^{-m+1})^{\lambda},$$

where M_{m-1} *is the* $(m-1)$*-dimensional content.*

The *α-dimensional content* $M_\alpha(E)$ of a set E is defined to be $\inf\sum_n r_n^\alpha$, with the infimum taken over all coverings of E consisting of countably many balls with radii r_n.

We also need the following estimate of harmonic measures **[19]**, which is first proved by Carleson **[1]** for the half plane. Again, the constants are described more precisely here.

THEOREM F. *Let* Ω *be a* BMO_1 *domain in* $\mathbb{R}^m$ $(m \geq 3)$, $C_0 > 1$, $A \in \Omega$ *and* E *be a closed set in* $\Omega \cap B(A, C_0\delta(A))$. *Let* $\mathscr{M}$ *be the family of positive measures* ν *on* E, *which satisfy, for each cube* Q *in* Ω *with* $16d(Q) \leq \delta(Q) \leq 256d(Q)$,

$$\nu(Q) \leq \operatorname{cap}(E \cap Q)l(Q);$$

and for each cube Q *in* $\mathbb{R}^m$ *that meets* $\partial\Omega$,

$$\nu(Q) < l(Q)^{m-1}.$$

Then there exist constants γ, $c > 0$, *depending only on* Ω *and* C_0 *so that*

$$\omega(A, E, \Omega\backslash E) \geq c \sup_{\mathscr{M}}(\nu(E)\delta(A)^{-m+1})^{\gamma}.$$

Here cap *is the Newtonian capacity.*

Let $\Phi: \mathbb{R}^m \to \mathbb{R}^m$ be a K-quasiconformal mapping. Following are some properties of Φ due to Gehring and Väisälä **[17]**; all constants depend on m and K only unless otherwise mentioned.

Lemma 1. *There exists $c_0 > 0$ so that if $0 < c < c_0$, B_1 and B_2 are balls with $d(B_1) < cd(B_2)$ and* $\operatorname{dist}(B_1, B_2) < cd(B_1)$, *then $d(\Phi(B_1)) < c_0 c^\alpha d(\Phi(B_2))$ for some $\alpha > 0$ depending only on K.*

Lemma 2. *Let B be a ball with center X; then there exist balls B' and B'' with center $\Phi(X)$, so that $B'' \subseteq \Phi(B) \subseteq B'$ and $d(B') < Cd(B'')$.*

The next theorem is due to Gehring **[6]**.

Theorem G. *The Jacobian of Φ is in $A_\infty(dx)$ on $\mathbb{R}^m$. Thus there exists $\alpha > 0$ so that*

$$\frac{|\Phi(F)|}{|\Phi(B)|} \leq C\left(\frac{|F|}{|B|}\right)^\alpha,$$

for any ball B and $F \subseteq B$.

Lemma 3. *There exists $a > 1$ depending on K so that if U is a ring $\{x : r < |x - x_0| < ar\}$ then $\Phi(U)$ contains a ring in the form $\{x : \rho < |x - \Phi(x_0)| < 2\rho\}$ for some $\rho > 0$.*

Proof. Let $B_1 = B(x_0, r)$ and $B_2 = B(x_0, ar)$. Then there exist balls B_1', B_1'', B_2', B_2'' centered at $\Phi(x_0)$ so that $B_1'' \subseteq \Phi(B_1) \subseteq B_1'$, $B_2'' \subseteq \phi(B_2) \subseteq B_2'$, $\operatorname{diam} B_1' \leq C \operatorname{diam} B_1''$ and $\operatorname{diam} B_2' \leq C \operatorname{diam} B_2''$. Because of Theorem G, $(\operatorname{diam} B_1''/\operatorname{diam} B_2') \leq Ca^{-\alpha}$. Hence $\operatorname{diam} B_1' \leq ca^{-\alpha} \operatorname{diam} B_2''$ and $\Phi(U)$ contains the ring $B_2'' \backslash \overline{B_1'}$ provided that a is sufficiently large.

Let $\Omega = \Phi(B(0, 1))$ and Φ^* be the *quasiconformal reflection* about $\partial\Omega$ defined by

$$\Phi^*(x) = \Phi\left(\frac{\Phi^{-1}(x)}{\|\Phi^{-1}(x)\|^2}\right). \tag{1.8}$$

Then Ω is an NTA domain **[10]**, and Φ^* is quasiconformal on $\{c^{-1} < |x - \Phi(0)| < c\}$. Denote by S^* the reflection $\Phi^*(S)$.

Lemma 4. *Given $c_1, c_2 > 1$ there exists $c = c(c_1, c_2, K) > 1$ so that if Q is a cube in $\{c_1^{-1} < |x - \Phi(0)| < c_1\}$ which does not meet $\partial\Omega$ and satisfies $c_2^{-1} < l(Q)/\delta(Q) < c_2$ then*

$$c^{-1} < \frac{d(Q^*)}{\delta(Q^*)} < c.$$

Moreover, there exists a ball $B \subseteq Q^*$ *so that*

$$d(Q^*) \cong l(Q) \cong d(B).$$

And if Q *is a cube in* $\{c_1^{-1} < |x - \Phi(0)| < c_1\}$ *that meets* $\partial\Omega$, *then* $d(Q^*) \leq cl(Q)$.

By $a \cong b$, we mean a/b is bounded above and below by positive constants.

This lemma is a simple consequence of Lemmas 1, 2 and 3.

LEMMA 5. *Let* $h > 3$ *and* H *be the circular right cylinder* $\{x : \sum_1^{m-1} x_j^2 < 1$ *and* $0 < x_m < h\}$. *Let* E *be the base* $\{x : \sum_1^{m-1} x_j^2 \leq 1$ *and* $x_m = 0\}$ *of* H, *and* A *be the point* $(0, 0, \ldots, 0, h-1)$. *Then there exists* $c > 0$ *depending on* m, h *and* K *only so that*

$$\omega(\Phi(A), \Phi(E), \Phi(H)) > c. \tag{1.9}$$

Proof. Note that each $\Phi(\{x : \sum_1^{m-1} x_j^2 < 1,\ j < x_m < j+2\})$ is a C-quasiball $(0 \leq j \leq h-2)$. Hence (1.9) follows from successive applications of the Harnack inequality.

2. Proof of Theorem 1. Constants in this section depend on Ω, L, D, P and $\operatorname{dist}(P, \Gamma)$.

Assume from now on that $\Gamma = \{x_m = 0\}$ and fix a partition $\mathscr{C} = \{S_j\}$ of $\Gamma \cap \Omega$ so that S_j's are $(m-1)$-dimensional closed dyadic squares on Γ with mutually disjoint interiors and that

$$0 < c < \frac{l(S_j)}{\delta(S_j)} \leq \frac{1}{10}. \tag{2.1}$$

Let Y_j be the center of S_j, $B_j = B(Y_j, \frac{1}{10} l(S_j))$ and $D_j = B_j \cap \Gamma$.

Let $\{S_j\}_J$ be any subcollection of $\mathscr{C}$. Because Ω is an NTA domain, it follows from (2.1) and the exterior corkscrew condition (1.2) that the disks $\{D_j\}_J$ are uniformly separated as in (0.3). It follows from Theorem C and the maximum principle that for any $x \in \Omega \setminus \bigcup_J D_j$,

$$\begin{aligned}\sum_J \omega(x, S_j, \Omega\setminus S_j) &\cong \sum_J \omega(x, D_j, \Omega\setminus D_j) \\ &\leq c\omega\left(x, \bigcup_J D_j, \Omega\setminus \bigcup_J D_j\right) \\ &\leq c\omega\left(x, \bigcup_J S_j, \Omega\setminus \bigcup_J S_j\right).\end{aligned}$$

The last two inequalities can easily be reversed; thus

$$\sum_J \omega(x, S_j, \Omega\backslash S_j) \cong \omega\left(x, \bigcup_J S_j, \Omega\backslash \bigcup_J S_j\right) \tag{2.2}$$

which is a weak substitute for the additivity and is essential in our proof.

Suppose that I is a dyadic square on Γ with center in $\Gamma \cap \Omega$ and that

$$I \cap \Omega = \bigcup_J S_j \quad \text{for some } \{S_j\}_J \subseteq C. \tag{2.3}$$

Then $\delta(I) \leq C_3 l(I)$ for some $c_3 > 1$, because $\delta(I) \leq \delta(S_j) \cong l(S_j) \leq l(I)$ for any $j \in J$. Let Z be a point on $\partial\Omega$ that satisfies $\operatorname{dist}(Z, I) = \delta(I)$, and let $B \equiv B(Z, 4C_3 d(I))$, $\Delta = B \cap \partial\Omega$. Clearly that $I \subseteq \frac{1}{2}B$. Because of (1.1), we may choose and fix a point $A \in \Omega\backslash\Gamma$ with

$$8c_3 l(I) \leq |A - Z| \leq cl(I)$$

and $\delta(A) \cong l(I)$. We claim that

$$\omega(P, S_j, \Omega\backslash S_j) \cong \omega(P, \Delta, \Omega)\omega(A, S_j, \Omega\backslash S_j) \tag{2.4}$$

for each $j \in J$. If S_j were on $\partial\Omega$, (2.4) would follow from Lemma 4.11 in **[10]**. Since S_j is interior to Ω, (2.4) can be obtained by modifying the proof of that lemma; or by applying it to the NTA domain $\Omega\backslash\overline{B_j}$ and then using the Harnack inequality.

Suppose that $F = \bigcup_{\widetilde{J}} S_j$ for some $\widetilde{J} \subseteq J$. It follows from (2.2) and (2.4) that

$$\begin{aligned} \omega(P, F, \Omega\backslash F) &\cong \sum_{\widetilde{J}} \omega(P, S_j, \Omega\backslash S_j) \\ &\cong \sum_{\widetilde{J}} \omega(P, \Delta, \Omega)\omega(A, S_j, \Omega\backslash S_j) \\ &\cong \omega(P, \Delta, \Omega)\omega(A, F, \Omega\backslash F). \end{aligned} \tag{2.5}$$

So far, only the NTA assumption on Ω is used; this part of the proof also applies to Theorem 2. *To localize the problem, we need the estimate* $\omega(P, I \cap \Omega, \Omega\backslash I) \cong \omega(P, \Delta, \Omega)$ *which may not hold even when* $\Omega \cap \Gamma$ *is a square* (*example in* §4).

Let

$$\mu(F) = \int_F \frac{G(x)}{\delta(x)}\, d\sigma(x) \quad \text{for } F \subseteq \Gamma \cap \Omega. \tag{2.6}$$

LEMMA 6. *There exist* $\alpha, \beta \in (0, 1)$ *so that if* I *is a closed square on* Γ *centered in* $\overline{L}$ *and* $F \subseteq I \cap L$ *then*

$$\frac{\sigma(F)}{\sigma(I \cap L)} > \alpha \Rightarrow \frac{\mu(F)}{\mu(I \cap L)} > \beta. \tag{2.7}$$

Proof. Suppose that I is a dyadic square. Then either $I \subseteq S_{j_0}$ for some $S_{j_0} \in \mathscr{C}$ or (2.3) holds.

When $I \subseteq S_{j_0}$, from the Harnack inequality, it follows that

$$\mu(F)/\mu(I \cap L) \cong \sigma(F)/\sigma(I \cap L);$$

and thus (2.7).

Proceed with the assumption (2.3) and assume as we may that $l(I) \leq 4\,\mathrm{diam}(L)$. Because L is a uniform domain on Γ and the center of I is in $\overline{L}$, there exists a square $S \subseteq I \cap L$ satisfying

$$l(I) \cong l(S) \cong \mathrm{dist}(S, \partial' L). \tag{2.8}$$

Notice that $\mathrm{dist}(S, \partial\Omega) \leq cl(I)$ and that in general they are not comparable. To get around this difficulty, we deduce from the additional assumption on L that there exists a cube $Q \subseteq \Omega$ so that Q has one face lying on S and $l(Q) \cong l(S)$. Let A_0 be the center of Q; thus $\delta(A_0) \cong l(Q) + \delta(S) \cong l(I)$.

It follows from (1.5), (1.6) and the Harnack inequality that

$$\begin{aligned}\omega(P, I \cap L, \Omega\backslash(I \cap L)) \geq \omega(P, S, \Omega\backslash S) \geq cG(P, A_0)\delta(A_0)^{m-2}\\ \cong \omega(P, \Delta, \Omega);\end{aligned}$$

and from Lemma 4.2 in **[10]** and $I \subseteq \frac{1}{2}B$ that

$$\omega(P, I \cap L, \Omega\backslash(I \cap L)) \leq c\omega(P, \Delta, \Omega).$$

Thus

$$\omega(P, I \cap L, \Omega\backslash(I \cap L)) \cong \omega(P, \Delta, \Omega). \tag{2.9}$$

Let $F = \bigcup_{\widetilde{J}} S_j$ for some $\widetilde{J} \subseteq J$. We deduce form (1.6), (2.5), and (2.9) and the Harnack inequality that

$$\begin{aligned}\mu(F) \cong \sum_{\widetilde{J}} G(P, Y_j)d(S_j)^{m-2} \cong \sum_{\widetilde{J}} \omega(P, S_j, \Omega\backslash S_j)\\ \cong \omega(P, I \cap L, \Omega\backslash(I \cap L))\omega(A, F, \Omega\backslash F).\end{aligned}$$

Note also from the Harnack inequality that

$$\omega(A, F, \Omega\backslash F) \cong \omega(A_0, F, \Omega\backslash F)$$

and that

$$\omega(A, I\cap L, \Omega\backslash(I\cap L)) \cong \omega(A_0, I\cap L, \Omega\backslash(I\cap L)) \geq 1/2m.$$

Thus,

$$\mu(F)/\mu(I\cap L) \cong \omega(A_0, F, \Omega\backslash F).$$

We note that

$$\begin{aligned}\omega(A_0, F, \Omega\backslash F) &\geq \omega(A_0, F, Q) \geq \omega(A_0, F\cap \tfrac{1}{2}(\partial Q\cap S), Q)\\ &\geq c\frac{\sigma(F\cap\frac{1}{2}(\partial Q\cap S))}{\sigma(\partial Q\cap S)}.\end{aligned}$$

Because $\sigma(\frac{1}{2}(\partial Q\cap S)) \geq c_4\sigma(I\cap L)$ for some $c_4 > 0$, we conclude

$$\frac{\sigma(F\cap\frac{1}{2}(\partial Q\cap S))}{\sigma(\frac{1}{2}(\partial Q\cap S))} > c_4$$

provided that $\sigma(F)/\sigma(I\cap L) > 1 - c_4/2$. This implies (2.7) when $F = \bigcup_{\widetilde{J}} S_j$.

Let α and β be the constants associated with (2.7) for all previously proved special cases.

In general, for $F \subseteq I\cap L$, we may write $F = \bigcup_{\widetilde{J}} F_j$ where $F_j \subseteq S_j$ and $\widetilde{J} \subseteq J$. Suppose that

$$\frac{\sigma(F)}{\sigma(I\cap L)} > \frac{1+\alpha}{2}.$$

Let $\widetilde{J}_1 = \{j \in \widetilde{J}: \sigma(F_j)/\sigma(S_j) > (1-\alpha)/2\}$ and $\widetilde{J}_2 = \widetilde{J}\backslash\widetilde{J}_1$. Then

$$\sum_{\widetilde{J}_2}\sigma(F_j) \leq \frac{1-\alpha}{2}\sum_{\widetilde{J}_2}\sigma(S_j) \leq \frac{1-\alpha}{2}\sigma(I\cap L).$$

Since $\sum_{\widetilde{J}_1}\sigma(F_j) \leq \sum_{\widetilde{J}_1}\sigma(S_j)$, we have $\sum_{\widetilde{J}_1}\sigma(S_j) \geq \alpha\sigma(I\cap L)$. Therefore $\sum_{\widetilde{J}_1}\mu(S_j) \geq \beta\mu(I\cap L)$. It follows from the Harnack inequality and the choice of $\widetilde{J}_1$ that

$$\mu(F) \geq \sum_{\widetilde{J}_1}\mu(F_j) \geq c\sum_{\widetilde{J}_1}\mu(S_j) \geq c\beta\mu(I\cap L).$$

This proves (2.7) for dyadic squares I.

For general I, (2.7) follows from the fact that L is a uniform domain and the following doubling property (2.10) of μ.

Doubling property: for any square I on Γ centered in $\overline{L}$,

$$\mu(2I\cap L) \cong \mu(I\cap L). \tag{2.10}$$

Again, we assume as we may that $l(I) \leq 4 \operatorname{diam} L$. Let I_1 be the union of the squares in $\{S_j\}$ that meet $2I$, and S be a square in $I \cap L$ that satisfies (2.8). Then $l(I_1) \cong l(I) \cong l(S)$. If $\delta(I_1) > l(I_1)$, (2.10) follows from the Harnack principle. Otherwise, let Z_1 be a point on $\partial\Omega$ that satisfies $\operatorname{dist}(Z_1, I_1) = \delta(I_1)$, $B_1 \equiv B(Z_1, 4c_3 d(I_1))$, $\Delta_1 = B_1 \cap \partial\Omega$ and A_1 be a point in Ω satisfying $8c_3 d(I_1) \leq |A_1 - Z_1| \leq cd(I_1)$ and $\delta(A_1) \cong d(I_1)$. Following the argument before, we conclude that

$$\begin{aligned}\mu(I_1 \cap L) &\cong \omega(P, I_1 \cap \Omega, \Omega \backslash I_1) \cong \omega(P, \Delta_1, \Omega) \\ &\cong G(P, A_1) l(I)^{m-2} \cong \omega(P, S, \Omega \backslash S) \leq c\mu(I \cap L).\end{aligned}$$

This proves (2.10) and Lemma 6.

The extension of $\frac{G(x)}{\delta(x)}|_L$ to Γ follows from the next lemma.

LEMMA 7. *Let L be a uniform domain in $\mathbb{R}^n$ and σ be the Lebesgue measure on $\mathbb{R}^n$. Let μ be a measure on L which is absolutely continuous with respect to σ, and satisfies the restricted doubling property on $\overline{L}$:*

$$\mu(2I \cap L) \leq c\mu(I \cap L)$$

for any cube I centered in $\overline{L}$, and the restricted A_∞ property on L: there exist $\alpha, \beta \in (0, 1)$ so that if I is a cube centered in $\overline{L}$ and $F \subseteq I$, then

$$\frac{\sigma(F)}{\sigma(I \cap L)} > \alpha \Rightarrow \frac{\mu(F)}{\mu(I \cap L)} > \beta.$$

Then μ can be extended to $\mathbb{R}^n$ so that $\mu \ll \sigma$, μ has the doubling property and $\mu \in A_\infty(d\sigma)$ on $\mathbb{R}^n$.

Proof. Let $\mathscr{C} = \{Q_k\}$ be a dyadic Whitney decomposition of L, $\mathscr{C}' = \{T_j\}$ be a dyadic Whitney decomposition of $\mathbb{R}^n \backslash \overline{L}$, and Q_1 be one of the largest cubes in $\mathscr{C}$. Following Jones ([**11**] and [**12**]), we define the reflection $\widetilde{T}_j$ of $T_j \in \mathscr{C}'$ as follows: If L is unbounded, $\widetilde{T}_j$ is chosen to be a cube Q_k in $\mathscr{C}$ nearest to T_j and that $l(Q_k) \geq l(T_j)$; if L is bounded, define $\widetilde{T}_j$ as above provided that $l(T_j) \leq l(Q_1)$, otherwise define $\widetilde{T}_j = Q_1$. Because L is a uniform domain, $\operatorname{dist}(T_j, \widetilde{T}_j) \leq cl(T_j)$ and that $l(T_j) \cong l(\widetilde{T}_j)$ unless $l(T_j) > l(Q_1)$. See [**11**] and [**12**] for detailed properties of this reflection.

Because L is a uniform domain, $\sigma(\partial L) = 0$ ([**12**]). Extend μ to $\mathbb{R}^m$ by defining $\mu(\partial L) = 0$ and

$$d\mu = \frac{\mu(\widetilde{T}_j)}{\sigma(\widetilde{T}_j)}\, d\sigma \quad \text{on } T_j.$$

The proof of the doubling property and the A_∞ property of μ is based on the following observation: let I be a dyadic cube that meets ∂L; then either $I \cap L$ or $I\backslash\overline{L}$ contains a large Whitney cube. More precisely, if $\frac{1}{3}I \cap \overline{L} \neq \varnothing$, then due to the fact that L is uniform, there exists a Whitney cube $Q_k \subseteq I \cap L$ with $l(Q_k) \geq cl(I)$; otherwise $\frac{1}{3}I \subseteq \mathbb{R}^n\backslash\overline{L}$, and hence there exists $T_j \in \mathscr{C}'$ so that $T_j \subseteq I\backslash\overline{L}$ and $l(T_j) \geq cl(I)$. The rest of the proof is routine verification.

3. Proof of Theorem 2. Let $\Omega = \Phi(B(0, 1))$, where $\Phi\colon \mathbb{R}^m \to \mathbb{R}^m$ is K-quasiconformal and $P = \Phi(0)$. When $m = 2$, Theorem 2 follows from Theorem B. We assume that $m \geq 3$ and constants depend on K, $\operatorname{dist}(P, \Gamma)$, and $\operatorname{dist}(P, \partial\Omega)$ only.

Assume $\Gamma = \{x_m = 0\}$ and $0 \in \Gamma \cap \Omega$. Let $\mathscr{C} = \{S_j\}$ be the partition of $\Gamma \cap \Omega$ in §2, M be the integer satisfying $32 \operatorname{diam}\Omega \leq 2^M < 64 \operatorname{diam}\Omega$, and D be the $(m-1)$-dimensional square on Γ centered at 0 with sides parallel to the axes and of length 2^{M+1}. Let $\Omega' = \mathbb{R}^m\backslash\overline{\Omega}$ and $\mathscr{C}' = \{R_j\}$ be a partition of $\Gamma\cap\Omega'$ by dyadic squares with mutually disjoint interiors so that

$$0 < c < \frac{l(R_j)}{\delta(R_j)} \leq \frac{1}{10}$$

and $D\backslash\overline{\Omega} = \bigcup_{K_0} R_j$ for a subcollection $\{R_j\}_{K_0}$ of $\mathscr{C}'$.

Let Φ^* be the quasiconformal reflection about $\partial\Omega$ defined in (1.8), X_j be the center of R_j and $X_j^* = \Phi(X_j)$. Define μ on Γ so that

$$\mu(S) = \begin{cases} \int_S \frac{G(x)}{\delta(x)}\, dx, & S \subseteq \Gamma \cap \Omega, \\ \sum_j \frac{G(P, X_j^*)}{d(R_j)} \sigma(S \cap R_j), & S \subseteq D \cap \Omega', \\ \omega(P, S, \Omega), & S \subseteq \Gamma \cap \partial\Omega, \\ \sigma(S), & S \subseteq \Gamma\backslash D. \end{cases}$$

Let $U_j = B(X_j, \frac{1}{10}l(R_j))$, $V_j = U_j \cap \Gamma$ and $\{R_j\}_K$ be a subcollection of $\{R_j\}_{K_0}$. We note that $\{V_j^*\}_K$ lie on a quasisphere; and claim that $\{V_j^*\}_K$ are uniformly separated, that is,

$$\inf_K \inf_{x \in V_j^*} \omega(x, \partial\Omega, \Omega_j'') > c > 0 \tag{3.1}$$

where $\Omega_j'' = \Omega\backslash \bigcup_{k\neq j, j\in K} V_k^*$.

To prove this, we fix $j \in K$ and recall that $\delta(R_j) \cong l(R_j) \leq C \operatorname{diam}\Omega$. Recall also that Ω is a quasiball thus an NTA domain and that $\operatorname{dist}(P, \partial\Omega) > c \operatorname{diam}\Omega$. From these facts and elementary geometry, we may find a circular cylinder $H_j \subseteq \mathbb{R}^m \backslash \Gamma$, whose base has radius $r_j \cong l(R_j)$ and whose height is $h_j r_j$ $(3 < h_j < C)$, joining U_j to Ω. Moreover, we may require one base E_j lying in Ω, and the point A_j which is on the axis of H_j and of distance r_j to the other base, lying in $U_j \backslash \Gamma$. Because $H_j \cap \Gamma = \varnothing$, we have $H_j^* \cap \Omega \subseteq \Omega_j''$. Applying Lemma 5 to Φ^*, H_j, h_j, we obtain from the maximum principle that

$$\begin{aligned}\omega(A_j^*, \partial\Omega, \Omega_j'') &\geq \omega(A_j^*, \partial\Omega \cap H_j^*, H_j^* \cap \Omega)\\ &\geq \omega(A_j^*, E_j^* \cap H_j^*, H_j^*) > c > 0.\end{aligned}$$

In view of Lemmas 1 and 5, we conclude (3.1) by applying the Harnack inequality to $\omega(x, \partial\Omega, \Omega_j'')$ on U_j^*.

Therefore Theorem C implies that

$$\begin{aligned}(3.2) \qquad &\sum_K \omega(x, V_j^*, \Omega \backslash V_j^*)\\ &\cong \omega\left(x, \bigcup_K V_j^*, \Omega \backslash \bigcup_K V_j^*\right) \quad \text{for } x \in \Omega \backslash \bigcup_K V_j^*.\end{aligned}$$

Also note from (3.2), Lemmas 1 and 5 and the Harnack inequality that

$$\begin{aligned}\mu(R_j) &\cong G(P, X_j^*)(d(R_j))^{m-2}\\ &\cong \omega(P, U_j^*, \Omega \backslash U_j^*) \cong \omega(P, V_j^*, \Omega \backslash V_j^*).\end{aligned}$$

The last equivalence relation holds because $\omega(x, V_j^*, \Omega \backslash V_j^*) > c > 0$ on U_j^*.

Let I be a dyadic square in D. Then either $I \subseteq S_{j_0}$ for some $S_{j_0} \in \mathscr{C}$ or $I \subseteq R_{j_1}$ for some $R_{j_1} \in \mathscr{C}'$ or

$$(3.3) \qquad I = (I \cap \partial\Omega) \cup \bigcup_J S_j \cup \bigcup_K R_j$$

for some $\{S_j\} \subseteq \mathscr{C}$ and $\{R_j\}_K \subseteq \mathscr{C}'$. In the first two cases, by the Harnack inequality,

$$\frac{\mu(F)}{\mu(I)} \cong \frac{\sigma(F)}{\sigma(I)} \quad \text{for any } F \subseteq I.$$

We proceed with the assumption of (3.3), and denote by

$$I_* = (I \cap \partial\Omega) \cup \bigcup_J S_j \cup \bigcup_K R_j^*.$$

Let Z be a point on $\partial\Omega$ so that $\operatorname{dist}(Z, I) = \delta(I)$. Because $\Omega = \Phi(B(0, 1))$, in view of Lemmas 1 and 5 we may find $c_5 > 0$ so that $I_* \cup I \subseteq B(Z, c_5 l(I))$; let $B \equiv B(Z, 4c_5 d(I))$, $\Delta = B \cap \partial\Omega$ and A be a point in Ω so that $\delta(A) \cong l(I)$ and $8c_5 d(I) \le |A - Z| \le Cl(I)$.

Since Ω is NTA, it follows from the argument for (2.4) that

$$\omega(P, V_j^*, \Omega\backslash V_j^*) \cong \omega(P, \Delta, \Omega)\omega(A, V_j^*, \Omega\backslash V_j^*). \tag{3.4}$$

We claim that there exist $\alpha, \beta \in (0, 1)$ so that if $F \subseteq I$,

$$\frac{\mu(F)}{\mu(I)} < \alpha \Rightarrow \frac{\sigma(F)}{\sigma(I)} < \beta. \tag{3.5}$$

Assume first that F is in one of the three forms: (1) $F \subseteq I \cap \partial\Omega$, (2) $F = \bigcup_{\widetilde{J}} S_j$ for some $\widetilde{J} \subseteq J$ or (3) $F = \bigcup_{\widetilde{K}} R_j$ for some $\widetilde{K} \subseteq K$.

If F is in the form (1) or (2), we deduce from theorems in **[9]** or arguments in §2 respectively, that

$$\mu(F) \cong \omega(P, F, \Omega\backslash F) \cong \omega(P, \Delta, \Omega)\omega(A, F, \Omega\backslash F).$$

If F is in the form (3), then it follows from (3.2), (3.4) and the Harnack inequality that

$$\begin{aligned} \mu(F) &\cong \sum_{\widetilde{K}} \omega(P, V_j^*, \Omega\backslash V_j^*) \\ &\cong \omega(P, \Delta, \Omega)\omega\left(A, \bigcup_{\widetilde{K}} V_j^*, \Omega\backslash \bigcup_{\widetilde{K}} V_j^*\right) \\ &\cong \omega(P, \Delta, \Omega)\omega(A, F^*, \Omega\backslash F^*) \\ &\cong \omega(P, \Delta, \Omega)\omega\left(A, \bigcup_{\widetilde{K}} U_j^*, \Omega\backslash \bigcup_{\widetilde{K}} U_j^*\right). \end{aligned} \tag{3.6}$$

Again the last two equivalence relations follow from

$$\omega\left(x, \bigcup_{\widetilde{K}} V_j^*, \Omega\backslash \bigcup_{\widetilde{K}} V_j^*\right) > c > 0$$

on F^* and on $\bigcup_{\widetilde{K}} U_j^*$. Similarly,

$$\mu(I) \cong \omega(P, \Delta, \Omega)\omega(A, I_*, \Omega\backslash I_*). \tag{3.7}$$

Thus

$$\frac{\mu(F)}{\mu(I)} \geq c_6\omega(A, F, \Omega\backslash F) \quad \text{or} \quad c_6\omega(A, F^*, \Omega\backslash F^*) \tag{3.8}$$

depending on $F \subseteq \overline{\Omega}$ or $F \subseteq \Omega'$.

If $F \subseteq I \cap \partial\Omega$ and $\mu(F)/\mu(I) < \alpha$, then $\omega(A, F, \Omega) < c_6^{-1}\alpha$. Following the proof that ω is A_∞ with respect to the surface measure on the boundary of a BMO_1 domain [10, p. 133], we obtain

$$\frac{\sigma(F)}{\sigma(I)} < 1 - c_7 + c_7^{-1}(c_6^{-1}\alpha)^\lambda,$$

where $0 < c_7 < 1$ and $\lambda > 0$ depend only on the BMO_1 constant of Ω. Thus, if α is sufficiently small, $\sigma(F)/\sigma(I) < 1 - c_7/2$.

In the case $F = \bigcup_{\widetilde{J}} S_j$, $\sigma(F) \cong M_{m-1}(F)$ because F is contained in an $(m-1)$-dimensional hyperplane Γ. In view of Theorem E, $\sigma(F)/\sigma(I) < c_7/4$ if $\mu(F)/\mu(I)$ is sufficiently small.

When $F = \bigcup_{\widetilde{K}} R_j$, (3.5) would follow from Theorem E if we could prove that

$$M_{m-1}(F^*) \geq c\sigma(F). \tag{3.9}$$

In view of the examples in [14], [16] and [18] on contents, it is not clear whether (3.9) is true. We shall apply Theorem F, and define a measure ν on $E \equiv \bigcup_{\widetilde{K}} U_j^*$ with support $\bigcup_{\widetilde{K}}\{X_j^*\}$, so that

$$\nu(\{X_j^*\}) = l(R_j)^{m-1}.$$

Clearly $\nu(\bigcup_{\widetilde{K}} U_j^*) \cong \sigma(F)$. We claim that $c\nu$ is in the class $\mathscr{M}$ defined in Theorem F.

In fact, let Q be a cube in Ω satisfying $16d(Q) \leq \delta(Q) \leq 256d(Q)$. If $X_j^* \in Q$ for some j, then by Lemma 4, $d(Q) \cong \delta(Q) \cong \delta(X_j^*) \cong d(U_j^*) \cong d(R_j)$. Since each U_j^* contains a ball of diameter comparable to $d(U_j^*)$, there are at most C distinct X_j^*'s in Q; thus $\nu(Q) \leq Cd(Q)^{m-1}$. Moreover, if $X_j^* \in Q$, then $\operatorname{cap}(Q \cap U_j^*) \cong d(U_j^*)^{m-2} \cong d(Q)^{m-2}$. Hence

$$\nu(Q) \leq c\operatorname{cap}(Q \cap E)l(Q).$$

Next, let Q be a cube that meets $\partial\Omega$, and note from Lemma 4 that $d(\Phi^*(Q)) \leq cd(Q)$. Note also that if $X_j^* \in Q$ then $X_j \in \Phi^*(Q \cap \Omega)$ and $\delta(R_j) \cong \delta(X_j) \leq d(\Phi^*(Q))$. Thus $\operatorname{dist}(R_j, \Phi^*(Q\cap\Omega)) \leq d(R_j) + d(\Phi^*(Q)) \leq c\delta(R_j) + d(\Phi^*(Q)) \leq cd(\Phi^*(Q)) < cd(Q)$. Therefore

$$\nu(Q) = \sum_{X_j^* \in Q} l(R_j)^{m-1} \leq cd(Q)^{m-1}.$$

This shows that $c\nu \in \mathscr{M}$ for some $c > 0$. We conclude from Theorem F that

$$(3.10) \qquad \omega(A, E, \Omega\backslash E) \geq c\left(\frac{\nu(E)}{\delta(A)^{m-1}}\right)^{\gamma} \geq c\left(\frac{\sigma(F)}{\sigma(I)}\right)^{\gamma}.$$

Recall from (3.6) that $\omega(A, E, \Omega\backslash E) \cong \omega(A, F^*, \Omega\backslash F^*)$. Thus in view of (3.8) and (3.10), $\sigma(F)/\sigma(I) < c_7/4$ if $\mu(F)/\mu(I)$ is sufficiently small.

To obtain (3.5) for general F, we follow the corresponding arguments in §2.

It follows from (3.5) that for dyadic $I \subseteq D$

$$(3.11) \qquad \omega(A, I_*, \Omega\backslash I_*) > c > 0.$$

We extend (3.5) to all squares $I \subseteq D$ by the *doubling property*: let I be a dyadic square in D,

$$(3.12) \qquad \mu(2I) \leq c\mu(I).$$

In fact, when $5I \cap \partial\Omega = \varnothing$, (3.12) follows from the Harnack inequality; when $5I \cap \partial\Omega \neq \varnothing$, (3.12) follows from (1.4), (3.7) and (3.11).

To obtain (3.5) for all squares $I \subseteq \Gamma$, we use the facts that $\mu(D) \cong 1$ and $d\mu/d\sigma \cong 1$ on $\mathbb{R}^m\backslash\frac{1}{4}D$. This completes the proof of Theorem 2.

4. The example. The construction is given in $\mathbb{R}^2$ for simplicity; it can easily be extended to $\mathbb{R}^m$, $m \geq 3$. If one is only interested in an example in $\mathbb{R}^2$, some steps can be further reduced.

Let $Y_{k,p}$ be the point $((p+\frac{1}{2})/2^k, \frac{3}{4}/2^k)$ in $\mathbb{R}^2$ and $B_{k,p}$ be the disk $B(Y_{k,p}, 2^{-k-10})$ for any integers k and p. Let

$$\Omega_0 = \{x: 0 < x_1 < 1, 0 < x_2 < 1\}\backslash \bigcup_{k,p} \overline{B}_{k,p}$$

and note that Ω_0 is an NTA domain. Note also that $\bigcup_{k,p} \overline{B}_{k,p}$ does not meet any line $x_2 = 2^{-k}$ or any line segment $\{x: x_1 = p/2^k$ and $0 \leq x_2 \leq 2^{-k}\}$.

Let sequences $\{\delta_n\}$ and $\{A_n\}$ be given so that $\{\delta_n\} \subseteq \{2^{-k}: k$ positive integer$\}$, $\lim \delta_n = 0$, $A_n > 0$ and $\lim A_n = \infty$. Let $\{\lambda_n\} \subseteq \{2^{-k}: k$ positive integer$\}$ be another sequence with $\lambda_n < \delta_n 2^{-10}$. We shall construct a domain $\Omega \subseteq R^2$, by adding another part in the lower half-plane and restoring some of the disks $\overline{B}_{k,p}$ which were originally removed. For each $n \geq 1$, let

$$S_n = \{(x_1, 0): 2^{-n} \leq x_1 \leq 2^{-n+1}\},$$
$$U_n = \{x: (x_1, 0) \in S_n,\ -\lambda_n 2^{-n} < x_2 < \delta_n 2^{-n}\},$$
$$V_n = \{x: (x_1, 0) \in (1-2\delta_n)S_n,\ \lambda_n 2^{-n} \leq x_2 \leq \delta_n 2^{-n-3}\},$$

where $(1-2\delta_n)S_n$ is the interval on $x_2 = 0$ concentric to S_n of length $(1-2\delta_n)2^{-n}$, and $W_n = U_n \backslash V_n$; and note that ∂W_n does not meet $\bigcup_{k,p} \overline{B}_{k,p}$. Let

$$\Omega = \text{interior of } \left(\Omega_0 \cup \bigcup_1^\infty W_n\right),$$

and P be the point $(\frac{1}{2}, \frac{9}{10})$. Then Ω is an NTA domain.

Denoting by $I_n = (1-2\delta_n)S_n$ and $J_n = (1-\delta_n)S_n \backslash I_n$, we have the following lemma.

Lemma 9. *Given $n \geq 1$, λ_n can be chosen sufficiently small depending on A_n and δ_n only, so that*

$$\omega(P, J_n, \Omega\backslash J_n) \geq A_n \omega(P, I_n, \Omega\backslash I_n).$$

Assume Lemma 9 for the moment and let $\Gamma = \{x_2 = 0\}$. Then $\Gamma \cap \Omega$ is the unit interval on Γ and $\delta(x) = \lambda_n 2^{-n}$ for $x \in I_n \cup J_n$. From the reasoning in §2, we note that $\omega(P, J_n, \Omega\backslash J_n) \cong \mu(J_n)$ and $\omega(P, I_n, \Omega\backslash I_n) \cong \mu(I_n)$ where μ is defined in (2.6). Thus

$$\mu(J_n) \geq (1 - CA_n^{-1})\mu(I_n \cup J_n),$$

while

$$\sigma(J_n) \leq 2\delta_n \sigma(I_n \cup J_n)$$

for all $n \geq 1$. Thus $\mu \notin A_\infty(d\sigma)$ on $\Gamma \cap \Omega$.

It remains to prove Lemma 9. Fix $n \geq 1$ and let $P_1 = (2^{-n}, 0)$ and $P_2 = (2^{-n+1}, 0)$ be the end points of S_n, and $P_3 = (2^{-n} + \delta_n 2^{-n}, 0)$, $P_4 = (2^{-n+1} - \delta_n 2^{-n}, 0)$, $P_5 = (2^{-n} + \delta_n 2^{-n-1}, 0)$ and $P_6 = (2^{-n+1} - \delta_n 2^{-n-1}, 0)$ be the end points of the two intervals in J_n. Note that $J_n = \overline{P_5 P_3} \cup \overline{P_4 P_6}$ and $I_n = \overline{P_3 P_4}$. Let $P_7 = P_5 - (0, \lambda_n 2^{-n})$, $P_8 = P_6 - (0, \lambda_n 2^{-n})$, $P_9 = P_5 + (0, \delta_n 2^{-n-1})$ and $P_{10} = P_5 + (0, \delta_n 2^{-n-1})$.

In view of the Markov property, it suffices to show that if λ_n is sufficiently small then

$$\omega(x, J_n, \Omega\backslash J_n) \geq A_n \omega(x, I_n, \Omega\backslash I_n) \tag{4.1}$$

for $x \in \overline{P_7P_9} \cup \overline{P_9P_{10}} \cup \overline{P_{10}P_8}$. Let D be the domain $\Omega \cap U_n$ and T be the domain $\Omega \cup \{x, x_1 \notin S_n\}$, and note that their configurations are independent of δ_j, A_j and λ_j for any $j \neq n$. In view of the maximum principle, it is enough to show that for sufficiently small λ_n,

$$\omega(x, J_n, D\backslash J_n) \geq A_n \omega(x, I_n, T\backslash I_n) \tag{4.2}$$

for $x \in \overline{P_7P_9} \cup \overline{P_9P_{10}} \cup \overline{P_{10}P_8}$.

Consider first $x \in \overline{P_5P_7}$; and let $P_{11} = P_1 - (0, \lambda_n 2^{-n})$, $P_{13} = P_3 - (0, \lambda_n 2^{-n})$, H be the rectangle $P_1P_3P_{13}P_{11}$ and M be the semi-infinite strip $\{x: 2^{-n} < x_1 < 2^{-n} + \delta_n 2^{-n},\ x_2 > -\lambda_n 2^{-n}\}$. It is easy to see that there exists ξ_n, $0 < \xi_n < \delta_n 2^{-10}$, depending only on δ_n and A_n, such that if $0 < \lambda_n \leq \xi_n$, then

$$\omega(x, \overline{P_5P_3}, H) \geq A_n \omega(x, \partial M \backslash \overline{P_{11}P_{13}}, M)$$

for $x \in \overline{P_5P_7}$. From the maximum principle, we obtain (4.2) for $x \in \overline{P_5P_7}$ provided that $0 < \lambda_n \leq \xi_n$. Similarly (4.2) holds on $\overline{P_6P_8}$ under the same assumptions.

Denote by $K = \overline{P_5P_9} \cup \overline{P_9P_{10}} \cup \overline{P_{10}P_6}$; it remains to prove (4.2) for $x \in K$. We note that

$$\omega(x, J_n, D\backslash J_n) > \tau_n > 0, \quad x \in K$$

for some τ_n depending only on δ_n.

Let γ_n be a number in the form 2^{-k} with $0 < \gamma_n < \delta_n 2^{-10}$, $P_{15} = P_3 + (\gamma_n 2^{-n}, 0)$ and $P_{16} = P_4 - (\gamma_n 2^{-n}, 0)$. The number γ_n can be chosen sufficiently small, depending on δ_n, A_n and ξ_n only, so that if $0 < \lambda_n \leq \xi_n$,

$$\omega(x, \overline{P_3P_{15}} \cup \overline{P_{16}P_4}, T\backslash(\overline{P_{13}P_{15}} \cup \overline{P_{16}P_4})) < \tau_n/(10A_n) \tag{4.3}$$

for $x \in K$. (First choose and fix γ_n so that (4.3) holds when $\lambda_n = \xi_n$; then extend (4.3) to $0 < \lambda_n < \xi_n$ by the maximum principle.)

To complete the proof, it remains to show that for sufficiently small λ_n,

$$\omega(x, \overline{P_{15}P_{16}}, T\backslash\overline{P_{15}P_{16}}) < \tau_n/(10A_n) \quad \text{on } K. \tag{4.4}$$

Assume that $\lambda_n < 2^{-10}\min\{\xi_n, \gamma_n\}$, and let $R_0 = \overline{P_{15}P_{16}} = \{(x_1, 0): a \leq x_1 \leq b\}$ where $a = 2^{-n} + \delta_n 2^{-n} + \gamma_n 2^{-n}$ and $b = 2^{-n+1} - \delta_n 2^{-n} - \gamma_n 2^{-n}$. For $k \geq 1$, let R_k be the rectangle $\{x: a - \lambda_n 2^{-n+k} \leq x_1 \leq b + \lambda_n 2^{-n+k}$ and $-\lambda_n 2^{-n} \leq x_2 \leq \lambda_n 2^{-n+k}\}$. We note that T is an

NTA domain. By the exterior corkscrew condition of T, there exists a constant ε, $0 < \varepsilon < 1$, independent of k, so that

$$\omega(x, \partial R_k \cap T, T \backslash R_k) < \varepsilon \quad \text{on } \partial R_{k+1} \cap T$$

provided that $2^{k+5} \leq \gamma_n \lambda_n^{-1}$. From the Markov property it follows that

$$\omega(x, \overline{P_{15}P_{16}}, T \backslash \overline{P_{15}P_{16}}) < \varepsilon^{\log_2(\gamma_n/\lambda_n)-6}$$

for $x \in K$. Therefore (4.4) holds if λ_n is sufficiently small, depending only on δ_n and A_n. This completes the proof of Lemma 9.

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