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NON-UNIQUENESS OF THE METRIC IN LORENTZIAN MANIFOLDS

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This paper is concerned with the correspondence between a Lorentzian metric and its Levi-Civita connection. Although each metric determines a unique compatible symmetric connection, it is possible for more than one metric to engender the same connection. This non-uniqueness is studied for metrics of arbitrary signature and for Lorentzian metrics is shown to arise either from a de Rham-Wu decomposition or a local parallel null vector field. A key ingredient in the analysis is the construct of a submersive connection in which a connection passes to a quotient space. Finally, two examples of metrics are given, the first of which shows that the metric may be non-unique even though a null vector field exists only locally. The second example indicates that for metrics of higher signature non-uniqueness need not result from the existence of a de Rham decomposition or parallel null vector fields.

The fundamental lemma of Riemannian or pseudo-Riemannian geometry asserts that a non-degenerate metric g determines a unique compatible symmetric connection, the so-called Levi-Civita connection. Nonetheless, it is possible for more than one metric to engender the same connection. For example, suppose that ∇ is the Levi-Civita connection of a metric g on a manifold M. (By the term "metric" shall be meant a symmetric type $(0, 2)$ non-degenerate tensor of arbitrary signature, although the main concern of this paper will be with Lorentzian metrics.) Suppose, further, that K is a vector field on M which is parallel with respect to ∇, that is, $\nabla_X K$ is zero for all vector fields X on M. Denote by α the 1-form dual to K by g and by $\alpha \odot \alpha$ its symmetric square. Then provided it is non-degenerate, $g + \lambda\alpha \odot \alpha$, where $\lambda \in \mathbb{R}$, is another metric which has ∇ as its Levi-Civita connection. The main concern of this article is to give a description of all possible metrics that are compatible with the Levi-Civita connection of a Lorentz metric.

Another situation in which there is non-uniqueness in the metric description is when (M, g) admits a de Rham decomposition. In that case M is diffeomorphic to a product $M_1 \times M_2$ of manifolds and g

decomposes as a sum $g_1 + g_2$ with g_1 and g_2 metrics on M_1 and M_2, respectively. Then provided it is non-degenerate, $\lambda_1 g_1 + \lambda_2 g_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$, yields the same connection as $g_1 + g_2$. In the case of Riemannian manifolds, that is, where g is positive definite, whenever the holonomy group of g leaves invariant a proper non-trivial subspace it leaves the complementary subspace invariant and hence (M, g) admits a local de Rham decomposition. It follows easily in the positive definite case that g is unique up to a multiplicative constant if it admits no local de Rham decomposition.

The situation for indefinite metrics, however, is different. If the holonomy group of an indefinite metric g leaves invariant a subspace that is non-degenerate then (M, g) will admit a de Rham decomposition as before. See **[13, 14]**. It may happen though that the holonomy group leaves invariant a degenerate subspace in which case the existence of a local de Rham decomposition is not guaranteed. One refers to these contrasting situations as non-degenerate and degenerate decomposability of the holonomy representation, respectively.

This article is mainly devoted to investigating the uniqueness problem for metrics in the context of Lorentzian manifolds. It will be apparent that many of the results will be applicable to metrics of other signatures, but the more general problem is considerably more complicated. The main result which will be proved is as follows:

THEOREM. *Let (M, g) be a simply connected Lorentzian manifold and ∇ its Levi-Civita connection. Suppose that (M, g) admits no local de Rham decomposition but admits an alternative metric g' that is not just a constant multiple of g. Then (M, g) has a parallel null vector field K and g' is of the form $g + \lambda\alpha \odot \alpha$, where $\lambda \in \mathbb{R}$ and α is the 1-form dual to K via g.*

It should be pointed out that other authors have studied the non-uniqueness problem for metrics **[1–5, 9, 12]**. Most noteworthy in this regard perhaps are the results of Hall and his co-workers **[1–4]**. For the most part Hall is concerned with 4-dimensional Lorentzian manifolds. He uses Schell's classification of the subalgebras of $o(3, 1)$ **[7]** to study the correspondence between metric and curvature. In particular Hall has already established the validity of the main result proved here in the special case of 4 dimensions. However, his techniques do not lend themselves readily to n dimensions let alone metrics of higher signature.

In §2 we state two results from linear algebra concerning a pair of quadratic forms. Section 3 gives the necessary background in differential geometry and studies in particular submersive connections. Section 4 gives the proof of the main theorem and considers briefly the uniqueness problem for metrics of higher signature. Throughout the paper it will be assumed that all geometric objects are smooth, that is, of class C^∞.

2. Linear algebra of a pair of quadratic forms. In this section the linear algebra of a pair of quadratic forms g and g' on a vector space V of dimension n of which g is Lorentzian, will be investigated. It is assumed that g' is not simply a multiple of g. A similar technique has been used in **[3]**.

LEMMA 2.1. *There exists a linear combination of g and g' which is degenerate.*

Proof. The space of non-degenerate quadratic forms of a fixed signature forms an open set in the space of all quadratic forms. Consider then the 1-parameter family of forms $g(t)$ given by $tg + (1-t)g'$. We may assume of course that both g and g' are non-degenerate and further that the signatures of g and g' are different; for if g and g' both are Lorentzian with signature $(n-1, 1)$ we may replace g' by $-g'$. Thus $g(0)$ and $g(1)$ are non-degenerate and have different signatures. Let t_0 be the supremum of values of t such that, for $0 \le t \le t_0$, $g(t)$ has the same signature as g. Then $g(t_0)$ will be degenerate.

The next piece of theory we shall require is in essence due to Weierstrass **[11]**. For a more recent reference see **[8, 9]**. The general problem is that of finding a simultaneous matrix normal form for a pencil of quadratic forms $g + \lambda g'$, which contains a non-degenerate form. For our purposes we may assume that g is Lorentzian. The result we shall need is as follows:

THEOREM 2.2. *Let g be a Lorentzian quadratic form on V^n and g' a second quadratic form. Then there exists a basis of V^n relative to which g and g' after scaling g and g' and adding an appropriate multiple of g to g' correspond to the following symmetric matrices, $\lambda_1, \dots, \lambda_{n-2}$, α, β, γ and δ being real,*

$$g = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & 0 & 0 \\ & & 0 & 0 & 1 \\ 0 & & 0 & 1 & 0 \end{bmatrix}, \qquad g' = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_{n-2} & 0 & 0 \\ & & 0 & \alpha & \beta \\ 0 & & 0 & \gamma & \delta \end{bmatrix}.$$

Furthermore, either (i) $\delta = -\alpha$, $\gamma = \beta$ *with* $\alpha^2 + \beta^2 \neq 0$ (ii) $\alpha = \beta = \gamma = 0$, $\delta = 1$ (iii) $\alpha = \beta = \gamma = \delta = 0$ *or* (iv) $\alpha = \beta = \gamma = \delta = 1$.

Denoting the contravariant inverse of g by G and the linear transformation obtained by contracting G with g' by T, the above theorem can be proved by putting T into real Jordan normal form and then casting G into normal form.

3. Submersive connections. In this section we provide the background differential geometry that will be needed in the proof of the main theorem. In particular we consider submersive connections, a construction originally introduced as a special case of submersive second order vector fields. See **[6]**. The account given here will be self-contained.

Again M will denote a smooth m-dimensional manifold and we let ∇ denote a given smooth connection on M, which need not be a Levi-Civita connection. Suppose further that D is a foliation on M whose leaves are of dimension $m-n$. Then ∇ is said to be submersive if the following condition holds: whenever the vector fields X and Y are projectable to D, the vector field $\nabla_X Y$ is also projectable to M/D. [Note that this condition is stronger than saying that D is totally geodesic which would give $\nabla_X Y \in D$ whenever $X, Y \in D$.] If also the quotient space M/D has the structure of a smooth Hausdorff manifold denoted by N, then we say that ∇ is *projectable.* In that case we can define the connection $\overline{\nabla}$ on N to which ∇ projects as follows. Let $\rho\colon M \to N$ be the projection map and let X and Y be vector fields on M which are projectable to $\overline{X}$ and $\overline{Y}$ on N, respectively, that is, $\rho_* X = \overline{X}$ and $\rho_* Y = \overline{Y}$. Then $\overline{\nabla}$ is determined by

$$\overline{\nabla}_{\overline{X}} \overline{Y} = \rho_*(\nabla_X Y). \tag{3.1}$$

One may check that $\overline{\nabla}$ thus defined is a symmetric connection on N.

The following two theorems are the main results about submersive connections. The curvature tensor of ∇ is denoted by R.

THEOREM 3.1. *A symmetric connection ∇ on M is submersive if and only if there exists a distribution D on M such that for vector fields V tangent to D and X and Y arbitrary,*

(i) $\nabla_X V \in D$,

(ii) $R(V, X)Y \in D$.

Proof. First of all, let ∇ be a connection which satisfies conditions (i) and (ii). It follows easily from (i) and the fact that ∇ is symmetric, that D is integrable. What we need to establish now is that the Lie bracket $[\nabla_X Y, V]$ is tangent to D, where V is tangent to D and X and Y are both projectable to the quotient space M/D. We may then use (3.1) to define $\overline{\nabla}$ on M/D, since $\nabla_X Y$ will pass to the quotient.

Using the fact that ∇ is symmetric, we have

$$\begin{aligned}(3.2)\quad [\nabla_X Y, V] &= \nabla_{\nabla_X Y} V - \nabla_V \nabla_X Y \\ &= \nabla_{\nabla_X Y} V - R(V, X)Y - \nabla_X \nabla_V Y - \nabla_{[V, X]} Y\end{aligned}$$

by the definition of R. Once again using the symmetry of R one obtains

$$\begin{aligned}(3.3)\quad [\nabla_X Y, V] = \nabla_{\nabla_X Y} V - R(V, X)Y - \nabla_X \nabla_Y V - \nabla_Y [V, X] \\ - \nabla_X [V, X] - [[V, X], Y].\end{aligned}$$

Now (3.3), conditions (i) and (ii) above and the fact that X and Y are projectable, imply that $[\nabla_X Y, V]$ is tangent to D.

Conversely, let $(N, \overline{\nabla})$ be a manifold with symmetric connection, $\rho: M \to N$ a submersion and suppose that ∇ and $\overline{\nabla}$ are related by (3.1). Then we must show that ∇ satisfies (i) and (iii) with D being the integrable distribution corresponding to the fibres of ρ. For (i) it is enough to show that $\rho_*(\nabla_X V) = 0$ in the cases where X is projectable and vertical, respectively. But in either case, X is projectable and (3.1) implies that $\rho_*(\nabla_X V)$ is zero.

Finally, (ii) follows from (3.1) and the identity (3.3).

THEOREM 3.2. *Let ∇ be the Levi-Civita connection of some metric g and suppose that D is a distribution satisfying hypotheses* (i) *and* (ii) *in Theorem* 3.1. *Then conditions* (i) *and* (ii) *hold for the orthogonal distribution $D^\perp$ and the distributions $D \cap D^\perp$ and $D + D^\perp = (D \cap D^\perp)^\perp$.*

Proof. Suppose that W is tangent to $D^\perp$. Then by definition, if V is tangent to D,

$$(3.4)\qquad g(V, W) = 0.$$

Now differentiate (3.4) along an arbitrary vector field X to find

$$g(\nabla_X V, W) + g(V, \nabla_X W) = 0, \tag{3.5}$$

whence by (i) of Theorem 3.1

$$g(V, \nabla_X W) = 0. \tag{3.6}$$

From (3.4) we have that $\nabla_X W$ is tangent to $D^\perp$ as required.

Next suppose that V and W are tangent to D and $D^\perp$, respectively, and let X and Y be arbitrary. Then one has the identity

$$g(R(W, X)Y, V) = g(R(V, Y)X, W) \tag{3.7}$$

from which it follows by condition (ii) of Theorem 3.1 that $R(W, X)Y$ is tangent to $D^\perp$. Thus conditions (i) and (ii) hold for $D^\perp$ and it is easy to see that they apply also to $D \cap D^\perp$ and $D + D^\perp$.

Next suppose that ∇ admits a parallel, uniformly degenerate metric h, by which we mean that h is degenerate at each point but that the degenerate distribution D is smooth and of constant dimension. We shall show that D is integrable and that ∇ submerses to the quotient space M/D.

Note then that V is tangent to D if and only if for arbitrary X,

$$h(V, X) = 0. \tag{3.8}$$

Computing the Lie derivative of (3.8) along a second field, Y gives

$$h(\nabla_Y V, X) = 0 \tag{3.9}$$

using (3.8) and the fact that h is compatible with ∇. From (3.9), we obtain condition (i) of Theorem (3.1) and hence also that D is integrable.

We note next that the following identity holds despite the fact that h is degenerate:

$$h(R(V, X)Y, Z) = h(R(Y, Z)V, X). \tag{3.10}$$

Taking V tangent to D, (3.10) shows that condition (ii) of Theorem (2.1) holds. Thus ∇ submerses to the quotient space M/D and indeed the connection on M/D is the Levi-Civita connection of the metric induced by h on M/D, assuming that M/D has the structure of a smooth manifold.

We remark finally in this section that Theorem 3.1 can be used to derive a proof of the local de Rham-Wu Theorem, which is simple and instructive.

4. Proof of the main theorem. We can now proceed to the proof of the main theorem as stated in §1. We may assume that g is a Lorentzian metric and that h is a uniformly degenerate metric on M. The degenerate distribution of h will be denoted by D. If (M, g) is not to be a de Rham product then D must be a degenerate distribution, that is, g restricted to D is singular. The one-dimensional distribution $D \cap D^{\perp}$ is parallel in the sense that the line field is invariant by parallel transport. However, something stronger is true.

PROPOSITION 4.1. *The distribution $D \cap D^{\perp}$ is spanned locally by a parallel, null vector field K and if M is simply connected K spans $D \cap D^{\perp}$ globally.*

Proof. If $D \cap D^{\perp}$ is spanned locally by K, it is sufficient to show that $R(X, Y)K$ is zero, where X and Y are arbitrary vector fields on M. For in that case K may be scaled by a function so as to obtain a parallel, null vector field.

In the neighborhood of some point choose a vector field L which satisfies $g(K, L) = 1$. Then L is complementary to $D + D^{\perp}$. It will suffice to show for X, Y and Z arbitrary that $g(Z, R(X, Y)K)$ is zero. This is certainly so if $Z \in D + D^{\perp}$ because by submersiveness, $R(X, Y)K \in D \cap D^{\perp}$. Similarly, if X or $Y \in D + D^{\perp}$ we have $g(Z, R(X, Y)K) = -g(K, R(X, Y)Z)$ and again this latter term is zero by submersiveness. Finally since $R(L, L)K$ is zero by skew-symmetry it follows generally that $R(X, Y)K$ is zero.

The condition that $R(X, Y)K$ is zero ensures that the line bundle $D \cap D^{\perp}$ can be spanned locally by a parallel null vector field. If in addition M is simply connected, then $D \cap D^{\perp}$ is orientable and a global parallel null field exists.

We consider now the endomorphism field $T = G \circ h$ (the contraction of G with h, G being the cometric dual to g). Clearly T is a parallel tensor field. We shall need the following lemma.

LEMMA 4.2. *The eigenvalues of T are constant.*

Proof. The tensor T induces an endomorphism of the k-vector bundle $\bigwedge^k(TM)$ where $1 \leq k \leq n$. Denote the corresponding tensor field on M by $\bigwedge^k T$. For any vector field X on M we have, since ∇ commutes with contractions,

$$X\left(\operatorname{Tr}\left(\bigwedge^k T\right)\right) = \operatorname{Tr}\left(\nabla_X\left(\bigwedge^k T\right)\right) \tag{4.1}$$

where Tr denotes the trace of an endomorphism field. Since $\bigwedge^k T$ is parallel and X arbitrary we conclude that $\mathrm{Tr}(\bigwedge^k T)$ is constant. It follows that the coefficients of the characteristic polynomial of T are constant and hence the eigenvalues of T are constant.

Note that a vector field X is in the kernel of T, that is, is an eigenvector field to the eigenvalue 0 if and only $X \in D$. Suppose next that λ is a non-zero real eigenvalue of T and denote the eigendistribution associated to λ by E_λ. Thus $X \in E_\lambda$ if and only if $TX = \lambda X$. If Y is an arbitrary vector field on M, we have since T is parallel

$$T(\nabla_Y X) = \lambda \nabla_Y X, \tag{4.2}$$

and hence E_λ is a parallel distribution. Furthermore, E_λ must be non-degenerate in the sense that g restricted to E_λ is non-singular; otherwise E_λ would contain a non-zero null vector field Y. However, since λ is non-zero and

$$\lambda g(E_\lambda, D) = h(E_\lambda, D) \tag{4.3}$$

we have that $E_\lambda \subset D^\perp$ and hence Y would be a multiple of K which would contradict the assumption that $Y \in E_\lambda$.

Suppose finally that $\gamma + i\delta$ with $\delta \neq 0$ is an eigenvalue of T. Associated to $\gamma + i\delta$ is an eigendistribution and we denote by $E_{\gamma,\delta}$ the corresponding real distribution. There are pairs of vector fields X, Y which span $E_{\gamma,\delta}$ and that satisfy

$$\begin{aligned} TX &= \gamma X + \delta Y, \\ TY &= -\delta X + \gamma Y. \end{aligned} \tag{4.4}$$

Since T is parallel, it follows easily that $E_{\gamma,\delta}$ is parallel. Furthermore, (4.4) is equivalent to the equality of the 1-forms

$$\begin{aligned} \gamma g(X, -) + \delta g(Y, -) &= h(X, -), \\ -\delta g(X, -) + \gamma g(Y, -) &= h(Y, -). \end{aligned} \tag{4.5}$$

Hence, since $\gamma^2 + \delta^2$ is not zero, one has that $X \in D^\perp$ and $Y \in D^\perp$. Thus a linear combination of X and Y is in $D^\perp$ and, if null, in $D \cap D^\perp$ from which it follows that $E_{\gamma,\delta}$ is non-degenerate.

From the foregoing discussion, if (M, g) is not to split as a de Rham product, we may assume that all the eigenvalues of T are zero. Now we invoke Theorem 2.2 with g and h in a single tangent space of M, though the Jordan normal form of T applies to the whole of M. That normal form corresponds to $G \circ g'$ with g and g' as in

Theorem 2.2 where also $\lambda_1, \ldots, \lambda_{n-2}$, α, β, γ are zero and $\delta = 1$. In particular we conclude that h must be of rank one.

Since h has rank one, we have that $D^\perp \subset D$ and that $D^\perp$ is spanned by the parallel null field K. Moreover h passes to the quotient space M/D. Consider now the 1-form α which is dual to K via g. Clearly α is parallel and thus closed. Also since α annihilates any vector field tangent to D, α must pass to the quotient M/D. It follows that h and $\alpha \odot \alpha$ can differ by only a multiplicative constant. Thus the proof of the theorem is complete. We remark also that for all values of λ, $g + \lambda\alpha \odot \alpha$ are Lorentz metrics all having the same Levi-Civita connection.

We next give an example to show that the vector field K appearing above may exist only locally if M is not simply connected. On $\mathbb{R} \times S^1 \times \mathbb{R}$ with coordinates (x, θ, r) consider the metric

$$g = 2\,dx\,dr + (r^2 + 1)(2 + \sin 2\theta)(d\theta)^2. \tag{4.6}$$

Then g is invariant under the involution $(x, \theta, r) \mapsto (-x, \theta+\pi, -r)$ and hence passes to the resulting quotient space as does

$$h = (dr)^2. \tag{4.7}$$

However, by construction the distribution $D^\perp$ is not orientable and $\frac{\partial}{\partial x}$ is only a local, parallel null vector field.

It is apparent that many of the conclusions about non-uniqueness of the metric carry over from the Lorentz to the higher signature case. Thus, denoting the original metric by g and an alternative metric compatible with the Levi-Civita connection ∇ of g by h, we can argue as before that the eigenvalues of T are all zero and in particular that h is degenerate. It is reasonable to conjecture that h is always associated to a set of k, parallel, null, orthogonal, vector fields of g of which the main theorem here corresponds to the case $k = 1$. Indeed such metrics g necessarily have the following local normal form (see **[10]**),

$$\delta_{ij}\,dx^i\,dz^j + A_{\alpha\beta}\,dy^\alpha\,dy^\beta + 2H_{\alpha i}\,dy^\alpha\,dz^i + B_{ij}\,dz^i\,dz^j. \tag{4.8}$$

Here the coordinates are (x^i, y^α, z^i) with Latin indices ranging from 1 through r Greek indices from $r+1$ through $n-r$ and the summation convention on repeated indices applies. Furthermore, $A_{\alpha\beta}$ and B_{ij} are symmetric matrices which, as well as $H_{\alpha i}$, are independent of the x^i. The parallel, null, orthogonal vector fields are the $\partial/\partial x^i$ and an alternative degenerate metric is given by $C_{ij}\,dz^i\,dz^j$ where C_{ij} is a constant symmetric matrix.

Finally, we give an example of a metric which provides a counterexample to the conjecture made in the previous paragraph. On $\mathbb{R}^4$ with coordinates (x^1, x^2, x^3, x^4) the metric g is given by

$$
\begin{aligned}
(4.9) \qquad & dx^1dx^3 + dx^2dx^4 + \left(x^2\frac{\partial\lambda}{\partial x^4} - x^1\frac{\partial\lambda}{\partial x^3}\right)(dx^3)^2 \\
& - 2\left(x^1\frac{\partial\lambda}{\partial x^4} + x^2\frac{\partial\lambda}{\partial x^3}\right)dx^3dx^4 \\
& + \left(x^1\frac{\partial\lambda}{\partial x^3} - x^2\frac{\partial\lambda}{\partial x^4}\right)(dx^4)^2
\end{aligned}
$$

and the metric h is given by $e^{2\lambda}((dx^3)^2 + (dx^4)^2)$, where λ is an arbitrary smooth function of x^3 and x^4. Thus g and h have signature $(2, 2)$ and $(2, 0)$, respectively, and h passes to the two-dimensional quotient space obtained by reducing along the $\partial/\partial x^1$ and $\partial/\partial x^2$ directions. In particular, λ may be chosen so as to make h correspond to any Riemannian metric on $\mathbb{R}^2$.

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