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OPTIMAL APPROXIMATION CLASS FOR MULTIVARIATE BERNSTEIN OPERATORS

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For the Bernstein polynomial approximation process on a simplex or a cube, the class of functions yielding optimal approximation will be given. That is, we will find the class of functions for which $\|B_n f - f\|_{C(S)} = O(n^{-1})$ in terms of the behaviour of a certain K-functional. Moreover, this is done in the context of direct and converse results which yields an improvement on such results as well.

1. Introduction. For the simplex S in R^d,

$$S \equiv \left\{ x = (x_1, \dots, x_d) \colon x_i \geq 0, \ |x| \equiv \sum_{i=1}^{d} x_i \leq 1 \right\}, \tag{1.1}$$

the Bernstein polynomial approximation is given by

$$B_n f = B_n(f, x) \equiv \sum_{\mu/n \in S} P_{n,\mu}(x) f\left(\frac{\mu}{n}\right), \qquad x \in S, \tag{1.2}$$

where $\mu = (m_1, \dots, m_d)$ with m_i integers, and

$$P_{n,\mu}(x) \equiv \frac{n!}{\mu!(n-|\mu|)!} x^{\mu}(1-|x|)^{n-|\mu|}, \tag{1.3}$$

$$|x| \equiv \sum_{i=1}^{d} x_i, \quad \left(|\mu| \equiv \sum_{i=1}^{d} m_i \right)$$

with the convention

$$\mu! = m_1! \cdots m_d! \quad \text{and} \quad x^{\mu} = x_1^{m_1} \cdots x_d^{m_d}.$$

For the cube Q in R^d,

$$Q \equiv \{x = (x_1, \dots, x_d) \colon 0 \leq x_i \leq 1 \text{ for } 1 \leq i \leq d\}, \tag{1.4}$$

the Bernstein polynomial approximation is given by

$$\overline{B}_n f = \overline{B}_n(f, x) \equiv \sum_{\mu/n \in Q} \overline{P}_{n,\mu}(x) f\left(\frac{\mu}{n}\right), \qquad x \in Q, \tag{1.5}$$

where

$$\overline{P}_{n,\mu}(x) \equiv \prod_{i=1}^{d} P_{n,m_i}(x_i) \quad \text{and} \tag{1.6}$$
$$P_{n,l}(t) \equiv \frac{n!}{l!(n-l)!} t^l (1-t)^{n-l}.$$

We note that both (1.2) and (1.5) reduce to the classical Bernstein polynomials in case $d = 1$.

The class of functions for which

$$\|B_n f - f\|_{C(S)} = O(n^{-\alpha}) \quad (\text{or } \|\overline{B}_n f - f\|_{C(Q)} = O(n^{-\alpha}))$$

for $0 < \alpha < 1$ was determined by the first author [7]. Some additional articles were written in the past few years about the rate of approximation of $B_n f$ or $\overline{B}_n f$ to f (see [8] and [13]). However, the determination of the class of functions for which the optimal rate of approximation is achieved, that is, $\|B_n f - f\|_{C(S)} = O(n^{-1})$ or $\|\overline{B}_n f - f\|_{C(S)} = O(n^{-1})$ eluded investigators of the subject (including [13], the billing of which in MR 89k41007 looked promising). It is clear that the rate $O(n^{-1})$ is optimal as both B_n and $\overline{B}_n$ satisfy conditions in [6] with $\sigma_n^2 = n^{-1}$ and hence $\|B_n f - f\|_{C(S)} = o(n^{-1})$ (or $\|\overline{B}_n f - f\|_{C(Q)} = o(n^{-1})$) implies that f is locally a solution of a certain elliptic partial differential equation given below in (2.2).

Recently, the rate of convergence of the related sequence of operators, that is, the Bernstein-Durrmeyer operators (see [1], [3] and [4]), was extensively and successfully investigated. One can only hope to match the success of the investigation of the Bernstein-Durrmeyer operators as those have properties like commutativity, self adjointness and simple expansion by orthogonal polynomials. Still, we were encouraged by the above mentioned success and, below, we have a saturation theorem for the rate of convergence of $B_n f - f$ (which is a much more difficult problem).

The result below will contain a characterization, that is, a necessary and sufficient condition on f so that $\|B_n f - f\|_{C(S)} = O(n^{-1})$ or $\|\overline{B}_n f - f\|_{C(Q)} = O(n^{-1})$. The saturation result will be a consequence of a set of direct and converse inequalities, and thus, the generally difficult problem of unifying the direct-converse theorem with the saturation theorem is handled.

It is hoped though, that stronger results will eventually emerge. We will explain the possibilities and make conjectures about further, more powerful results, in §8. We suspect that it will take quite some time as well as the introduction of new techniques before these conjectures are settled.

Here, we rely heavily on the investigation of best polynomial approximation on the simplex or on the cube [**11**, Chapter 12], an introduction of a three-term K-functional and on some recent results on multivariate Bernstein and Markov inequalities [**8**]. The use of these techniques, combined with some hard work and a new notation that overcomes the need to solve the problem in two dimensions first (see [**6**] and [**2**]), led us to the solution of the problems below on which we have worked for the last few years.

2. Further notations and the main result. We first introduce the elliptic differential operators that will be crucial in our investigation. For a polytope A (that for us will be either the simplex S of (1.1) or the cube Q of (1.4)), we denote by V_A the set of unit vectors in the directions of the edges of A where e and $-e$ are considered to be the same vector. We define, for a convex set A, a direction ξ, and a point $x \in A$,

$$\begin{aligned} \varphi_\xi(x)^2 &\equiv \tilde{d}(\xi, x) \equiv \tilde{d}(A, \xi, x) \\ &\equiv \operatorname*{Inf}_{\substack{x+\lambda\xi \notin A \\ \lambda>0}} d(x, x+\lambda\xi) \operatorname*{Inf}_{\substack{x+\lambda\xi \notin A \\ \lambda<0}} d(x, x+\lambda\xi) \end{aligned} \tag{2.1}$$

where $d(x, y)$ is the Euclidean distance between x and y in R^d. The differential operators are now given by

$$\begin{aligned} P(D) &\equiv \sum_{\xi \in V_S} \tilde{d}(S, \xi, x) \left(\frac{\partial}{\partial \xi}\right)^2, \\ \overline{P}(D) &\equiv \sum_{\xi \in V_Q} \tilde{d}(Q, \xi, x) \left(\frac{\partial}{\partial \xi}\right)^2 \end{aligned} \tag{2.2}$$

where S and Q are given by (1.1) and (1.4), respectively. As $\tilde{d}(S, e_i, x) = x_i(1-|x|)$, $\tilde{d}(S, (e_i-e_j)/\sqrt{2}, x) = 2x_ix_j$, $\tilde{d}(Q, e_i, x) = x_i(1-x_i)$ and for $\xi = (e_i-e_j)/\sqrt{2}$, $\partial/\partial\xi = \frac{1}{\sqrt{2}}(\partial/\partial x_i - \partial/\partial x_j)$, we may write

$$(2.3)\qquad P(D) = \sum_{i=1}^{d} x_i(1-|x|)\left(\frac{\partial}{\partial x_i}\right)^2 + \sum_{1\le i\le j\le d} x_i x_j \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2,$$

$$\overline{P}(D) = \sum_{i=1}^{d} x_i(1-x_i)\left(\frac{\partial}{\partial x_i}\right)^2 .$$

The K-functionals used in the present paper are now given by

$$(2.4)\qquad K_S(f, t) = \operatorname*{Inf}_{g\in C^3(S)} \Big(\|f-g\|_{C(S)} + t^2\|P(D)g\|_{C(S)} + t^3 \operatorname*{Sup}_{\xi\in V_S} \|\varphi_\xi^3(\partial/\partial\xi)^3 g\|_{C(S)}\Big)$$

and

$$(2.5)\qquad K_Q(f, t) = \operatorname*{Inf}_{g\in C^3(Q)} \Big(\|f-g\|_{C(Q)} + t^2\|\overline{P}(D)g\|_{C(Q)} + t^3 \operatorname*{Sup}_{\xi\in V_Q} \|\varphi_\xi^3(\partial/\partial\xi)^3 g\|_{C(Q)}\Big)$$

where $\varphi_\xi(x)$ of (2.4) and (2.5) is given by (2.1) with $A = S$ and $A = Q$, respectively.

The main result of this paper can now be stated.

THEOREM 2.1. *For $B_n f$ and $K_S(f, t)$, given in* (1.2) *and* (2.4), *respectively, we have*

$$(2.6)\qquad \|B_n f - f\|_{C(S)} \le M(K_S(f, n^{-1/2}) + n^{-3/2}\|f\|_{C(S)})$$

and

$$(2.7)\qquad K_S(f, n^{-1/2}) \le M n^{-3/2} \sum_{k=1}^{n} k^{1/2}\|B_k f - f\|_{C(S)}$$

with M independent of f and n, and hence for $0 < \alpha \le 1$

$$(2.8)'\qquad \|B_n f - f\|_{C(S)} = O(n^{-\alpha}) \Leftrightarrow K_S(f, n^{-1/2}) = O(n^{-\alpha}).$$

We observe that in the above theorem, (2.6) and (2.7) are the direct and converse results respectively and (2.8) for $\alpha = 1$ is the saturation result.

The analogous result for the cube is given by:

THEOREM 2.2. *For $\overline{B}_n f$ and $K_Q(f, t)$ given by* (1.5) *and* (2.5), *respectively, we have*

$$(2.6)' \qquad \|\overline{B}_n f - f\|_{C(Q)} \le M(K_Q(f, n^{1/2}) + n^{-3/2}\|f\|_{C(Q)})$$

and

$$(2.7)' \qquad K_Q(f, n^{-1/2}) \le M n^{-3/2} \sum_{k=1}^{n} k^{1/2}\|\overline{B}_k f - f\|_{C(Q)}$$

with M independent of f and n, and hence for $0 < \alpha \le 1$

$$(2.8)' \qquad \|\overline{B}_n f - f\|_{C(Q)} = O(n^{-\alpha}) \Leftrightarrow K_Q(f, n^{-1/2}) = O(n^{-\alpha}).$$

We will prove the more difficult Theorem 2.1 in more detail in §§4, 5 and 6 and will comment on the necessary changes in the proof of Theorem 2.2 in §7. It is clear that (2.8) follows from (2.6) and (2.7) and that (2.8)′ follows from (2.6)′ and (2.7)′.

3. Results about polynomials. This section will be dedicated to results on polynomials. Modifications of earlier results and rephrasing for the new notation are given for a somewhat more general situation than is needed for this paper. We hope these points will be useful elsewhere too.

We denote the set of polynomials of total degree n by Π_n.

In [9], it was proved that for a bounded convex set A,

$$(3.1) \qquad \|\tilde{d}(A, \xi, \cdot)^{r/2}(\partial/\partial\xi)^r P\|_{L_p(A)} \le Cn^r\|P\|_{L_p(A)} \quad \text{for } P \in \Pi_n$$

where r is an integer, $0 < p \le \infty$, $\xi \in R^d$, $\|\xi\| = 1$, and $\tilde{d}(A, \xi, x)$ is given in (2.1). For the set S, we can prove:

THEOREM 3.1. *For S given in* (1.1), *$0 < p \le \infty$, and $\xi \in V_S$,*

$$(3.2) \qquad \|w(\cdot)\tilde{d}(\xi, \cdot)^{1/2}(\partial/\partial\xi)P\|_{L_p(S)} \le Cn\|wP\|_{L_p(S)}, \qquad P \in \Pi_n,$$

where $w(x) = x_1^{\alpha_1} \cdots x_d^{\alpha_d}(1 - |x|)^{\alpha_{d+1}}$ with $\alpha_i \ge 0$ for $p = \infty$ and $\alpha_i > -1/p$ for $p < \infty$.

For Theorem 3.1, we may deduce the following result by repeating (3.2) using $\frac{\partial}{\partial\xi}P \in \Pi_n$ and the fact that $w(x)(\tilde{d}(\xi_1, x) \cdots \tilde{d}(\xi_j, x))^{1/2}$ satisfies conditions on $w(x)$ in Theorem 3.1.

COROLLARY 3.2. *For $w(x)$, S and p of Theorem 3.1 and $\xi_i \in V_S$, we have*

$$\left\| w(\cdot)\,\tilde{d}(\xi_1,\cdot)^{1/2}\cdots\tilde{d}(\xi_k,\cdot)^{1/2}\frac{\partial}{\partial\xi_1}\cdots\frac{\partial}{\partial\xi_k}P\right\|_{L_p(S)} \le Cn^k\|wP\|_{L_p(S)}, \qquad P\in\Pi_n. \tag{3.3}$$

Proof of Theorem 3.1. In fact, we only have to prove our result for $\xi = e_1$. If $\xi = e_i$, it is clear that renaming i is sufficient. If $\xi = (e_i - e_j)/\sqrt{2}$, we write the polynomial $P(x)$ and the weight $w(x)$ as a polynomial and weight in x_r for $r \ne j$ and $1-|x|$ for the j variable. We now observe that $\partial/\partial\xi$ acts on the new variables like $2^{-1/2}\partial/\partial x_i$ and that $\tilde{d}(\xi,\cdot)^{1/2}$ contributes a factor of $\sqrt{2}$ multiplied by $d(e_i,\cdot)^{1/2}$ in the new variables. We note also that this transformation was used extensively in [7] and later in [8] and [2] for similar purposes. Using iterated integration and the notation $|\tilde{x}| \equiv x_2 + \cdots + x_d$, it is clear that all that we have to show is

$$\operatorname*{Sup}_{0\le x_1\le 1-|\tilde{x}|}\left|x_1^{\alpha+1/2}(1-|x|)^{\beta+1/2}\frac{\partial}{\partial x_1}P(x_1,\dots,x_d)\right| \le Cn \operatorname*{Sup}_{0\le x_1\le 1-|\tilde{x}|} x_1^{\alpha}(1-|x|)^{\beta}|P(x_1,\dots,x_d)| \tag{3.4}$$

and

$$\int_0^{1-|\tilde{x}|}\left|x_1^{\alpha+1/2}(1-|x|)^{\beta+1/2}\frac{\partial}{\partial x_1}P(x_1,\dots,x_d)\right|^p dx_1 \le Cn^p\int_0^{1-|\tilde{x}|}\left|x_1^{\alpha}(1-|x|)^{\beta}\frac{\partial}{\partial x_1}(x_1,\dots,x_d)\right|^p dx_1 \tag{3.5}$$

for $0<p<\infty$. We regard $x_2,\dots,x_d$ as constants and make the change of variable $y = x_1/(1-|\tilde{x}|)$ and hence

$$1-y = (1-|x|)/(1-|\tilde{x}|) \qquad ([0,1-|\tilde{x}|]\to[0,1])$$

to obtain the result for $w^*(y)\frac{\partial}{\partial y}P^*(y)$ from [**11**, Theorem 8.4.7] for $1\le p\le\infty$ and [**14**, Theorem 5] for $0<p<\infty$. (Both references have the interval $[-1,1]$ rather than $[0,1]$ as the underlying interval but that does not create any problems.) $\square$

We now recall a result about best polynomial approximation that will be crucial for the present paper. The rate of best polynomial approximation on a set S is denoted by

$$E_n(f)_{L_p(S)} \equiv \operatorname*{Inf}_{P\in\Pi_n}\|f-P\|_{L_p(S)}. \tag{3.6}$$

We further define K-functionals on the simple polytope S. We recall that a polytope (convex hull of finitely many points) in R^d is simple if it has an interior point and each vertex is joined to other vertices by exactly d edges.

$$(3.7)\quad K_{r,S}(f, t^r)_p = \operatorname*{Inf}_{g\in C^r(S)} \left(\|f-g\|_{L_p(S)} + t^r \operatorname*{Sup}_{\xi\in V_S} \|\varphi_\xi^r (\partial/\partial\xi)^r g\|_{L_p(S)} \right)$$

where $\varphi_\xi(x)$ is given by (2.1) (with $S = A$) and V_S is the set of unit vectors in the directions of the edges of S. For much of this paper, S can be regarded as a simplex or a cube which are simple polytopes. We now restate part of Theorem 12.2.3 of **[11]** in the following way which will be needed later.

THEOREM 3.3 (*Ditzian-Totik*). *Suppose* $f \in L_p(S)$, $1 \le p \le \infty$ *or* $f \in C(S)$ (*in which case the* $L_\infty(S)$ *norm is used*). *Suppose further, that* S *is a simple polytope,* (*in particular,* S *is a simple* (1.1) *or a cube* (1.4)), $E_n(f)_{L_p(S)}$ *is given by* (3.6) *and* $K_{r,S}(f, t^r)_p$ *is given by* (3.7). *Then*

$$(3.8)\qquad E_n(f)_{L_p(S)} \le C(K_{r,S}(f, n^{-r})_{L_p(S)} + n^{-r}\|f\|_{L_p(S)})$$

and

$$(3.9)\qquad K_{r,S}(f, t^r)_p \le M t^r \sum_{0\le k\le 1/t} (k+1)^{r-1} E_k(f)_{L_p(S)}.$$

Proof. The proof consists mainly of relating the present notation with the notation of **[11**, Chapter 12] to show that in fact, the theorem was proved there. We can write

$$(3.10)\qquad K_{r,S}(f, t^r)_{L_p(S)} \sim \operatorname*{Sup}_{\xi\in V_S} \operatorname*{Sup}_{0<h\le t} \|\Delta^r_{h\varphi_\xi\xi} f\|_{L_p(S)} \equiv \overline{\omega}^r_S(f, t)_p$$

where $\Delta^r_{\eta e}$ is given by

$$(3.11)\quad \Delta^r_{\eta e} f(x) = \begin{cases} \displaystyle\sum_{k=0}^{r} \binom{r}{k} (-1)^r f(x + ((r/2) - k)\eta e), & x \pm \frac{r}{2}\eta e \in S, \\ 0, & \text{otherwise.} \end{cases}$$

The equivalence (3.10) follows the one-dimensional analogue **[11**, Chapter 2] and is actually implied in **[11**, Chapter 12] for an essentially identical situation. We now recall the modulus $\omega_S^r(f, t)_p$ from [**11**, p. 202] which is given by

$$\omega_S^r(f, t)_p \equiv \operatorname*{Sup}_{\xi \in V_S} \operatorname*{Sup}_{0<h\le t} \|\Delta^r_{h\tilde{d}_S(e,\cdot)^{1/2}\xi} f(\cdot)\|_{L_p(S)} \tag{3.12}$$

where

$$\tilde{d}_S(e, x) \equiv \left(\inf_{x+\lambda e \notin S} d(x, x+\lambda e)\right)\left(\max_{x+\lambda_i e \in S} d(x+\lambda_1 e, x+\lambda_2 e)\right).$$

(Note that (3.12) looks somewhat simpler but is exactly what is given in [**11**, p. 202].) We further observe that for $\tilde{d}(S, e, x)$ given in (2.1), we have

$$\frac{1}{2}\tilde{d}_S(e, x) \le \tilde{d}(S, e, x) \le \tilde{d}_S(e, x)$$

and hence

$$\omega_S^r(f, t/\sqrt{2}) \le \overline{\omega}_S^r(f, t) \le \omega_S^r(f, t) \tag{3.13}$$

where $\overline{\omega}_S^r(f, t)$ and $\omega_S^r(f, t)$ are given in (3.10) and (3.12). We now note that

$$\overline{\omega}_S^r(f, \alpha t)_p \le C(\alpha^r + 1)\overline{\omega}_S^r(f, t)_p \tag{3.14}$$

which again follows from its analogue for one dimension [**11**, Theorem 4.1.2] or directly from (3.10) and is implied in [**11**, Chapter 12]. Combining (3.10), (3.13) and (3.14) with (12.2.3) and (12.2.4) of [**11**], we obtain (3.8) and (3.9). Note that the "proof" above consists mainly of matching and slightly modifying notations. □

Further results which are corollaries of Theorem 3.3 will be used later for $p = \infty$ and where the simple polytope is a simplex or a cube.

THEOREM 3.4. *For $K_{r,S}(f, t^r)_p$ given by* (3.7), $1 \le p \le \infty$, $r \in N$, *and a simple polytope S, we have*

$$K_{r+1,S}(f, t^{r+1})_p \le C(K_{r,S}(f, t^r)_p + t^r\|f\|_{L_p(S)}) \tag{3.15}$$

and

$$\begin{aligned} K_{r,S}(f, t^r)_p & \\ &\le C\left[t^r \sum_{1\le k\le 1/t} k^{r-1}K_{r+1,S}(f, k^{-r-1})_p + t^r\|f\|_{L_p(S)}\right]. \end{aligned} \tag{3.16}$$

Before we go into the proof which actually consists of substituting the results of Theorem 3.3 in an appropriate manner, we make some observations.

(a) Often the K-functional uses two norms rather than a norm and a seminorm for its definition. In this case, we define

(3.17)
$$K^*_{r,S}(f, t^r)_p = \operatorname*{Inf}_{g\in C^r(S)} \left(\|f-g\|_{L_p(S)} + t^r\left(\|g\|_{L_p(S)} + \max_{\xi\in V_S} \|\varphi^r_\xi(\partial/\partial\xi)^r g\|_{L_p(S)}\right)\right)$$

instead of (3.7) and we can rewrite (3.15) and (3.16) as

$$K^*_{r+1,S}(f, t^{r+1})_p \le CK^*_{r,S}(f, t^r)_p \tag{3.15$'$}$$

and

$$K^*_{r,S}(f, t^r)_p \le Ct^r \sum_{1\le k\le 1/t} k^{r-1}K^*_{r+1,S}(f, k^{-r-1})_p \tag{3.16$'$}$$

which appear to be somewhat nicer but carry the same contents as (3.15) and (3.16).

(b) Using the equivalence in (3.10) and (3.13), we can restate (3.15) and (3.16) as

$$\omega^{r+1}_S(f, t)_p \le C(\omega^r_S(f, t)_p + t^r\|f\|_{L_p(S)}) \tag{3.15$''$}$$

and

$$\omega^r_S(f, t)_p \le Ct^r \left(\int_t^1 \frac{\omega^{r+1}_S(f, u)_p}{u^{r+1}}\, du + \|f\|_{L_p(S)}\right). \tag{3.16$''$}$$

(The second inequality follows from $\int_{(k+1)^{-1}}^{k^{-1}} du/u^{r+1} \sim k^{r-1}$.) We note that (3.16), (3.16)$'$ and (3.16)$''$ are forms of the Marchaud-type inequality.

Proof of Theorem 3.4. We use (3.9) (with $r+1$) and then (3.8) (with r) to obtain

$$\begin{aligned} K_{r+1,S}(f, t^{r+1})_p &\le Mt^{r+1} \sum_{1\le k\le t^{-1}} k^r E_k(f)_{L_p(S)} + Mt^{r+1}\|f\|_{L_p(S)} \\ &\le M_1 t^{r+1} \sum_{1\le k\le t^{-1}} k^r[K_{r,S}(f, k^{-r})_p + k^{-r}\|f\|_{L_p(S)}]. \end{aligned}$$

The definition of $K_{r,S}(f, t^r)_p$ in (3.7) implies

$$K_{r,S}(f, (At)^r)_p \le A^r K_{r,S}(f, t^r)_p \quad \text{for } A \ge 1 \tag{3.18}$$

which, with $A = 1/kt$ $(k \le 1/t)$, completes the proof of (3.15). To prove (3.16), we just use (3.9) (with r) and then (3.8) (with $r+1$). □

4. Bernstein inequality for Bernstein polynomials. In this section, we will prove the Bernstein inequality estimate and other estimates of derivatives of Bernstein polynomials that will be crucial for our paper.

THEOREM 4.1. *For $\xi \in V_S$, $B_n f$ defined by (1.2), $\nu = 0, 1$ and for $r = 0, 1, 2, \ldots$, we have*

$$\|\varphi_\xi^{r+\nu}(\partial/\partial\xi)^{r+\nu} B_n f\|_{C(S)} \le C n^{\nu/2} \|\varphi_\xi^r (\partial/\partial\xi)^r f\|_{C(S)}. \tag{4.1}$$

We observe that for $\nu = 1$, (4.1) yields the Bernstein-type inequality and for $\nu = 0$ and $r > 0$, (4.1) yields the analogues of the inequalities in §9.7 of **[11]**. The reader should note that we save a substantial amount of work by not proving the inequality (4.1) for $\nu = 2, 3, \ldots$ which would have come in handy. The use of such estimates for higher ν is replaced in what follows by applying (4.1) to iterates of $B_n f$, that is, $B_n^l f$ and it will be shown that such estimates are sufficient for the proof of our main result.

Proof of Theorem 4.1. First, we recall from [7] the transformation that will allow us to consider (4.1) for $\xi = e_1$ only in which case $\partial/\partial\xi = \partial/\partial x_1$. It is clear that if $\xi = e_i$, we may just rename the coordinates. The transformation

$$(u_1, \ldots, u_d) = T(x_1, \ldots, x_d), \qquad \begin{aligned} &u_l = x_l \quad \text{for } l \ne j, \\ &u_j = 1 - x_1 - \cdots - x_d, \end{aligned} \tag{4.2}$$

introduced in [7], satisfies

$$\begin{gathered} T^2 = I, \quad T: S \to S \text{ onto}, \\ \frac{\partial}{\partial u_l} = \frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_j} \quad \text{for } l \ne j, \qquad \frac{\partial}{\partial u_j} = -\frac{\partial}{\partial x_j}; \end{gathered} \tag{4.3}$$

it maps the point e_j onto $(0, \ldots, 0)$ and a vector of V_S onto a vector (not necessarily of Euclidean norm 1) in the direction of some edge of S. Also, we have for any $\xi \in V_S$,

$$\|\tilde{d}(\xi, \cdot)^{r/2}(\partial/\partial\xi)^r f(\cdot)\| = \|\tilde{d}(\eta, \cdot)^{r/2}(\partial/\partial\eta)^r f_T(\cdot)\| \tag{4.4}$$

where $f_T(u) \equiv f(Tx)$, $r \in N$, $\eta = T\xi/\|T\xi\|_2$ and $\|T\xi\|_2$ is the Euclidean norm in R^d. (Note that if $\xi = (e_i - e_j)/\sqrt{2}$, $\eta = e_i$ but $\tilde{d}(\xi, x) = 2x_i x_j = 2u_i(1 - u_1 - \cdots - u_j) = 2\tilde{d}(\eta, Tx)$.) For the Bernstein polynomials, we have

$$(4.5) \qquad B_n(f, x) = B_n(f_T, Tx), \quad B_n(f, Tx) = B_n(f_T, x).$$

Suppose we proved (4.1) for $\xi = e_1$ and hence for $\xi = e_i$; the above implies for $\xi = (e_i - e_j)/\sqrt{2}$ and $\eta = e_i$,

$$\begin{aligned}
&\|\varphi_\xi(x)^{r+\nu}(\partial/\partial\xi)^{r+\nu} B_n(f, x)\| \\
&\quad = \|\varphi_\eta(Tx)^{r+\nu}(\partial/\partial\eta)^{r+\nu} B_n(f_T, Tx)\| \\
&\quad = \|\varphi_\eta(u)^{r+\nu}(\partial/\partial\eta)^{r+\nu} B_n(f_T, u)\| \\
&\quad \le Cn^{\nu/2}\|\varphi_\eta(u)^r(\partial/\partial\eta)^r f_T(u)\| \\
&\quad = Cn^{\nu/2}\|\varphi_\xi(x)^r(\partial/\partial\xi)^r f(x)\|.
\end{aligned}$$

Note that this type of argument is used repeatedly in the present paper and elsewhere and was given in detail here to utilize T of [7] on the new, more efficient notations of the present paper. We now prove (4.1) for $\xi = e_1$.

For $P_{n,\beta}(x)$ given by (1.3) and $P_{n,\beta}(x) = 0$ for $\beta/n \notin S$, we have

$$\frac{\partial}{\partial x_1} P_{n,\beta}(x) = n(P_{n-1,\beta-e_1}(x) - P_{n-1,\beta}(x)).$$

We now denote the forward difference by

$$(4.6) \quad \vec{\Delta}^r_{eh} f(x) \equiv \begin{cases} \displaystyle\sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(x + keh), & x, x + reh \in S, \\ 0, & \text{otherwise.} \end{cases}$$

The routine calculation iterating the above implies now

$$\begin{aligned}
(4.7) \qquad &\left(\frac{\partial}{\partial x_1}\right)^{r+\nu} B_n(f, x) \\
&\quad = n(n-1)\cdots(n-r+1) \\
&\qquad \cdot \sum_{\beta/(n-r)\in S} \vec{\Delta}^r_{e_1/n} f\left(\frac{\beta}{n}\right) \left(\frac{\partial}{\partial x_1}\right)^{\nu} P_{n-r,\beta}(x).
\end{aligned}$$

Recall that $\beta/(n-r) \in S$ implies β/n, $\beta/n + re_1/n \in S$. We now write (recalling $\tilde{d}(e_1, x) = \varphi_{e_1}(x)^2 = x_1(1 - |x|)$ and $\beta =$

$(k_1, \dots, k_d))$ for $\beta/(n-r) \in S$,

$$\begin{aligned}
&\left|\vec{\Delta}^r_{e_1 n^{-1}} f\left(\frac{\beta}{n}\right)\right| \\
&\quad = \int_0^{1/n} \cdots \int_0^{1/n} \left(\frac{\partial}{\partial x_1}\right)^r f\left(\frac{\beta}{n} + e_1(t_1 + \cdots + t_r)\right) dt_1 \cdots dt_r \\
&\quad \le I(r) \int_0^{1/n} \cdots \int_0^{1/n} \left[\left(\frac{k_1}{n} + \sum_{i=1}^r t_i\right)\left(1 - \frac{|\beta|}{n} - \sum_{i=1}^r t_i\right)\right]^{-r/2} dt_1 \cdots dt_r \\
&\quad \le I(r) \left(\int_0^{1/n} \left[\left(\frac{k_1}{n} + t\right)\left(1 - \frac{|\beta| + r - 1}{n} - t\right)\right]^{-1/2} dt\right)^r
\end{aligned}$$

where $I(r) = \|\varphi^r_{e_1}(\partial/\partial x_1)^r f\|_{C(S)}$. For $\beta/(n-r) \in S$ (and hence $0 \le k_1 \le |\beta| \le n - r$), we have

$$\begin{aligned}
&\int_0^{1/n} \left(\frac{k_1}{n} + t\right)^{-1/2} \left(1 - \frac{|\beta| + (r-1)}{n} - t\right)^{-1/2} dt \\
&\quad \le \left(1 - \frac{|\beta| + (r-1)}{n} - \frac{1}{2n}\right)^{-1/2} \int_0^{1/2n} \left(\frac{k_1}{n} + t\right)^{-1/2} dt \\
&\quad\quad + \left(\frac{k_1}{n} + \frac{1}{2n}\right)^{-1/2} \int_{1/2n}^{1/n} \left(1 - \frac{|\beta| + r - 1}{n} - t\right)^{-1/2} dt \\
&\quad \le C\left(\left(\frac{k_1 + 1}{n}\right)^{-1/2} \left(\frac{n - |\beta| - r + 1}{n}\right)\right)^{-1/2} n^{-1}.
\end{aligned}$$

Combining the above considerations, we have

$$\begin{aligned}
&\tilde{d}(e_1, x)^{(r+\nu)/2} |(\partial/\partial x_1)^{r+\nu} B_n(f, x)| \\
&\quad \le C_1 \|\tilde{d}(e_1, \cdot)^{r/2} (\partial/\partial x_1)^r f(\cdot)\|_{C(S)} J_{n,\nu}(x)
\end{aligned}$$

where

$$\begin{aligned}
J_{n,\nu}(x) &= (x_1(1 - |x|))^{(r+\nu)/2} \\
&\quad \cdot \sum_{\beta/(n-r)\in S} \left(\frac{k_1 + 1}{n}\right)^{-r/2} \left(1 - \frac{|\beta| + r - 1}{n}\right)^{-r/2} \\
&\quad \cdot \left|\left(\frac{\partial}{\partial x_1}\right)^\nu P_{n-r,\beta}(x)\right|.
\end{aligned}$$

Using

$$\frac{\partial}{\partial x_1} P_{l,\beta}(x) = \left(\frac{k_1(1 - |x|) - (l - |\beta|)x_1}{x_1(1 - |x|)}\right) P_{l,\beta}(x) \tag{4.8}$$

for $l = n - r$ and the Cauchy-Schwarz inequality, we have

$$J_{n,\nu}(x) = \left\{ (x_1(1-|x|))^r \cdot \sum_{\beta/(n-r)\in S} \left(\frac{k_1+1}{n}\right)^{-r} \left(\frac{n-\beta-r+1}{n}\right)^{-r} P_{n-r,\beta}(x) \right\}^{1/2}$$

$$\times \left\{ (x_1(1-|x|))^{-\nu} \cdot \sum_{\beta/(n-r)\in S} (k_1(1-|x|) - (n-r-|\beta|)x_1)^{2\nu} P_{n-r,\beta}(x) \right\}^{1/2}$$

$$\equiv L_n(x)^{1/2} I_{n,\nu}(x)^{1/2}.$$

For $\nu = 0$, it is clear that $I_{n,\nu}(x) = 1$. For $\nu = 1$, one may calculate $I_{n,1}(x)$ and obtain

$$I_{n,1}(x) = (n-r)(1-x_2-\cdots-x_d).$$

We omit this calculation as (3.8) of **[4]** implies $I_{n,1} + J \le (n-r)d$ where J (given there) is positive and hence $I_{n,1} \le (n-r)d \le Cn$. To calculate $L_n(x)$, we follow Lemma 3.2 of **[5]** to write

$$\text{(4.9)} \qquad x_1^r(1-|x|)^r P_{n-r,\beta}(x) = \frac{(n-r)!}{(n+r)!}\frac{(k+r)!}{k_1!}\frac{(n-|\beta|)!}{n!(|\beta|-r)!} \cdot P_{n+r,\beta+re_1}(x)$$

which implies that $L_n(x)$ is bounded as

$$\frac{(n-r)!}{(n+r)!}\frac{(k_1+r)!}{k_1!}\frac{(n-|\beta|)!}{(n-|\beta|-r)!}\left(\frac{n}{k_1+1}\right)^r \left(\frac{n}{n-|\beta|-r+1}\right)^r \le M$$

with M independent of n. □

5. An estimate for Bernstein polynomial approximation of polynomials. In several articles (see **[11**, §9.3], **[12]** and **[3]**), extensive use is made of approximation of polynomials first which is later used to estimate approximation of other functions applying the result to polynomials of best approximation. This will be followed here for our present, more involved problem. We prove the necessary estimates on polynomials in this section and utilize the results in §6. It should be noted that we will be advancing the use of polynomials of best approximation somewhat further than earlier results and we hope that the techniques of this and the next section will prove fruitful as guidance for other situations as well.

The main result of this section is a strong Voronovskaja result for Bernstein polynomials on polynomials.

Theorem 5.1. *For a polynomial* $P \in \Pi_m$, $m \le \sqrt{n}$, *we have*

$$\left| B_n(P, x) - P(x) - \frac{1}{2n} \sum_{\xi \in V_S} \varphi_\xi(x)^2 \left(\frac{\partial}{\partial \xi} \right)^2 P(x) \right| \le M n^{-2} m^4 \|P\|_{C(S)} . \tag{5.1}$$

Proof. We follow [7] and divide the simplex S into the regions $V_i = \{x \in S : x_i > 1/2d\}$, $i = 1, \dots, d$ and $V_0 = \{x \in S : 1 - |x| > 1/2d\}$. Obviously, $\bigcup_{i=0}^d V_i = S$, and with the transformation T of (4.2), we have $T: V_j \to V_0$ (onto) where we consider j as generic. Furthermore, it is sufficient to prove (5.1) for $x \in V_0$. This follows as in the case $x \in V_j$, we can set $u = Tx$ and $P_T(x) \equiv P(Tx)$. From the result (5.1) for $u \in V_0$ and the polynomial P_T, we will obtain (5.1) for $x \in V_j$ and $P(x)$. To prove the above implication we write, for $x \in V_j$, $u = Tx \in V_0$ and $\eta = T\xi / \|T\xi\|_2$,

$$\begin{aligned} &\left| B_n(P, x) - P(x) - \frac{1}{2n} \sum_{\xi \in V_S} \varphi_\xi(x)^2 \left(\frac{\partial}{\partial \xi} \right)^2 P(x) \right| \\ &\quad = \left| B_n(P_T, u) - P_T(u) - \frac{1}{2n} \sum_{\eta \in V_S} \varphi_\eta(u)^2 \left(\frac{\partial}{\partial \eta} \right)^2 P_T(u) \right| \\ &\quad \le M n^{-2} m^4 \|P_T\|_{C(S)} = M n^{-2} m^4 \|P\|_{C(S)} . \end{aligned}$$

We further note that since the left-hand side of (5.1) is a polynomial of degree $m \le \sqrt{n}$, it is sufficient to estimate it on $S_{1/n}$ (or $S_{1/n} \cap V_0$) given by

$$S_{1/n} \equiv \left\{ x \in S : d(x, S^c) \ge \frac{1}{n} \right\} .$$

This follows $S_{1/m^2} \subset S_{1/n}$ and hence Theorem 3.1 of [9] implies for $Q \in \Pi_m$,

$$\|Q\|_{C(S)} \le A \|Q\|_{C(S_{1/n})} .$$

We now need the following computational lemma which will be proved after the completion of the proof of Theorem 5.1.

Lemma 5.2. *For* $\psi_i(x) = x_i$, $i = 1, \dots, d$, *we have*

(a) $B_n(\psi_i - x_i, x) = 0$,
(b) $B_n((\psi_i - x_i)^2, x) = x_i(1 - x_i)/n$,
(c) $B_n((\psi_i - x_i)^3, x) = (1/n^2) x_i (1 - x_i)(1 - 2x_i)$,

(d) $B_n((\psi_i - x_i)^{2r}, x) \le C(A)n^{-r}(x_i(1-x_i))^r$ *for* $A/n \le x_i \le 1 - A/n$,

(e) $B_n((\psi_i - x_i)(\psi_j - x_j), x) = -(1/n)x_i x_j$ *for* $i \ne j$,

(f) $B_n((\psi_i - x_i)(\psi_j - x_j)(\psi_l - x_l), x) = (2/n^2)x_i x_j x_l$ *for* $i \ne j \ne l \ne i$,

and

(g) $B_n((\psi_i - x_i)(\psi_j - x_j)^2, x) = (1/n^2)(2x_j - 1)x_i x_j$ *for* $i \ne j$.

We continue with the proof of Theorem 5.1 using Lemma 5.2. We expand $P(\frac{\beta}{n})$ using Taylor's formula by

$$
\begin{aligned}
(5.2)\quad P\left(\frac{\beta}{n}\right) &= P(x) + \sum_{i=1}^{d} \left(\frac{k_i}{n} - x_i\right) \frac{\partial}{\partial x_i} P(x) \\
&\quad + \frac{1}{2} \sum_{i,j=1}^{d} \left(\frac{k_i}{n} - x_i\right)\left(\frac{k_j}{n} - x_j\right) \frac{\partial^2}{\partial x_i \partial x_j} P(x) \\
&\quad + \frac{1}{6} \sum_{i,j,l=1}^{d} \left(\frac{k_i}{n} - x_i\right)\left(\frac{k_j}{n} - x_j\right)\left(\frac{k_l}{n} - x_l\right) \\
&\qquad\qquad \cdot \frac{\partial^3}{\partial x_i \partial x_j \partial x_l} P(x) + R\left(\frac{\beta}{n}, x\right) \\
&\equiv P(x) + I_1\left(\frac{\beta}{n}, x\right) + I_2\left(\frac{\beta}{n}, x\right) \\
&\quad + I_3\left(\frac{\beta}{n}, x\right) + R\left(\frac{\beta}{n}, x\right)
\end{aligned}
$$

where

$$
\begin{aligned}
(5.3)\qquad R\left(\frac{\beta}{n}, x\right) &= \frac{1}{6} \int_0^1 t^3 F^{(4)}(t)\, dt, \\
F(t) &= P\left(\frac{\beta}{n} + t\left(x - \frac{\beta}{n}\right)\right).
\end{aligned}
$$

As B_n is a linear operator on functions of $\frac{\beta}{n}$ (where x is considered a constant), we have

$$B_n(P(\cdot), x) = B_n(P(x), x) + \sum_{r=1}^{3} B_n(I_r(\cdot, x), x) + B_n(R(\cdot, x), x).$$

Using $B_n(1, x) = 1$, we have

$$B_n(P(x), x) = P(x).$$

Using Lemma 5.1(a), we have $B_n(I_1(\cdot\,, x), x) = 0$. Using Lemma 5.1(b) and (e) and a rearrangement of terms, we have

$$
\begin{aligned}
&B_n(I_2(\cdot\,, x), x) \\
&= \frac{1}{2n}\left[\sum_{i=1}^{d} x_i(1-x_i)\left(\frac{\partial}{\partial x_i}\right)^2 P(x) - 2\sum_{1\le i<j\le d} x_i x_j \frac{\partial^2}{\partial x_i\,\partial x_j} P(x)\right] \\
&= \frac{1}{2n}\left[\sum_{i=1}^{d} x_i(1-|x|)\left(\frac{\partial}{\partial x_i}\right)^2 + \sum_{1\le i<j\le d} x_i x_j \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2\right] P(x) \\
&= \frac{1}{2n}\sum_{\xi\in V_S} \varphi_\xi(x)^2 \left(\frac{\partial}{\partial \xi}\right)^2 P(x).
\end{aligned}
$$

Combining the above identities, it is evident that we will complete the proof when we estimate $|B_n(I_3(\cdot\,, x), x)|$ and $|B_n(R(\cdot\,, x), x)|$ by $Cn^{-2}m^4\|P\|_{C(S)}$ for $x \in V_0 \cap S_{1/n}$.

To estimate $B_n(I_3(\cdot\,, x), x)$, we use (c), (f) and (g) of Lemma 5.2 to write

$$
\begin{aligned}
B_n(I_3(\cdot\,, x)) \le{}& \frac{1}{6n^2}\sum_{i=1}^{d} x_i(1-x_i)|1-2x_i| \left|\left(\frac{\partial}{\partial x_i}\right)^3 P(x)\right| \\
&+ \frac{C}{n^2}\sum_{i<j<l} x_i x_j x_l \left|\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l} P(x)\right| \\
&+ \frac{C}{n^2}\sum_{i\ne j} |2x_j - 1| x_i x_j \left|\frac{\partial^2}{\partial x_j^2}\frac{\partial}{\partial x_i} P(x)\right|.
\end{aligned}
$$

Recalling that $|1-2x_i| \le 1$ and $0 \le 1 - x_i \le 1$ for $x \in S$ and that $x_i \le 2d\varphi_{e_i}(x)^2$ for $x \in V_0$, we now have, for $P \in \Pi_m$,

$$
\begin{aligned}
&\left|x_i(1-x_i)|1-2x_i|\left(\frac{\partial}{\partial x_i}\right)^3 P(x)\right| \le 2d\left|\varphi_{e_i}(x)^2\left(\frac{\partial}{\partial x_i}\right)^2 \frac{\partial}{\partial x_i} P(x)\right| \\
&\quad \le C_1 m^2 \|\partial/\partial x_i P\|_{C(S)} \le C_2 m^4 \|P\|_{C(S)}.
\end{aligned}
$$

For the second inequality above, we use Corollary 3.2 with $p = \infty$, $w(x) = 1$, $\xi_1 = \xi_2$ and $k = 2$, and for the third inequality we use the multivariate Markov inequality

$$\|(\partial/\partial\xi)P(x)\|_{C(S)} \le Cm^2\|P\|,$$

proved in Theorem 4.1 of [9] (for instance). For $x \in V_0$, the estimates

$$\begin{aligned} x_i x_j x_l \left| \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} P(x) \right| &\le (2d)^{3/2} \left\| \varphi_{e_i} \varphi_{e_j} \varphi_{e_l} \frac{\partial^3}{\partial x_i\, \partial x_j\, \partial x_l} P \right\| \\ &\le C m^3 \|P\| \end{aligned}$$

and

$$\begin{aligned} & x_i x_j |2x_j - 1| \left| \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_j} \right)^2 P(x) \right| \\ & \quad \le (2d)^{3/2} \left\| \varphi_{e_i} \varphi_{e_j}^2 \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_j} \right)^2 P \right\| \le C m^3 \|P\| \end{aligned}$$

follow from Corollary 3.2 with $p = \infty$, $w = 1$, $k = 3$ and ξ_l chosen appropriately.

To estimate $B_n(R(\cdot, x), x)$, we recall that

$$\begin{aligned} F^{(4)} = \sum_{i_1=1}^{d} \cdots \sum_{i_4=1}^{d} \prod_{j=1}^{4} \left(x_{i_j} - \frac{k_{i_j}}{n} \right) \\ \times \left. \frac{\partial^4}{\partial x_{i_1}\, \partial x_{i_2}\, \partial x_{i_3}\, \partial x_{i_4}} P(\zeta) \right|_{\zeta = \beta/n + t(x - \beta/n)} . \end{aligned} \tag{5.4}$$

We now use the binomial expansion identity in the form

$$\sum_{\beta/n \in S,\ |\beta|/n = 1} P_{n,\beta}(x) = |x|^n$$

to obtain, using the multivariate Markov inequality of [9, Theorem 4.1], for $x \in V_0$,

$$\begin{aligned} & \sum_{\beta/n \in S,\ |\beta|/n = 1} P_{n,\beta}(x) \left| R\left(\frac{\beta}{n}, x \right) \right| \\ & \quad \le C |x|^n \max_{1 \le i_j \le d} \left\| \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_4}} P \right\|_{C(S)} \\ & \quad \le C_1 \left(1 - \frac{1}{2d} \right)^n m^8 \|P\|_{C(S)} \\ & \quad \le C_2 n^{-2} m^4 \|P\|_{C(S)} . \end{aligned} \tag{5.5}$$

For $0 \le t \le 1$, $x \in S$ and $\beta/n \in S$, we have

$$\frac{t|x_i - k_i/n|}{k_i/n + t(x_i - k_i/n)} \le \frac{|x_i - k_i/n|}{x_i} \tag{5.6}$$

which is trivial in case $x_i < k_i/n$ and follows from the fact that $t\alpha/(A+t\alpha)$ is increasing for $A>0$, $t>0$ and $\alpha>0$ in case $x_i \ge k_i/n$. Furthermore,

$$\text{(5.7)}\qquad \begin{aligned}\frac{1}{1-(|\beta|/n+t(|x|-|\beta|/n))} &\le \frac{1}{1-|x|}+\frac{1}{1-|\beta|/n}\\ &\le C\frac{1}{1-|\beta|/n}\quad \text{for } x\in V_0.\end{aligned}$$

In fact, we needed the estimate (5.6) because (5.7) is meaningless for $|\beta|=n$. Using (5.4) and (5.5), we only have to estimate

$$\begin{aligned}I=&\sum_{\beta/n\in S,\ |\beta|/n<1} P_{n,\beta}(x)\int_0^1 t^3\left(\prod_{j=1}^4\left|x_{i_j}-\frac{k_{i_j}}{n}\right|\right)\\ &\qquad\times\left|\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_4}}P(\zeta)\right|_{\zeta=\beta/n+t(x-\beta/n)}\\ \le&\left\|\varphi_{e_{i_1}}\cdots\varphi_{e_{i_4}}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_4}}P\right\|\sum_{\beta/n\in S,\ |\beta|/n<1}P_{n,\beta}(x)\\ &\times\int_0^1 t^3\prod_{j=1}^4\left(\left|x_{i_j}-\frac{k_{i_j}}{n}\right|\Big/\varphi_{e_{i_j}}\left(\frac{\beta}{n}+t\left(x-\frac{\beta}{n}\right)\right)\right)dt.\end{aligned}$$

Using Corollary 3.2 (with $w=1$), (5.6) and (5.7), we have

$$\begin{aligned}I\le& Cm^4\|P\|_{C(S)}\sum_{\beta/n\in S,\ |\beta|/n<1}P_{n,\beta}(x)\left(\prod_{j=1}^4\frac{|x_{i_j}-k_{i_j}/n|}{x_{i_j}^{1/2}}\right)\frac{1}{(1-|\beta|/n)^2}\\ \le& Cm^4\|P\|_{C(S)}\prod_{j=1}^4\frac{1}{x_{i_j}^{1/2}}\left(\sum_{\beta/n\in S,\ |\beta|/n<1}P_{n,\beta}(x)\frac{|x_{i_j}-k_{i_j}/n|^4}{(1-|\beta|/n)^2}\right)^{1/4}\\ \le& Cm^4\|P\|_{C(S)}\prod_{j=1}^4\frac{1}{x_{i_j}^{1/2}}\left(\sum_{\beta/n\in S}P_{n,\beta}(x)\left|x_{i_j}-\frac{k_{i_j}}{n}\right|^8\right)^{1/8}\\ &\cdot\left(\sum_{\beta/n\in S,\ |\beta|/n<1}\frac{P_{n,\beta}(x)}{(1-|\beta|/n)^4}\right)^{1/8}.\end{aligned}$$

We now use (d) of Lemma 5.2 with $r=4$, where we need to recall that $x\in S_{4n}$ implies $x_i\ge 1/dn$ ($A=1/d$), and the estimate $1-x_i\le 1$, to obtain

$$\frac{1}{x_{i_j}^{1/2}}\left(\sum_{\beta/n\in S} P_{n,\beta}(x)\left|x_{i_j}-\frac{k_{i_j}}{n}\right|^8\right)^{1/8} \le \frac{C}{n^{1/2}}.$$

For $|\beta|<n$, we have

$$\begin{aligned}(1-|x|)^4 P_{n,\beta}(x)&\left(1-\frac{|\beta|}{n}\right)^{-4}\\ &\le \frac{n!}{(n+4)!}\frac{(n+4-|\beta|)!}{(n-|\beta|)!}\frac{n^4}{(n-|\beta|)^4}P_{n+4,\beta}(x)\le MP_{n+4,\beta}(x)\end{aligned}$$

where M is independent of n and hence for $x\in V_0$,

$$\sum_{\beta/n\in S,\ \beta/n<1} P_{n,\beta}(x)\left(1-\frac{|\beta|}{n}\right)^{-4}\le M_1$$

which, together with the above, completes the proof of Theorem 5.1 pending the proof of Lemma 5.2. □

Proof of Lemma 5.2. Parts (a), (b), (c) and (d) follow from summing first on the other indices ($\neq i$) and observing that what we have is the rth moment of the univariate Bernstein polynomial in x_i. The exact expressions (a), (b) and (c) are known from Lemma 9.4.3 of **[11]** (for example) and the estimate (d) from Lemma 9.4.4 of **[11]**. We now calculate

$$\begin{aligned}B_n(\psi_i\psi_j, x)&=\frac{x_ix_j}{n^2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\left(z+\sum_{i=1}^d x_i\right)^n\Bigg|_{z=1-|x|}\\ &=x_ix_j\left(1-\frac{1}{n}\right)\quad\text{for } i\neq j,\\ B_n(\psi_i\psi_j\psi_l, x)&=\frac{x_ix_jx_l}{n^2}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_l}\left(z+\sum_{i=1}^d x_i\right)^n\Bigg|_{z=1-|x|}\\ &=x_ix_jx_l\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\quad\text{for } i\neq j\neq l\neq i,\end{aligned}$$

and

$$\begin{aligned}B_n(\psi_i\psi_j^2, x)&=\frac{x_ix_j}{n^3}\frac{\partial}{\partial x_j}x_j\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\left(z+\sum_{i=1}^d x_i\right)^n\Bigg|_{z=1-|x|}\\ &=x_ix_j^2\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\frac{x_ix_j}{n}\left(1-\frac{1}{n}\right)\quad\text{for } i\neq j.\end{aligned}$$

Simple arithmetic now implies (e), (f) and (g). □

6. Proof of the main result. First, we apply the strong Voronovskaja estimate for polynomials given in Theorem 5.1 to polynomials of best approximation to a function f. In this section, $\| \; \| = \| \; \|_{C(S)}$, $K_{r,S}(f, t^r) = K_{r,S}(f, t^r)_\infty$ and $E_n(f) = E_n(f)_{C(S)}$ with a simplex S of (1.1).

THEOREM 6.1. *For $f \in C(S)$ and $P_m \in \Pi_m$ $(m = [\sqrt{n}])$ satisfying $\|P_m - f\| \le ME_m(f)$ and for $K_{3,S}(f, t^3)$ given by (3.7), we have*

$$\begin{aligned} \|B_n P_m - P_m - (2n)^{-1}P(D)P_m\| \\ \le (K_{3,S}(f, n^{-3/2}) + n^{-3/2}\|f\|). \end{aligned} \tag{6.1}$$

Proof. We choose P_j, $P_j \in \Pi_j$, satisfying $\|P_j - f\| \le ME_j(f)$ and expand P_m by

$$P_m = P_m - P_{2^l} + \sum_{j=1}^{l}(P_{2^j} - P_{2^{j-1}}) + P_1, \qquad l = \max\{j\colon 2^j < m\}. \tag{6.2}$$

We recall $B_n P_1 - P_1 = P(D)P_1 = 0$ and utilize Theorem 5.1 to write (for $m = [\sqrt{n}]$)

$$\begin{aligned} I(n) &= \|B_n P_m - P_m - (2n)^{-1}P(D)P_m\| \\ &\le C_1 n^{-2}\left(m^4\|P_m - P_{2^l}\| + \sum_{j=1}^{l} 2^{4j}\|P_{2^j} - P_{2^{j-1}}\|\right) \\ &\le C_2 n^{-2}\left(m^4 E_{2^l}(f) + \sum_{j=1}^{l} 2^{4j} E_{2^{j-1}}(f)\right). \end{aligned}$$

Applying now (3.8) of Theorem 3.3, we have

$$I(n) \le C_3 n^{-2}\sum_{j=0}^{l} 2^{4j}(K_{3,S}(f, 2^{-3j}) + 2^{-3j}\|f\|).$$

Using the definition of $K_{3,S}(f, t^3)$, we have

$$K_{3,S}(f, (At)^3) \le A^3 K_{3,S}(f, t^3) \quad \text{for } A \ge 1. \tag{6.3}$$

We now choose in (6.3) $t = 2^{-l}$, with l of (6.2), and $A = 2^{l-j}$ and then $t = n^{-1/2}$ and $A = 2^{-l}/n^{1/2}$ and obtain

$$\begin{aligned} I(n) &\le C_4 n^{-2} \sum_{j=0}^{l} 2^{4j} 2^{-3j} (2^{3l} K_{3,S}(f, 2^{-3l}) + \|f\|) \\ &\le C_5 (n^{-2} 2^{4l} (K_{3,s}(f, 2^{-3l}) + n^{-3/2} \|f\|)) \\ &\le C(K_{3,S}(f, n^{-1/2}) + n^{-3/2}\|f\|). \end{aligned}$$

□

We are now able to prove the main result.

Proof of Theorem 2.1. Using the definition of $K_S(f, t)$ in (2.4), there exists a function $g \in C^3(S)$ such that

$$\begin{aligned} \|f - g\| &+ n^{-1}\|P(D)g\| \\ &+ n^{-3/2} \max_{\xi \in V_S} \|\varphi_\xi^3 (\partial/\partial\xi)^3 g\| \le 2K_S(f, n^{-1/2}). \end{aligned} \tag{6.4}$$

We observe that the K-functional $K_{3,S}(f, t^3)$ given by (3.7) satisfies

$$K_{3,S}(f, n^{-3/2}) \le K_S(f, n^{-1/2}) \quad \text{and} \quad K_{3,S}(g, n^{-3/2}) \le 2K_S(f, n^{-1/2})$$

for all $f \in C(S)$ and $g \in C^3(S)$ satisfying (6.4). For a polynomial $P_m \equiv P_m(g)$ satisfying

$$\|P_m - g\| = E_m(g), \quad P_m \in \Pi_m \quad \text{and} \quad m = [\sqrt{n}],$$

we have, using Theorem 6.1,

$$\begin{aligned} \|B_n P_m - P_m\| &\le \|B_n P_m - P_m - (2n)^{-1} P(D) P_m\| \\ &\quad + (2n)^{-1} \|P(D) P_m\| \\ &\le C(K_{3,S}(g, n^{-3/2}) + n^{-3/2}\|g\| + n^{-1}\|P(D)P_m\|) \\ &\le C_1(K_S(f, n^{-1/2}) + n^{-1}\|P(D)P_m\| + n^{-3/2}\|f\|). \end{aligned}$$

Furthermore,

$$\begin{aligned} \|P_m - g\| &\le M E_m(g) \le M_1(K_{3,S}(g, m^{-3}) + m^{-3}\|g\|) \\ &\le M_2(K_S(f, n^{-1/2}) + n^{-3/2}\|f\|) \end{aligned}$$

and as g satisfies (6.4),

$$\|P_m - f\| \le M_3(K_S(f, n^{-1/2}) + n^{-3/2}\|f\|). \tag{6.5}$$

Hence,

$$\begin{aligned} \|B_n f - f\| &\le \|B_n P_m - P_m\| + 2M_3(K_S(f, n^{-1/2}) + n^{-3/2}\|f\|) \\ &\le C_2(K_S(f, n^{-1/2}) + n^{-1}\|P(D)P_m\| + n^{-3/2}\|f\|). \end{aligned}$$

To prove (2.6), it remains only to show that

$$n^{-1}\|P(D)P_m\| \le L(K_S(f, n^{-1/2}) + n^{-3/2}\|f\|). \tag{6.6}$$

Following (6.4), it is sufficient to estimate $n^{-1}\|P(D)(g-P_m)\|$. We write

$$g - P_m = P_{2^l} - P_m + \sum_{j=l}^{\infty}(P_{2^{j+1}} - P_{2^j}), \qquad l = \min\{j\colon 2^j > m\} \tag{6.7}$$

and use Corollary 3.2 to obtain

$$\begin{aligned} n^{-1}\|P(D)(g-P_m)\| &\le L_1 n^{-1}\left(m^2 E_m(g) + \sum_{j=1}^{\infty} 2^{(j+1)2} E_{2^j}(g)\right) \\ &\le L_2 n^{-1}\left(m^2 K_{3,S}(g, m^{-3}) + \sum_{j=1}^{\infty} 2^{2j} K_{3,S}(g, 2^{-3j})\right. \\ &\qquad \left. + \left(m^2 m^{-3} + \sum_{j=l}^{\infty} 2^{-j}\right)\|g\|\right). \end{aligned}$$

We recall from (3.7) and (6.4) that

$$K_{3,S}(g, t^3) \le t^3 \operatorname*{Sup}_{\xi\in V_S} \|\varphi_\xi^3(\partial/\partial\xi)^3 g\| \le 2t^3 n^{3/2} K_S(f, n^{-1/2})$$

and hence with the choices $t = m^{-1}$ and $t = 2^{-j}$, we obtain (6.5), and the proof of (2.6) is complete.

To prove (2.7), we define

$$B_k^r(f, x) \equiv B_k(B_k^{r-1}f, x), \quad B_k^1(f, x) = B_k(f, x) \tag{6.8}$$

and obtain, using the definition of $K_{4,S}(f, t^4)$,

$$K_{4,S}(f, t^4) \le \|f - B_k^4 f\| + t^4 \operatorname*{Sup}_{\xi\in V_S} \|\varphi_S^4(\partial/\partial\xi)^4 B_k^4 f\|. \tag{6.9}$$

The elementary estimate yields

$$\|f - B_k^4 f\| \le \sum_{j=0}^{3} \|B_k^j(B_k f - f)\| \le 4\|B_k f - f\|. \tag{6.10}$$

Theorem 4.1 with $\nu = 0$ and $r = 4$ repeated 4 times yields, for $g \in C^4$,

$$\operatorname*{Sup}_{\xi\in V_S} \|\varphi_\xi^4(\partial/\partial\xi)^4 B_n^4 g\| \le \operatorname*{Sup}_{\xi\in V_S} \|\varphi_\xi^4(\partial/\partial\xi)^4 g\|. \tag{6.11}$$

Theorem 4.1 with $\nu = 1$, used with $r = 3$, then $r = 2$, etc. yields

$$\text{(6.12)} \qquad \operatorname*{Sup}_{\xi \in V_S} \|\varphi_\xi^4 (\partial/\partial\xi)^4 B_k^4 f\| \le Ck^2 \|f\|.$$

A combination of (6.10), (6.11) and (6.12) yields

$$\text{(6.13)} \qquad K_{4,S}(f, t^4) \le M(\|f - B_k f\| + t^4 k^2 K_{4,S}(f, k^{-1/2})).$$

Following the inequality (6.13), we use a technique by V. Totik **[15]** used also in Theorem 9.3.4 of **[11]** and in §5 of **[3]** to obtain

$$\text{(6.14)} \qquad K_{4,S}(f, t^4) \le Ct^\rho \left(\sum_{1 \le k \le t^{-2}} k^{(\rho/2)-1} \|B_k f - f\| + \|f\| \right)$$

with any ρ satisfying $0 < \rho < 4$.

We now substitute (6.14) in the Marchaud type estimate (3.16) with $r = 3$ to obtain, with $3 < \rho < 4$,

$$\begin{aligned} K_{3,S}(f, t^3) &\le C_1 t^3 \left(\sum_{1 \le k \le 1/t} k^2 k^{-\rho} \sum_{1 \le l \le k^2} l^{(\rho/2)-1} \|B_l f - f\| \right) \\ &\quad + C_1 t^3 \|f\| \\ &\le C_1 t^3 \sum_{1 \le l \le 1/t^2} l^{(\rho/2)-1} \|B_l f - f\| \sum_{k \ge \sqrt{l}}^{\infty} k^{2-\rho} + C_1 t^3 \|f\| \\ &\le C_2 t^3 \left(\sum_{1 \le l \le 1/t^2} l^{1/2} \|B_l f - f\| + \|f\| \right). \end{aligned}$$

For a given n, we choose n_0 satisfying $n/2 < n_0 \le n$ such that

$$\|B_{n_0} f - f\| = \min_{n/2 < k \le n} \|B_k f - f\|.$$

For $P_m \in \Pi_m$, $\|P_m - f\| = E_m(f)$ and $m = [\sqrt{n_0}]$, we have

$$\begin{aligned} K_S(f, m^{-1}) \le \|f - P_m\| &+ m^{-2} \|P(D) P_m\| \\ &+ m^{-3} \max_{\xi \in V_S} \|\varphi_\xi^3 (\partial/\partial\xi)^3 P_m\|. \end{aligned}$$

We use (3.8) with m and prove

$$\text{(6.15)} \qquad m^{-3} \max_{\xi \in V_S} \|\varphi_\xi^3 (\partial/\partial\xi)^3 P_m\| \le C(K_{3,S}(f, m^{-3}) + m^{-3} \|f\|)$$

and

$$\text{(6.16)} \quad m^{-2} \|P(D) P_m\| \le C(K_{3,S}(f, m^{-3}) + m^{-3} \|f\|) + \|B_{n_0} f - f\|.$$

We now use the estimate of $K_{3,S}(f, t)$ for $t = m^{-1}$ and the inequalities (6.15) and (6.16) to obtain

$$\begin{aligned} K_S(f, n^{-1/2}) &\le K_S(f, m^{-1}) \\ &\le \|B_{n_0}f - f\| + Cm^{-3} \sum_{1\le l\le n_0} l^{1/2}\|B_l f - f\| + Cm^{-3}\|f\| \\ &\le C_1\left(n^{-3/2}\sum_{1\le l\le n} l^{1/2}\|B_l f - f\| + n^{-3/2}\|f\|\right). \end{aligned}$$

As $K_S(f-P_1, t) = K_S(f, t)$ and $E_1(f) = \|f-P_1\| \le \|B_1 f - f\|$, the term $\|f\|$ on the right-hand side of the above is redundant. Hence, we only have to show (6.15) and (6.16). To prove (6.16), we write, using (6.1) for n_0 $(m = [\sqrt{n_0}])$,

$$\frac{1}{m^2}\|P(D)P_m\| \le C(K_{3,S}(f, m^{-3}) + m^{-3}\|f\|) + \|B_{n_0}P_m - P_m\|.$$

We now estimate $\|B_{n_0}P_m - P_m\|$ by

$$\begin{aligned} \|B_{n_0}P_m - P_m\| &\le \|B_{n_0}f - f\| + 2E_m(f) \\ &\le \|B_{n_0}f - f\| + C(K_{3,S}(f, m^{-3}) + m^{-3}\|f\|) \end{aligned}$$

which completes the proof of (6.16). To prove (6.15), we follow the proof of Theorem 7.3.1 of **[11]** almost verbatim to show that

$$\|\varphi_\xi^4(\partial/\partial\xi)^4 P_m\| \le Lm^4(K_{3,S}(f, m^{-3}) + m^{-3}\|f\|). \tag{6.17}$$

We then observe that it is sufficient to examine $\xi = e_1$. The rest follows from the one-dimensional theorem applied to x_1 at the coordinates $(x_2, \dots, x_d) = \tilde{x}$. We obtain, using the mapping $y = x_1/1-|\tilde{x}|$ and the notations $\varphi(y)^2 = y(1-y)$, $f_{\tilde{x}}(y) = f(x_1, x_2, \dots, x_d)$ and $P_{m,\tilde{x}}(y) = P_m(x_1, x_2, \dots, x_d)$,

$$\begin{aligned} &\operatorname*{Sup}_{0\le x_1\le 1-|\tilde{x}|} |\varphi_{e_1}(x_1, \tilde{x})^r(\partial/\partial x_1)^r P_m(x_1, \tilde{x})| \\ &\quad = \operatorname*{Sup}_{0\le y\le 1} |\varphi(y)^r(\partial/\partial y)^r P_{m,\tilde{x}}(y)| \end{aligned}$$

and hence

$$\begin{aligned}
&\sup_{0\le x_1\le 1-|\tilde{x}|} |\varphi_{e_1}(x_1, \tilde{x})^3(\partial/\partial x_1)^3 P_m(x, \tilde{x})| \\
&\quad = \sup_{0\le y\le 1} |\varphi(y)^3(\partial/\partial y)^3 P_{m,\tilde{x}}(y)| \\
&\quad \le C\Bigg(m^3 \sup_{0<h\le 1/m} |\Delta^3_{h\varphi} f_{\tilde{x}}(y)| \\
&\qquad\qquad + m^{-1}\sup_{0\le y\le 1} |\varphi(y)^4(\partial/\partial y)^4 P_{m,\tilde{x}}(y)|\Bigg) \\
&\quad \le Cm^3\omega^3_S(f, 1/m) + m^{-1}\sup_{\xi\in V_S} \|\varphi^4_\xi(\partial/\partial\xi)^4 P_m\|
\end{aligned}$$

which, together with (6.17), completes the proof of (6.15). This in turn, completes the proof of (2.7). □

7. The proof of the main results for $\overline{B}_n$. In this section, we point out some changes that are needed for the proof of the saturation, direct and converse results for $\overline{B}_n$, that is, the proof of Theorem 2.2. We first need a form of Theorem 3.1 and Corollary 3.2 on the cube whose proof is simpler.

THEOREM 7.1. *For the cube Q given in* (1.4), $0 < p \le \infty$ *and* $\xi \in V_Q$, *we have, for* $P \in \Pi_n$,

$$\left\|w(\cdot)\tilde{d}(Q, \xi_1, \cdot)^{1/2}\cdots\tilde{d}(Q, \xi_k, \cdot)^{1/2}\frac{\partial}{\partial\xi_1}\cdots\frac{\partial}{\partial\xi_k}P\right\|_{L_p(Q)} \le Cn^k\|wP\|_{L_p(Q)} \tag{7.1}$$

where $w(x) \equiv x_1^{\alpha_1}\cdots x_d^{\alpha_d}(1-x_1)^{\beta_1}\cdots(1-x_d)^{\beta_d}$, $\alpha_i, \beta_i \ge 0$ *for* $p = \infty$ *and* $\alpha_i, \beta_i > 1/p$ *for* $p < \infty$.

We note that the rest of the results in §3 are for a simple polytope S which applies to Q as a special case.

The analogues of Theorems 4.1 and 5.1 follow easily. In the statements, we replace S by Q. We do not need the elaborate scheme of dividing Q into subdomains and using a transformation like T. In fact, we treat the whole domain Q in the same way as we treated V_0 in Theorems 4.1 and 5.1. We observe that $\overline{B}_n(f, x)$ is a polynomial of degree nd (not n) but if $P \in \Pi_{m\le n}$, $\overline{B}_n(P, x)$ is a polynomial of degree m. We further note that in the analogue of Lemma 5.2, (e), (f) and (g) are changed in the following manner.

LEMMA 7.2. *For* $\psi_i(x) = x_i$, $i = 1, \dots, d$, (a), (b), (c) *and* (d) *of Lemma 5.2 hold with* $\overline{B}_n$ *replacing* B_n. *Furthermore, we have*

(e)′ $\overline{B}_n((\psi_i - x_i)(\psi_j - x_j), x) = 0$ *for* $i \neq j$,

(f)′ $\overline{B}_n((\psi_i - x_i)(\psi_j - x_j)(\psi_l - x_l), x) = 0$ *for* $i \neq j \neq l \neq i$,

and

(g)′ $\overline{B}_n((\psi_i - x_i)(\psi_j - x_j)^2, x) = 0$ *for* $i \neq j$.

Theorem 6.1 for $\overline{B}_n$ follows verbatim and the same is true for the rest of §6.

8. Comparisons, further questions and conjectures. When the results in [7], [8] and [11] are examined, it is clear that, for $0 < \alpha < 2$,

$$\|B_n f - f\|_{C(S)} = O(n^{-\alpha/2}) \Leftrightarrow K_{2,S}(f, t^2)_\infty = O(t^\alpha) \Leftrightarrow E_n(f)_{C(S)} = O(n^{-\alpha}). \tag{8.1}$$

In [2], the investigation of $K_{2,S}(f, t^2)_\infty$ (in our notation) is pursued but, in view of the present results, there is no hope to include saturation in the context of $K_{2,S}(f, t^2)_\infty$.

For $K_S(f, t)$, given in (3.7),

$$K_S(f, t) \geq K_{3,S}(f, t^3).$$

As Theorem 3.4 yields, for $\alpha < 2$, the implication

$$K_{3,S}(f, t^3) = O(t^\alpha) \quad \text{implies } K_{2,S}(f, t^2) = O(t^\alpha),$$

we also have (for $\alpha < 2$)

$$K_{3,S}(f, t^3) = O(t^\alpha) \quad \text{implies } K_S(f, t) = O(t^\alpha).$$

It is now clear that $K_{3,S}(f, t^3)$ and $K_S(f, t)$ can replace $K_{2,S}(f, t^2)$ in (8.1). Of course, (8.1) does not include results at the saturation rate n^{-1} or close to it as $n^{-1}(\log n)^\alpha$ or $n^{-1}(\log n)^\alpha(\log\log n)^\beta$ for example. One may be led to believe that the behaviour of $K_{3,S}(f, t^3)$ is sufficient. However, it fails already in the univariate case.

To concentrate on possible generalization, we define

$$K_S^*(f, t^2)_\infty = \operatorname*{Inf}_{g\in C^2(S)} (\|f - g\| + t^2\|P(D)g\|) \tag{8.2}$$

with $P(D)$ of (3.2). Now, we may ask if a result utilizing $K_S^*(f, t^2)_\infty$ which contains saturation, direct and converse theorems is possible. We believe that the answer will be affirmative. (If the requirement that the saturation theorem will be included is dropped, the present technique and discussion is sufficient.)

We believe that an even stronger result is valid. We conjecture that a strong converse inequality of type A in the terminology of **[10]**, which will imply that

$$\|B_n f - f\| \sim K_S^*(f, n^{-1})_\infty , \tag{8.3}$$

will be proved one day. We recall that even for the univariate Bernstein polynomial, (8.3) is open. For $d > 1$, even the weakest "strong converse inequality of type D" (in the terminology of **[10]**), that is,

$$\operatorname*{Sup}_{k \geq n} \|B_k f - f\| \sim K_S^*(f, n^{-1}) \tag{8.4}$$

is not known. However, for Bernstein-Durrmeyer operators (see **[4]**), the analogue of (8.3) is known for $1 < p < \infty$ and all dimensions d (and for $d = 1, 2, 3$ for $1 \leq p \leq \infty$). There are some possible results between (8.3) and the result of this paper, but as we are in the business of making conjectures, we might as well be brave.

Another interesting question is to find out the actual behaviour of f from the K-functional or from the rate of convergence. Here, results in the multivariate case are very scant. We conjecture that

$$\|P(D)f\| \leq M \quad \text{implies } \|h^{-2}\Delta^3_{h\varphi_\xi \xi} f\| \leq M_1 \quad \text{for } \xi \in V_S \tag{8.5}$$

where M_1 depends on M but not on h or f.

References

[1] H. Berens and Y. Xu, *Bernstein-Durrmeyer polynomials with Jacobi weights, Approximation Theory and Functional Analysis*, C. K. Chui ed., Academic Press, 1990, 25–46.

[2] ——, *K-moduli, moduli of smoothness, and Bernstein polynomials on simplices*, (manuscript).

[3] W. Chen and Z. Ditzian, *Best polynomial and Durrmeyer approximation in* $L_p(S)$, Indag. Math. (N.S.), **2** (1991), 437–452.

[4] W. Chen, Z. Ditzian and K. Ivanov, *Strong converse inequality for the Bernstein-Durrmeyer operator*, J. Approx. Theory, (to appear).

[5] Z. Ditzian, *A global inverse theorem for combinations of Bernstein polynomials*, J. Approx. Theory, **26** (1979), 277–292.

[6] ——, *Saturation of approximation of n-dimensional functions*, Constructive function theory, Proceedings of conference held in Varna, Bulgaria, 1981, Sofia, Bulgaria, 1983, pp. 295–300.

[7] ——, *Inverse theorems for multi-dimensional Bernstein operators*, Pacific J. Math., **121** (1986), 293–319.

[8] ——, *Best polynomial approximation and Bernstein polynomial approximation on a simplex*, Indag. Math., **92** (1989), 243–256.

[9] ——, *Multivariate Bernstein and Markov inequalities*, J. Approx. Theory, **70** (1992), 273–283.

[10] Z. Ditzian and K. Ivanov, *Strong converse inequality*, J. d'Analise Mat., (to appear).

[11] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer-Verlag, 1987.

[12] Z. Ditzian and X. Zhou, *Kantorovich-Bernstein polynomials*, Constructive Approximation, **6** (1990), 421–435.

[13] R. Q. Jia and Z. C. Wu, *Bernstein polynomial on simplices* (Chinese), Acta Math. Sinica, **31** (1988), 510–522.

[14] P. G. Nevai, *Bernstein inequality in* L^p, $0 < p < 1$, J. Approx. Theory, **27** (1979), 239–243.

[15] V. Totik, *An interpolation theorem and its applications to positive operators*, Pacific J. Math., **111** (1984), 447–481.

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