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A SPECTRAL THEORY FOR SOLVABLE LIE ALGEBRAS OF OPERATORS

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The main objective of this paper is to develop a notion of joint spectrum for complex solvable Lie algebras of operators acting on a Banach space, which generalizes Taylor joint spectrum (T.J.S.) for several commuting operators.

I. Introduction. We briefly recall the definition of Taylor spectrum. Let $\bigwedge(\mathbb{C}^n)$ be the complex exterior algebra on n generators $e_1, \dots, e_n$, with multiplication denoted by $\bigwedge$. Let $\dot{E}$ be a Banach space and $a = (a_1, \dots, a_n)$ be a mutually commuting n-tuple of bounded linear operators on E(m.c.o.). Define $\bigwedge_k^n(E) = \bigwedge_k(\mathbb{C}^n) \otimes_{\mathbb{C}} E$, and for $k \geq 1$, D_{k-1} by:

$$D_{k-1}\colon \bigwedge_k^n(E) \to \bigwedge_{h-1}^n(E)$$

$$D_{k-1}(x \otimes e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j+1} x \cdot a_{i_j} \otimes \cdots \otimes e_{i_1} \wedge \cdots \wedge \tilde{e}_{i_j} \wedge \cdots \wedge e_{i_k}$$

where $\sim$ means deletion. Also define $D_k = 0$ for $k \leq 0$.

It is easily seen that $D_k D_{k+1} = 0$ for all k, that is, $\{\bigwedge_k^n(E), D_k\}_{k \in Z}$ is a chain complex, called the Koszul complex associated with a and E and denoted by $R(E, a)$. The n-tuple a is said to be invertible or nonsingular on E, if $R(E, a)$ is exact, i.e., $\operatorname{Ker} D_k = \operatorname{ran} E_{k+1}$ for all k. The Taylor spectrum of a on E is $\operatorname{Sp}(a, E) = \{\lambda \in \mathbb{C}^n : a - \lambda$ is not invertible$\}$.

Unfortunately, this definition depends very strongly on $a_1, \dots, a_n$ and not on the vector subspace of $L(E)$ generated by then $(= \langle a \rangle)$.

As we consider Lie algebras, and then naturally involve geometry, we are interested in a geometrical approach to spectrum which depends on L rather than on a particular set of operators.

This is done in II. Given a solvable Lie subalgebra of $L(E)$, L, we associate to it a set in L^*, $Sp(L, E)$.

This object has the classical properties. $\mathrm{Sp}(L, E)$ is compact. If L' is an ideal of L, then $\mathrm{Sp}(L', E)$ is the projection of $\mathrm{Sp}(L, E)$ in L'^*. $\mathrm{Sp}(L, E)$ is non-empty.

Besides, it satisfies other interesting properties.

If $x \in L^2$, then $\mathrm{Sp}(x) = 0$. If L is nilpotent, one has the inclusion

$$\mathrm{Sp}(L, E) \subset \{f \in [L, L]^{\perp} | \forall x \in L, \ |f(x)| \leq \|x\|\}.$$

However the spectral mapping property is ill behaved.

II. The joint spectrum for solvable Lie algebras of operators. First of all, we establish a proposition which will be used in the definition of $\mathrm{Sp}(L, E)$.

From now on, L denotes a complex finite dimensional solvable Lie algebra, and $U(L)$ its enveloping algebra.

Let f belong to L^* such that $f([L, L]) = 0$, i.e., f is a character of L. Then f defines a one dimensional representation of L denoted by $\mathbb{C}(f)$. Let $\varepsilon(f)$ be the augmentation of $U(L)$ defined by f:

$$\begin{aligned} &\varepsilon(f) \colon U(L) \to \mathbb{C}(f), \\ &\varepsilon(f)(x) = f(x) \qquad (x \in L). \end{aligned}$$

Let us consider the pair of spaces and maps $V(L) = (U(L) \otimes \bigwedge L, \overline{d}_{p-1})$, where $\overline{d}_{p-1}$ is the map defined by:

$$\overline{d}_{p-1} \colon U(L) \otimes \bigwedge^{p} L \to U(L) \otimes \bigwedge^{p-1} L.$$

If $p \geq 1$

$$\begin{aligned} \overline{d}_{p-1}\langle x_{i_1} \cdots x_{i_p} \rangle = &\sum_{k=1}^{p} (-1)^{k+1} (x_{i_k} - f(x_{i_k})) \langle x_i, \hat{x}_{i_k} x_{i_p} \rangle \\ &+ \sum_{1 \leq k \leq l \leq p} (-1)^{k+l} \langle [x_{i_k}, x_{i_l}] x_{i_1} \hat{x}_{i_k} \hat{x}_{i_l} x_{i_p} \rangle \end{aligned}$$

where $\hat{}$ means deletion. If $p \leq 0$, we also define $\overline{d}_p = 0$. Then

PROPOSITION 1. *The pair of spaces and maps $V(L)$ is a chain complex. Furthermore, with the augmentation $\varepsilon(f)$, the complex $V(L)$ is a $U(L)$-free resolution of $\mathbb{C}(f)$ as a left $U(L)$ module.*

We omit the proof of Proposition 1 because it is a straightforward generalization of Theorem 7.1 of [**3**, XIII, 7].

Let L be as usual, from now on, E denotes a Banach space on which L acts as right continuous operators, i.e., L is a Lie subalgebra of $L(E)$ with the opposite product. Then, by [**3**, XIII, 1], E is a right $U(L)$ module.

If f is a character of L, by Proposition 1 and elementary homological algebra, the q-homology space of the complex, $(E \otimes \bigwedge L, d(f))$ is $\operatorname{Tor}_q^{U(L)}(E, \mathbb{C}(f))$ $(= H_q(L, E \otimes \mathbb{C}(f))$.

We now state our definition.

Definition 1. Let L and E be as above the set $\{f \in L^*, f(L^2) = 0 | H_*((L, E \otimes \mathbb{C}(f)))$ is non-zero$\}$, is the spectrum of L acting on E, and is denoted by $\operatorname{Sp}(L, E)$.

By Proposition 1 and Definition 1, it is clear that, if L is a commutative algebra $\operatorname{Sp}(L, E)$ reduces to Taylor joint spectrum.

Let us see an example. Let $(E, \| \ \|)$ be $(\mathbb{C}^2, \| \ \|_2)$ and a, b the operators

$$a = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

It is easily seen that $[b, a] = b$, and then, the vector space $\mathbb{C}(b) \oplus \mathbb{C}(a) = L$ is a solvable Lie subalgebra of $L(\mathbb{C}^2)$.

Using Definition 1, a standard calculation shows that $\operatorname{Sp}(L, E) = \{f \in (\mathbb{C}^2)^* | f(b) = 0;\ f(a) = \frac{1}{2},\ f(a) = -\frac{3}{2}\}$.

Observe that, $\|a\| = \frac{1}{2}$; however, $\operatorname{Sp}(L, E)$ is not contained in $\{f \in (\mathbb{C}^2)^* | \forall x \in \mathbb{C}^2\ |f(x)| \le \|x\|\}$.

III. Fundamental properties of the spectrum. In this section, we shall see that the most important properties of spectral theory are satisfied by our spectrum.

Theorem 2. *Let L and E be as usual. Then $\operatorname{Sp}(L, E)$ is a compact set of L^*.*

Proof. Let us consider the family of spaces and maps $(E \otimes \bigwedge^i L, d_{i-1}(f))$ $f \in L^{2^\perp}$, where $L^{2^\perp} = \{f \in L^* | f(L^2) = 0\}$. This family is a parameterized chain complex on $L^{2^\perp}$. By Taylor [**6**, 2.1] the set $\{f \in L^{2^\perp} | (E \otimes \bigwedge^i L, d_{i-1}(f))$ is exact$\} = \operatorname{Sp}(L, E)^c$ is an open set in $L^{2^\perp}$. Then, $\operatorname{Sp}(L, E)$ is closed in $L^{2^\perp}$ and hence in L^*.

To verify that $\operatorname{Sp}(L, E)$ is a compact set we consider a basis of L^2 and we extend it to a basis of L, $\{X_i\}_{1 \le i \le n}$. If $K = \dim L^2$ and $n = \dim L$ let L_i be the ideal generated by $\{X_j\}_{1 \le j \le n, j \ne i}$, $i \ge K+1$.

Let f be a character of L and represent it in the dual basis of $\{X_i\}_{1 \le i \le n}$, $\{f_i\}_{1 \le i \le n}$ $f = \sum_{i=K+1}^n \xi_i f_i$. For each i, there is a positive number r_i such that if $\xi_i \geq r_i$,

$$\operatorname{Tor}_p^{U(L)}(E, C(f)) = H_p\left(E \otimes \bigwedge^i L, d_{i-1}(f)\right) = 0 \quad \forall p.$$

To prove our last statement, we shall construct an homotopy operator for the chain complex $(E \otimes \bigwedge^p L, d_{p-1}(f))$ $(f(L^2) = 0)$.

First of all we observe that

$$E \otimes \bigwedge^p L = \left(E \otimes \bigwedge^p L_i\right) \oplus \left(E \otimes \bigwedge^{p-1} L_i\right) \bigwedge \langle X_i \rangle.$$

As L_i is an ideal of L, $d_{p-1}(E \otimes \bigwedge^p L_i) \subseteq E \otimes \bigwedge^{p-1} L_i$. On the other hand, there is a bounded operator L_{p-1} such that

$$\begin{aligned} &d_{p-1}(f)(a \wedge \langle X_i \rangle) \\ &\quad = (d_{p-1}(f)a) \wedge \langle X_i \rangle + (-1)^p L_{p-1} a \qquad \left(a \in E \otimes \bigwedge^{p-1} L_i\right). \end{aligned}$$

It is easy to see that, for each p, there is a basis of $\bigwedge^p L_i$, $\{V_j^p\}$ $1 \leq j \leq \dim \bigwedge^p L_i$, such that if we decompose

$$E \otimes \bigwedge^p L_i = \bigoplus_{1 \leq j \leq \dim \bigwedge^p L_i} E\langle V_j \rangle,$$

then L_p has the following form

$$L_{p_{ij}^{ii}} = \begin{cases} \alpha_{ij}^p & i < j, \\ X_i - \xi_i + \alpha_{jj}^p & i = j, \\ 0 & i > j \quad \text{where } \alpha_{ij} \in \mathbb{C}. \end{cases}$$

Besides, let K_p be a positive real number such that

$$\bigcup_{1 \leq j \leq \dim \bigwedge^p L_i} \operatorname{Sp}(X_i + \alpha_{jj}^p) \subseteq B[0, K_p]$$

and $N_i = \max_{0 \leq p \leq n-1}\{K_p\}$. Then, as L_p has a triangular form, a standard calculation shows that L_p is a topological isomorphism of Banach spaces if $\xi_i \geq N_i$.

Outside $B[0, N_i]$ we construct our homotopy operator

$$\mathrm{Sp}\colon E \otimes \bigwedge^{p} L \to E \otimes \bigwedge^{p+1} L,$$

$$\mathrm{Sp}\colon E \otimes \bigwedge^{p-1} L_i \wedge \langle X_i \rangle = 0,$$

$$\mathrm{Sp}\colon E \otimes \bigwedge^{p} L_i \to E \otimes \bigwedge^{p} L_i \wedge \langle X_i \rangle$$

$$\mathrm{Sp} = (-1)^{p+1} L_p^{-1} \wedge \langle X_i \rangle .$$

From the definition of L_p, we have the following identity:

$$(-1)^{p+2} S_{p-1} d_{p-1}(f) L_p = d_{p-1}(f) \wedge \langle X_i \rangle .$$

The above identity and a standard calculation shows that Sp in an homotopy operator, i.e., $d_p S_p + S_{p-1} d_{p-1} = I$ and then $S_p(L, E)$ is a compact set.

THEOREM 3 (*Projection property*). *Let L and E be as usual, and I an ideal of L. Let π be the projection map from L^* onto I^*, then*

$$\mathrm{Sp}(I, E) = \pi(\mathrm{Sp}(L, E)).$$

Proof. By [**2**, **5**, **3**], there is a Jordan Hölder sequence of L such that I is one of its terms. Then, by means of an induction argument, we can assume $\dim(L/I) = 1$.

Let us consider the connected simply connected complex Lie group $G(L)$ such that its Lie algebra is L [**5**, LG, V].

Let Ad^* be the coadjoint representation of $G(L)$ in L^*: $\mathrm{Ad}^*(g)f = f\,\mathrm{Ad}(g^{-1})$, where $g \in G(L)$, $f \in L^*$ and Ad is the adjoint representation of $G(L)$ in L.

Let f belong to $\mathrm{Sp}(I, E)$. Then, as I is an ideal of L, by [**7**, 2.13.4], $\mathrm{Ad}^*(g)f$ belongs to I^*; besides, it is a character of I. Then, one can restrict the coadjoint action of $G(L)$ to I^*. Moreover, $\mathrm{Sp}(I, E)$ is invariant under the coadjoint action of $G(L)$ in I^*, i.e.: if $f \in \mathrm{Sp}(I, E)$, $\mathrm{Ad}^*(g)f \in \mathrm{Sp}(I, E)$ $\forall g \in G(L)$.

In order to prove this fact, it is enough to see:

$$\text{(I)} \qquad \mathrm{Tor}_*^{U(I)}(E, C(f)) \cong \mathrm{Tor}_*^{(U(I))}(E, C(h))$$

where $h = \mathrm{Ad}^*(g)f$, $g \in G(L)$.

Let Γ be the ring $U(I)$ and φ the ring morphism

$$\varphi = U(\mathrm{Ad}\, g)\colon U(I) \to U(I).$$

Let us consider the augmentation modules $(C(f), E(f))$ and $(C(h), E(h))$.

Then, a standard calculation shows that the hypothesis of [**3**, VIII, 3.1] are satisfied, which implies (I).

Thus, if $f \in \mathrm{Sp}(I, E)$, the orbit $G(L) \cdot f \subseteq \mathrm{Sp}(I, E)$. However, $\mathrm{Sp}(I, E)$ is a compact set of I^*.

As the only bounded orbits for an action of a complex connected Lie group on a vector space are points; $G(L) \cdot f = f$.

Let $\overline{f}$ be an extension of f to L^*, and consider $\alpha = G(L) \cdot \overline{f}$, the orbit of $\overline{f}$ under the coadjoint action of $G(L)$ in L^*.

As $G(L) \cdot f = f$, as an analytic manifold

$$\dim \alpha \leq 1 . \tag{II}$$

Now suppose $\overline{f}$ is not a character of L: i.e., $\overline{f}(L^2) \neq 0$.

Let $L^\perp$ be the following set: $L^\perp = \{x \in L | \overline{f}([X, L]) = 0\}$, and let n be the dimension of L.

As I is an ideal of dimension $n-1$, $f(I^2) = 0$ and $f(L^2) \neq 0$, by [**2**, **5**, **3**], [**1**, IV, 4.1] and [**4**, 1, 1.2.8], we have: $L^\perp \subset I$, and $\dim L^\perp = n - 2$.

Let us consider the analytic subgroup of $G(L)$ such that its Lie algebra is $L^\perp$.

As the Lie algebra of the subgroup $G(L)_{\overline{f}} = \{g \in GL | \mathrm{Ad}^*(g)\overline{f} = \overline{f}\}$ is $L^\perp$, the connected component of the identity of $G(L)_{\overline{f}}$ is $G(L^\perp)$.

However, by [**7**, 2.9.1, 2.9.7] $\alpha = G(L) \cdot \overline{f}$ satisfies the following properties: $\alpha \cong G(L)/G(L)_{\overline{f}}$, and $\dim \alpha = \dim G(L) - \dim G(L)_{\overline{f}} = \dim G(L) - \dim(G(L^\perp)) = \dim L - \dim L^\perp = 2$, which contradicts (II).

Then $\overline{f}$ is a character of L.

Thus, any extension $\overline{f}$ of an f in $\mathrm{Sp}(I, E)$ is a character of L.

However, as in [**6**], there is a short exact sequence of complexes

$$0 \to \left(\bigwedge^* I \otimes E, d(f)\right)$$
$$\to \left(\bigwedge^* L \otimes E, d(\overline{f})\right) \to \left(\bigwedge^* I \otimes E, d(f)\right) \to 0 .$$

As $U(I)$ is a subring with unit of $U(L)$ and the complex involved in Definition 1 differs from the one of [**6**] by a constant term, Taylor's argument of [**6**, 13, 3.1] still applies and then $\mathrm{Sp}(I, E) = \Pi(\mathrm{Sp}(L, E))$.

As a consequence of Theorem 3 we have

THEOREM 4. *Let L and E be as usual. Then* $\mathrm{Sp}(L, E)$ *is non-void.*

IV. Some consequences. In this section we shall see some consequences of the main theorems.

Let E be a Banach space and L a complex finite dimensional solvable Lie algebra acting on E as bounded operators.

One of the well known properties of Taylor spectrum for an n-tuple of m.c.o. acting on E is $\mathrm{Sp}(a, E) \subseteq \Pi B[0, \|a_i\|]$. In the noncommutative case, as we have seen in §II, this property fails.

However, if the Lie algebra is nilpotent, it is still true.

PROPOSITION 5. *Let L be a nilpotent Lie algebra which acts as bounded operators on a Banach space E.*

Then, $\mathrm{Sp}(L, E) \subset \{f \in L^* \mid |f(x)| \leq \|x\|,\ x \in L\}$.

Proof. We proceed by induction on $\dim L$. If $\dim L = 1$, we have nothing to verify.

We suppose true the proposition for every nilpotent Lie algebra L' such that $\dim L' < n$.

If $\dim L = n$, by [**2**, **4**, **1**], there is a Jordan Hölder series $S = (L_i)_{0\leq i\leq n}$, such that $[L, L_i] \subseteq L_{i-1}$.

Let $\{X_i\}_{1\leq i\leq n}$ be a basis of L such that $\{X_j\}_{1\leq j\leq i}$ generates L_i.

Let L'_{n-1} be the vector subspace generated by $\{X_i\}_{1\leq i\leq n}$. As $[L, L'_{n-1}] \subseteq L_{n-2} \subset L'_{n-1}$, L'_{n-1} is an ideal. Besides, $L_{n-1} + L'_{n-1} = L$.

Then, by means of Theorem 4 and the inductive hypothesis, we complete the inductive argument and the proposition.

Now, we deal with some consequences of the projection property.

PROPOSITION 6. *Let L and E be as usual.*

If I is an ideal contained in L^2, then $\mathrm{Sp}(I, E) = 0$. *In particular* $\mathrm{Sp}(L^2, E) = 0$.

Proof. By the projection property, $\mathrm{Sp}(I, E) = \Pi(\mathrm{Sp}(L, E))$, where Π is the projection from L^* on I^*. However, as $\mathrm{Sp}(L, E)$ is a subset of characters of L, $f|_I = 0$, if $I \subseteq L^2$.

PROPOSITION 7. *Let L and E be as in Proposition* 5.

If $\mathrm{Sp}(L, E) = 0$, *then* $\mathrm{Sp}(x) = 0$ $\forall x \in L$.

Proof. By means of an induction argument, the ideals L_{n-1}, L'_{n-1} of Proposition 5 and Theorem 3, we conclude the proof.

PROPOSITION 8. *Let L and E be as usual. Then, if $x \in L^2$*: $\mathrm{Sp}(x) = 0$.

Proof. First of all, recall that if L is a solvable Lie algebra, L^2 is a nilpotent one. Then by Proposition 6 $\mathrm{Sp}(L^2, E) = 0$, and by Proposition 7 $\mathrm{Sp}(x) = 0 \ \ \forall x \in L^2$.

V. Remark about the spectral mapping theorem. Note that the example of §II shows that the projection property fails for subspaces which are not ideals (take $I = \langle x \rangle$). Clearly this implies that the spectral mapping theorem also fails in the noncommutative case.

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