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SOME APPLICATIONS OF BELL'S THEOREM TO WEAKLY PSEUDOCONVEX DOMAINS

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In this paper, we study some problems of holomorphic maps in weakly pseudoconvex domains. For example, we consider the boundary version of the rigidity properties both for automorphisms and for self-holomorphic maps, the existence of the interior fixed points for some automorphisms, and the minimal property of the rank of the Levi form at boundary orbit accumulation points of the automorphism groups.

0. Introduction. Suppose Ω is a bounded domain in $\mathbf{C}^n$, and σ_n $(n = 1, 2, \ldots)$, $\sigma \in \operatorname{Aut}(\Omega)$. It is a classical theorem of Cartan which states that if σ_n converges to σ pointwise, then $\sigma_n \to \sigma$ in the topology of $C^\infty(K)$ for any $K \Subset \Omega$.

A natural question to ask is under what circumstances one can conclude furthermore that $\sigma_n \to \sigma$ in $C^\infty(\overline{\Omega})$.

Greene-Krantz [4] solved this problem affirmatively for Ω strongly pseudoconvex by using Fefferman's work on the asymptotic expansion of Bergman kernel function. More recently, Bell [2] proved the following

THEOREM (*Bell*). *Let Ω be a bounded pseudoconvex domain of finite type* (*in the sense of D'angelo*), *and let σ_n, $\sigma \in \operatorname{Aut}(\Omega)$ be such that $\sigma_n \to \sigma$ on compacta. Then $\sigma_n \to \sigma$ in $C^\infty(\overline{\Omega})$. Moreover, if $\sigma_n \to p \in \partial\Omega$* (*i.e., $\{\sigma_n\}$ converges to the constant map $c(z) \equiv p$ on compacta*), *and if $\sigma_n^{-1} \to q \in \partial\Omega$, then $\sigma_n \to p$ in $C^\infty(\overline{\Omega} - \{q\})$.*

As noted in Bell's paper, the first part of this theorem is actually valid for pseudoconvex domains which satisfy condition R. It would be very interesting to extend it to all smooth pseudoconvex domains. However, in this paper, we are only concerned with some applications of this important result and their related developments. For example, the following kinds of problems will be studied:

(A) *The boundary version of the rigidity theorem.* A fundamental theorem of Cartan says that if Ω is a bounded domain in $\mathbf{C}^n$, $p \in \Omega$, and if $f \in \mathrm{Hol}(\Omega, \Omega)$, so that $f(z) = z + o(|z-p|)$ (as $z \to p$), then $f \equiv \mathrm{id}$.

In general, when $p \in \partial\Omega$, the result may not be true. But with some extra-conditions, Krantz [6] could prove the following:

THEOREM (*Krantz*). *Suppose that* Ω *is a strongly pseudoconvex domain not biholomorphic to the ball. Let* $\sigma \in \mathrm{Aut}(\Omega)$, *and let* $p \in \partial\Omega$. *If* $\sigma(z) = z + o(|z-p|)$ $(z \to p)$, *then* $\sigma \equiv \mathrm{id}$.

Our first theorem is a generalization of the above one.

THEOREM 1. *Suppose that* Ω *is a bounded pseudoconvex domain satisfying condition* R, *and suppose that* $\sigma \in \mathrm{Aut}(\Omega)$, $p \in \partial\Omega$. *If* σ *is elliptic and satisfies* $\sigma(z) = z + o(|z-p|)$, *then* $\sigma \equiv \mathrm{id}$.

(B) *The fixed points of an automorphism.* From the Brouwer fixed point theorem, any continuous self map of a closed domain diffeomorphic to the closed ball must have a fixed point. Usually we cannot guarantee that the fixed points have to be interior points even if the map is a holomorphic automorphism. What we may hope is that when the map has some compactness properties, then this should be the case. In fact, Ma [9] proved the following beautiful result:

THEOREM (*Ma*). *Suppose* Ω *is a strongly pseudoconvex domain not biholomorphic to but diffeomorphic to the ball. Then every* $\sigma \in \mathrm{Aut}(\Omega)$ *fixes a point in* Ω.

The second thing we want to do in this paper is to combine Bell's theorem and Ma's idea to prove the following:

THEOREM 2. *Suppose* Ω *is a pseudoconvex domain satisfying condition* R *such that* $\overline{\Omega}$ *is diffeomorphic to the closed ball. If* $\sigma \in \mathrm{Aut}(\Omega)$ *is elliptic, then it must have an interior fixed point.*

(C) *The rank of the Levi form at the boundary accumulation point of the automorphism group.* The current state of the theory in several complex variables indicates that most of the bounded domains in $\mathbf{C}^n$ have compact or even trivial automorphism groups. When $\mathrm{Aut}(\Omega)$ is non-compact, the complex structure on Ω should be determined by its properties near boundary orbit accumulation points. The famous Wong-Rosay theorem ([7]) asserts that if the rank of the Levi form of Ω attains the maximal value $n-1$ at some boundary accumulation point, then Ω has the same complex structure as the unit ball. Our philosophy is that the boundary orbit accumulation point should be the worst point in some sense. Along these lines, we will prove the following:

THEOREM 4. *Let Ω be a bounded pseudoconvex domain of finite type in $\mathbf{C}^n$ and let $p \in \partial\Omega$ be a boundary orbit accumulation point of* $\mathrm{Aut}(\Omega)$. *Then either*

(a) *the rank of the Levi form at p is minimal over $\partial\Omega$; or*

(b) *there exists only one point $p_0 \in \partial\Omega$ where the rank of Levi form is less than that at p. In this case, p_0 is an orbit accumulation point and the common fixed point of* $\mathrm{Aut}(\Omega)$.

The paper is organized as follows: §1 is devoted to the detailed proofs of the above theorems. A boundary version of the classical Carathéodory-Cartan-Kaup-Wu theorem is also studied. Section 2 gives possible extensions of the previous theorems to more general situations. Some further questions are presented.

Most of the notation in this paper is adapted from [2], [7], and [10].

1. Detailed proofs of theorems in §0. In this section, the symbol Ω will denote a bounded C^∞ pseudoconvex domain in $\mathbf{C}^n$ satisfying condition R.

DEFINITION 1. An element $\sigma \in \mathrm{Aut}(\Omega)$ is called elliptic if the closed subgroup of $\mathrm{Aut}(\Omega)$ generated by σ is compact.

DEFINITION 2. A point $p \in \partial\Omega$ is called a point of uniqueness if for every elliptic element $\sigma \in \mathrm{Aut}(\Omega)$ satisfying $\sigma(z) = z + o(|z-p|)$ $(z \to p)$, we can deduce $\sigma \equiv \mathrm{id}$.

REMARKS (1.a). From the definition, when $\mathrm{Aut}(\Omega)$ is compact then every automorphism is elliptic. It is also easy to see that when σ has an interior fixed point, then it must be elliptic, and that σ is not elliptic if and only if $\sigma^{n_k} \to \partial\Omega$ (i.e., $\{\sigma^{n_k}\}$ compactly diverges).

(1.b) The following example (due to S. Krantz for the case of spheres) shows that we cannot expect, in general, that the uniqueness theorem holds for all automorphisms.

Let

$$H(m_1, \dots, m_n) = \{(z_1, \dots, z_n, \omega) : \operatorname{Im}\omega + |z_1|^{2m_1} + \cdots + |z_n|^{2m_n} < 0\}.$$

It is biholomorphic to the egg domain

$$E(m_1, \dots, m_n) = \{(z_1, \dots, z_n, \omega): |\omega|^2 + |z_1|^{2m_1} + \cdots + |z_n|^{2m_n} < 0\}$$

by a rational map f. Define $\sigma_0 \in \mathrm{Aut}(H)$ by

$$\sigma_0(z_1, \dots, z_n, \omega) = (\omega/(1+\omega), z_1/(1+\omega)^{m_1}, \dots, z_n/(1+\omega)^{m_n}).$$

The $\sigma_0(0) = 0$, $d\sigma_0 = \mathrm{id}$ at 0, $\sigma \neq \mathrm{id}$. Let $\sigma = f \circ \sigma_0 \circ f^{-1} \in \mathrm{Aut}(E)$, and denote by $p = f^{-1}(0) \in \partial E$. Then $\sigma = z + o(|z-p|)$, but $\sigma \neq \mathrm{id}$.

LEMMA. *Suppose $H \subset \mathrm{Aut}(\Omega)$ is a compact subgroup. Then the map $D^\alpha L: H \times \overline{\Omega} \to \mathbf{C}^n$, defined by $D^\alpha L(\sigma, z) = \partial^\alpha \sigma / \partial^\alpha z(z)$ for any multi-index α, is continuous.*

Proof. Let $\sigma_n \to \sigma$, $z_n \to z$. By Bell's theorem $\sigma_n \to \sigma$ in $C^\infty(\overline{\Omega})$. Hence for α fixed, $\mathrm{Sup}\,|D^\alpha \sigma_n - D^\alpha \sigma| \to 0$ (as $n \to \infty$), where $D^\alpha = \partial^\alpha / \partial^\alpha z$. Noting that $\sigma \in C^\infty(\overline{\Omega})$, we have

$$\begin{aligned} |D^\alpha L(\sigma_n, z_n) - D^\alpha L(\sigma, z)| &= |D^\alpha \sigma_n(z_n) - D^\alpha \sigma(z)| \\ &\leq |D^\alpha \sigma_n(z_n) - D^\alpha \sigma(z_n)| + |D^\alpha \sigma(z_n) - D^\alpha \sigma(z)| \to 0, \\ &\qquad \text{as } n \to \infty. \quad \square \end{aligned}$$

Let $I: H \times \overline{\Omega} \times \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}$ be defined by $I(\sigma, z, X, Y) = \langle d_z\sigma(X), d_z\sigma(Y)\rangle = \sum D^{\alpha_i} L(\sigma, z)\overline{D^{\alpha_i} L}(\sigma, z) x_i \overline{y_i}$, where $\alpha_i = (0, \dots, 0, 1, 0, \dots, 0)$, $X = (x_i)$, $Y = (y_i)$. For any differential operator $\overline{D}$ with respect to variables z, X, Y, by the above lemma, $\overline{D}(I)$ depends continuously on σ, z, X, Y. Let $d\mu$ be the Haar measure on H. By an elementary argument, it follows that

$$g_z(X, Y) = \int_H I(\sigma, z, X, Y)\, d\mu(\sigma)$$

depends smoothly on z, X, Y. Obviously, $g_z(\cdot, \cdot)$ defines a smooth Riemannian metric on $\overline{\Omega}$, which is invariant under H.

Similarly, let ρ be a smooth function on $\overline{\Omega}$, and let $\rho^*: H \times \overline{\Omega} \to \mathbf{C}$ by $\rho^*(\sigma, z) = \rho(\sigma(z))$. Then for any differential operator D with respect to z, $D\rho(\sigma(z))$ depends continuously on σ and z. Hence

$$h(z) = \int_H \rho^*(\sigma, z)\, d\mu(\sigma)$$

is smooth on $\overline{\Omega}$ and invariant under H.

We may now use the ideas in [7] and [9] to complete the proof of Theorem 1 and Theorem 2.

Proof of Theorem 1. Let $\sigma \in \mathrm{Aut}(\Omega)$ be elliptic, and let H be the compact subgroup generated by σ. Then σ is an isometry of the Riemannian metric g_z. Denote by ν the outward normal vector to $\partial\Omega$ at p, and define $U_\varepsilon = \{\omega \in T_p\overline{\Omega} | \langle \nu , \omega \rangle \leq -\varepsilon\}$ for some small $\varepsilon > 0$. By differential equations, we may find a $\delta > 0$ so that for any $\omega \in U_\varepsilon$ there exists a unique geodesic $\lambda_\omega(t)$ $(0 \leq t \leq \delta)$ with $\lambda_\omega(0) = p$, and $\lambda'_\omega(0) = \omega$. Furthermore, the set of all such geodesics fills out an open subset V of Ω. Since σ is an isometry, it sends geodesics to geodesics. From $d_p\sigma = \mathrm{id}$, we may conclude that $\sigma|_U = \mathrm{id}$ in U. Hence $\sigma = \mathrm{id}$ by the uniqueness property of holomorphic functions. □

Proof of Theorem 2. Let $\rho(z)$ be a smooth defining function of Ω. Then we claim that $h(z)$ is still a defining function. Obviously $h(z) < 0$ if $z \in \Omega$, $h(z) = 0$ if $z \in \partial\Omega$. For any $p \in \partial\Omega$ and outward vector ν_p, since $d_p\sigma(\nu_p)$ is still an outward vector and since $\nu_p(\rho(z)) > 0$, $\nu_p(h(z))$ must not vanish. Now by Morse theory, for $\delta > 0$ small enough, $h^{-1}_{-\delta} = \{z \in \Omega | h(z) \leq -\delta\}$ is diffeomorphic to $\overline{\Omega}$, hence diffeomorphic to the closed ball. Note that σ sends $h^{-1}_{-\delta}$ to itself; hence by the Brouwer fixed point theorem, σ has a fixed point inside $h^{-1}_{-\delta}$. □

REMARKS (2.a). When Ω has a non-compact automorphism group and has a finite type boundary, then from Greene-Krantz [3], Ω is diffeomorphic to the ball. So, in case $\sigma \in \mathrm{Aut}(\Omega)$ is elliptic, it must have an interior fixed point. In other words, $\sigma \in \mathrm{Aut}(\Omega)$ has no interior fixed point if and only if $\sigma^{n_k} \to \partial\Omega$ for some $\{n_k\}$.

(2.b) From Theorem 1, every boundary point of Ω is a point of uniqueness.

(2.c) Although the example in (1.b) shows that the uniqueness theorem may fail for nonelliptic elements, we may still use the direct characterization of the automorphism groups of egg domains to conclude the following:

PROPOSITION 1. *Let* $E(m_1, \dots, m_n)$ *be the egg domain in* $\mathbf{C}^{n+1}$ *defined as in* (1.b). *If* $\sigma \in \mathrm{Aut}(E)$, *and if* $p \in \partial E$ *is so that* $\sigma(z) = z + o((z-1)^2)$ *as* $z \to p$, *then* $\sigma(z) \equiv z$.

However, the situation is quite subtle when we deal with holomorphic maps instead of automorphisms. We will come back to this problem in the next section.

The following result is the boundary version of the Carathéodory-Cartan-Kaup-Wu theorem.

THEOREM 3. *Suppose that* Ω *has a finite type boundary,* $p \in \partial\Omega$, $\sigma \in \mathrm{Aut}(\Omega)$, *and suppose that* $\sigma(p) = p$.

(a) *If* σ *is elliptic, then all the eigenvalues of* $d_p\sigma$ *have modulus* 1, *and in this case there is a linear change of coordinates so that* $d_p\sigma$ *is diagonal.*

(b) *If* σ *is not elliptic, then* σ *cannot have more than two fixed point; and in case* σ *has exactly two fixed points, the modulus of eigenvalues of* $d_p\sigma$ *are either all* > 1 *or all* < 1.

Proof. (a) The argument is very similar to the classical one. For example, let $\xi \in \mathbf{C}^n$ not equal to 0, and let $\lambda \in \mathbf{C}$ be such that $d_p\sigma(\xi) = \lambda\xi$. Choose $\sigma^{n_k} \to \tau \in \mathrm{Aut}(\Omega)$ in $C^\infty(\overline{\Omega})$. Since $d_p\sigma^{n_k}(\xi) = \lambda^{n_k}\xi \to d_p\tau(\xi) \neq 0$, $|\lambda|$ must be 1.

(b) Noting that Ω is taut and that there is no nontrivial analytic variety in $\partial\Omega$, we may assume, by Bell's theorem, that $\sigma^{n_k} \to q_0 \in \partial\Omega$, in $C^\infty(\overline{\Omega} - \{q_1\})$, and $\sigma^{-n_k} \to q_1$ in $C^\infty(\overline{\Omega} - \{q_0\})$ for some subsequence $\{n_k\}$. Suppose that the first assertion of (b) is false. Then σ has two more fixed points p_0, p_1 (besides p). Without loss of generality, assume that $p_0 \neq q_0$; then $q_1 = \lim \sigma^{-n_k}(p_0) = p_0$. Hence $q_0 = \lim \sigma^{n_k}(p) = \lim \sigma^{n_k}(p_1)$, and consequently $p = p_1$. That is a contradiction! Now suppose that σ has only two fixed points p and r. Then, by the above discussion, we can still see that $p \neq q_0$ or $p \neq q_1$. Say $p \neq q_0$; then $p \in \overline{\Omega} - \{q_0\}$ and hence $d_p\sigma^{-n_k} \to 0$. Now the argument in (a) indicates that, in this case, the modulus of the eigenvalues must be strictly bigger than 1. Similarly, if $r \neq q_0$, then the modulus of the eigenvalues will be strictly less that 1. □

REMARKS (3.a). If the σ in Theorem 3 (b) has only one fixed point p, then the eigenvalues of $d_p\sigma$ may have modulus 1 as we saw in (1.b) (actually in that case, $d_p\sigma$ is the identical map).

(3.b) Let D be the unit disc in $\mathbf{C}$, and let $\sigma \in \mathrm{Aut}(D)$ defined by $\sigma(z) = (z-a)/(1-az)$ with $a \in (-1, 1) - \{0\}$. Then σ has two fixed points 1 and -1, and $d_1\sigma = (1+\lambda)/(1-\lambda)$. So a suitable choice of λ can make $d_1\sigma$ any number different from -1, 1.

(3.c) As a consequence of this theorem, every $\sigma \in \mathrm{Aut}(\Omega)$ with three distinct fixed points on $\partial\Omega$ must be elliptic. Hence the proof of the following generalized Hayden-Suffridge theorem [8] becomes a direct application of Theorem 2.

PROPOSITION 2. *Suppose that* $\overline{\Omega}$ *is diffeomorphic to the closed ball and has a finite type boundary. If* $\sigma \in \mathrm{Aut}(\Omega)$ *fixes three distinct points of* $\partial\Omega$, *then* σ *fixes a point of* Ω.

Denote the rank of the Levi form at $z \in \partial\Omega$ by $r(z)$. The following facts are obvious.

(1) $r(-)$ as a function on $\partial\Omega$ is lower-semicontinuous; i.e., if $z_n \to z$, then $\overline{\lim}\, r(z_n) \geq r(z)$.

(2) $r(-)$ is invariant under the automorphism group, i.e., for every $\sigma \in \mathrm{Aut}(\Omega)$, $r(\sigma(z)) = r(z)$.

Proof of Theorem 4. Let p as in the theorem. Without loss of generality, we may choose a series of automorphisms $\{\sigma_n\}$ so that $\sigma_n \to p$, and $\sigma_n^{-1} \to q$. If $p_0 \in \partial\Omega$ with $r(p_0) < r(p)$, then we claim that $p_0 = q$. Actually if that is not the case, by Bell's theorem, $\sigma_n(p_0) \to p$, and hence $r(p) \leq \overline{\lim}\, r(\sigma_n(p_0)) = r(p_0)$. Contradiction! So when $r(p)$ is not minimal, p_0 is the only boundary point with $r(p_0) < r(p)$ and in this instance, by the above fact (2), p_0 has to be the fixed point of $\mathrm{Aut}(\Omega)$. □

REMARK (4.a). If we further assume that Ω is circular in Theorem 4, then (a) is clearly the only possible case; namely, for every boundary accumulation point p of the orbits of $\mathrm{Aut}(\Omega)$, $r(p)$ is minimal over $\partial\Omega$.

2. Some related problems. 1. As we noted in (2.c), the analogue of Theorem 1 for general self-holomorphic maps may be much more complicated. In fact, it was only after the recent work of Burns-Krantz that the situations for strongly pseudoconvex cases were clarified.

THEOREM (*Burns-Krantz* [3]). *Let* $\varphi\colon \Omega \to \Omega$ *be a holomorphic map from the strongly pseudoconvex domain* Ω *to itself, and let* $p \in \partial\Omega$ *so that*

$$\varphi(z) = z + o((z-p)^3)$$

as $z \to p$. *Then* $\varphi(z) \equiv z$.

However, for the weakly pseudoconvex domains, our knowledge is almost nonexistent except for the following conjecture proposed by Burns-Krantz:

Conjecture. Let Ω be a pseudoconvex domain of finite type in $\mathbf{C}^n$, and let $p \in \partial\Omega$. Then there exists a number m_p depending only on the geometric property $\partial\Omega$ at p so that for every $f \in \mathrm{Hol}(\Omega, \Omega)$, if $f(z) = z + o((z-p)^{m_p})$ as $z \to p$, then $f(z) \equiv z$.

The next theorem is a verification of this conjecture for egg domains. Unfortunately, it does not seem that the method in the following discussion can be extended to the general cases (see also the discussion in the §7 of [**3**]).

THEOREM 5. *Let* $E_m = \{(z_1, z_2): |z_1|^2 + |z_2|^{2m} < 1\}$ *be the egg domain in* $\mathbf{C}^2$, *and let* $f \in \mathrm{Hol}(E_m, E_m)$ *so that*

$$f(z) = z + o(|z - \vec{1}|^{3m}) \quad as\ z \to \vec{1}$$

(*where* $\vec{1} = (1, 0)$, m *is a nature number*). *Then* f *must be the identical map.*

The key point of the proof is to push E_m to the ball B by the standard covering map $\pi: E_m \to B$, defined by $\pi(z_1, z_2) = (z_1, z_2^m)$, then to the Siegel domain $H = \{(z_1, z_2) | \operatorname{Re} z_1 > |z_2|^2\}$ by the biholomorphism $\Psi: B \to H$, defined by

$$\Psi(z_1, z_2) = \left(\frac{z_1 + 1}{1 - z_1}, \frac{z_2}{1 - z_1}\right),$$

where we have a lot of tools to work with.

Before turning to the proof, let us observe the following facts and notation:

(a) Ψ is a biholomorphism which sends the point $\vec{1} \in \partial E_m$ to the infinity of the Siegel domain H.

(b) For every point $b = (b_1, b_2) \in \partial H$, the dilation φ_b defined by $\varphi_b(z_1, z_2) = (z_1 + 2z_2\overline{b_2} + b_1, z_2 + b_2)$ is an automorphism of H.

(c) For every point $a \in D = \{\tau \in \mathbf{C} | |\tau| < 1\}$, define $D_a = \{\tau \in \mathbf{C} | |\tau|^2 + |a|^2|1-\tau|^2 < 1\}$, which is still a disc and hence has a smooth boundary. Define the map $j_a \in \mathrm{Hol}(D_a, \mathbf{C}^2)$ by

$$j_a(\tau) = (\tau, a^{1/m}(1-\tau)^{1/m}).$$

Then it is easy to check that j_a is actually a proper holomorphic embedding from D_a to E_m.

(d) Denote $\mathbf{C}^+ = \{\tau \in \mathbf{C} | \operatorname{Re}\tau > 0\}$, and denote by i the first projection map from $\mathbf{C}^2$ to $\mathbf{C}$, i.e., $i(z_1, z_2) = z_1$. Then i sends H to $\mathbf{C}^+$.

Proof of Theorem 5. For every point $a \in D$, set $b = (|a|^2, -a)$. Consider the following compositions of maps:

$$D_a \xrightarrow{j_a} E_m \xrightarrow{f} E_m \xrightarrow{\pi} B \xrightarrow{\Psi} H \xrightarrow{\varphi_b} H \to \mathbf{C}^+.$$

Then

$$\begin{aligned} f \circ j_a(\tau) &= f((\tau, a^{1/m}(1-\tau)^{1/m})) \\ &= (\tau, a^{1/m}(1-\tau)^{1/m}) + o((j_a(\tau) - \vec{1})^{3m}) \\ &= (\tau, a^{1/m}(1-\tau)^{1/m}) + o((\tau-1)^3), \quad \text{as } \tau \to \vec{1}; \end{aligned}$$

$$\begin{aligned} &\pi \circ f \circ j_a(\tau) \\ &\quad = (\tau + o((\tau-1)^3), (a^{1/m}(1-\tau)^{1/m} + o((\tau-1)^3))^m) \\ &\quad = (\tau + o((\tau-1)^3), a(1-\tau) + o((\tau-1)^{3+1/m})); \end{aligned}$$

$$\begin{aligned} &\Psi \circ \pi \circ f \circ j_a(\tau) \\ &\quad = \left(\frac{1+\tau+o((\tau-1)^3)}{1-\tau+o((\tau-1)^3)}, \frac{a(1-\tau)+o((\tau-1)^{3+1/m})}{1-\tau+o((\tau-1)^3)}\right) \\ &\quad = \left(\frac{1+\tau}{1-\tau} + h_1(\tau), a + h_2(\tau)\right), \end{aligned}$$

where h_1, h_2 are holomorphic functions in D_a with vanishing differentials at the boundary point 1;

$$\varphi_b \circ \Psi \circ \pi \circ f \circ j_a(\tau) = \left(\frac{1+\tau}{1+\tau} - |a|^2 - 2ah_2 + h_1, h_2\right); \quad \text{and}$$

$$i \circ \varphi_b \circ \Psi \circ \pi \circ f \circ j_a(\tau) = \frac{1+\tau}{1-\tau} - |a|^2 - 2ah_2 + h_1.$$

Let $g = i \circ \varphi_b \circ \Psi \circ \pi \circ j_a$, $\mathbf{C}_\varepsilon^+ = \{z \in \mathbf{C} | \operatorname{Re} z > \varepsilon\}$ for every positive ε, and let $D_a^\varepsilon = g^{-1}(\mathbf{C}_\varepsilon^+)$. From the above argument and a direct computation, it follows immediately that

(a) $h \triangleq h_1 - 2ah_2$ is continuous on $\overline{D}_a^\varepsilon$,

(b) $\bigcup_{\varepsilon>0} D_a^\varepsilon = D_a$, and

(c) $\operatorname{Re}(\frac{1+\tau}{1-\tau} - |a|^2) = \varepsilon$ on ∂D_a^ε except at $\tau = 1$.

Hence, the fact that $\operatorname{Re}(\frac{1+\tau}{1-\tau} - |a|^2 + h(\tau)) > 0$ on $\partial D_a^\varepsilon - \{1\}$ means exactly that $\operatorname{Re}(h(\tau)) > -\varepsilon$ on ∂D_a^ε. By the maximal principle of harmonic functions and by letting $\varepsilon \to 0^+$, we see that $\operatorname{Re} h \geq 0$. Since $\operatorname{Re} h(0)$ attains its minimal value 0 at 1, and since $\frac{d \operatorname{Re} h}{d\tau}(1) = 0$, the classic Hopf lemma therefore enforces that $h \equiv 0$, i.e., $h_1 \equiv 2ah_2$. On the other hand, on ∂D_a^ε,

$$|h_2|^2 < \operatorname{Re}\left(\frac{1+\tau}{1-\tau} - |a|^2 + h(\tau)\right) = \varepsilon.$$

So the analogous discussion shows that $h_2 \equiv 0$, and consequently, $h_1 \equiv 0$. Passing to f, this implies that $\pi \circ f \circ j_a = \pi \circ j_a$, and thereafter $f \equiv \mathrm{id}$ on $j_a(D_a)$. Finally, noting that $\bigcup_{a \in D} j_a(D_a)$ contains a small neighborhood of the original point of E_m, thus by the uniqueness theorem of holomorphic functions, we now come to the conclusion that $f(z) \equiv z$. □

REMARKS (5.a). The above argument can be used word by word to prove the following:

THEOREM 5′. *Let* $E^*(m_1, \ldots, m_n) = \{(z_1, \ldots, z_n) \in \mathbf{C}^n : |z_1|^{2m_1} + \cdots + |z_n|^{2m_n} < 1\}$ *with* $m_1, \ldots, m_n$ *being positive integers and* $1 \le m_1 \le \cdots \le m_n$. *If* $f \in \mathrm{Hol}(E^*(m_1, \ldots, m_n), E^*(m_1, \ldots, m_n))$ *satisfies that* $f(z) = z + o((z - \vec{1})^{3m_n})$ *as* $z \to \vec{1}$, *then* $f \equiv z$.

(5.b) Let $H_m = \{(z_1, z_2) \in \mathbf{C}^2 | \operatorname{Re} z_1 > |z_2|^{2m}\}$, and define the biholomorphism $\Psi_m : E_m \to H_m$ by sending z to

$$\left(\frac{1 + z_1}{1 - z_1}, \frac{z_2}{(1 - z_1)^{1/m}}\right).$$

For any holomorphic function h from E_m to $\mathbf{C}^+$, we now construct holomorphic maps f_h, f by defining $f_h(z) = \Psi_m(z) + (h(z), 0)$, and $f = \Psi_m^{-1} \circ f_h$. Obviously, $f \in \mathrm{Hol}(E_m, E_m)$. A direct calculation shows that $f(z) = z + o((z - \vec{1})^2)$ if and only if $h(z) = O(z - \vec{1})$. However, this kind of h exists in huge numbers; for example, h may be chosen as $k(1 - z_1)$ for any positive k. This indicates that we do have a lot of holomorphic maps g's from E_m to itself so that $g(z) \neq z$, but $g(z) = z + o((z - \vec{1})^2)$ as $z \to \vec{1}$.

2. Let $P = \{p_0, \ldots, p_n\}$ be a set of $n + 1$ positive integers with $p_j > 1$ for $j \neq 0$. Define

$$D_P = \left\{(z, w) \in \mathbf{C}^{p_0} \times \mathbf{C}^n \,\middle|\, |z|^2 + \sum_\alpha C\alpha |z^\alpha|^2 < 1\right\},$$

where

$$C\alpha = \binom{P}{\alpha} = \prod_j \binom{P_j}{\alpha_j},$$

$|\cdot|$ is the standard euclidean norm, and the sum is taken over those multi-indices such that

$$\sum_j \frac{\alpha_j}{p_j} = 1.$$

Let $\rho(z) = |z|^2 + \sum C_\alpha |z^\alpha|^2 - 1$, the defining function of D_P. A direct computation indicates that the Levi form of ρ attains its minimal value $p_0 - 1$ at points of $S = \{(z, w) | |z| = 1\}$. Since $\mathrm{Aut}(D_P)$ is noncompact and acts transitively on S (that is because the unitary subgroup of the automorphism group of the ball $B = \{z \in \mathbf{C}^{p_0} | |z| < 1\}$ acts transitively on S), it follows from (4.a) that

PROPOSITION 3. *S is the set of boundary accumulation points of* $\mathrm{Aut}(D_P)$; *i.e., "the worst points" of ∂D_P are precisely those accumulated by the orbits of* $\mathrm{Aut}(D_P)$.

A remarkable fact is that the above domains are essentially the only known examples of the finite type domains with noncompact automorphism groups. In fact, Bedford-Pincuk's theorem (**[1]**) asserts that they are precisely those examples in dimension 2. These lead me to conjecture that the first part of Theorem 4 can be the only possible case.

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REFERENCES

[1] E. Bedford and S. Pinchuk, *Domains in $\mathbf{C}^2$ with noncompact groups of automorphisms*, Math. Sb., Ser. **135** (1988), No. 2, 147–157.
[2] S. Bell, *Compactness of Families of Holomorphic Mappings up to the Boundary*, Springer LNM 1268, 29–42.
[3] D. Burns and S. Krantz, *A new rigidity property for holomorphic mappings*, preprint.
[4] R. Greene and S. Krantz, *The automorphism groups of strictly pseudoconvex domain*, Math. Ann., **261** (1982), 425–446.
[5] ——, *Invariant of Bergman geometry and results concerning the automorphism groups of domains in* $\mathbf{C}^n$, (to be published).
[6] S. Krantz, *A new compactness principle in complex analysis*, Univ. Antonoma de Madrid.
[7] ——, *Function Theory of Several Complex Variables*, Wiley, New York, 1982.
[8] T. L. Hayden and T. J. Suffridge, *Biholomorphic maps in Hilbert space have a fixed point*, Pacific J. Math., **38** (1971), 419–422.
[9] D. Ma, *Invariant Metrics on Manifolds*, Ph. D. Thesis, Washington Univ., 1990.
[10] W. Rudin, *Function Theory in the Unit Ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.

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