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FIXED POINTS OF BOUNDARY-PRESERVING MAPS OF SURFACES

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Let X be a compact 2-manifold with nonempty boundary ∂X. Given a boundary-preserving map $f\colon (X, \partial X) \to (X, \partial X)$, let $MF_\partial[f]$ denote the minimum number of fixed points of all boundary-preserving maps homotopic to f as maps of pairs and let $N_\partial(f)$ be the relative Nielsen number of f in the sense of Schirmer [S]. Call X *boundary-Wecken*, bW, if $MF_\partial[f] = N_\partial(f)$ for all boundary-preserving maps of X, *almost bW* if $MF_\partial[f] - N_\partial(f)$ is bounded for all such f, and *totally non-bW* otherwise. We show that if the euler characteristic of X is non-negative, then X is bW. On the other hand, except for a relatively small number of cases, we demonstrate that the 2-manifolds of negative euler characteristic are totally non-bW. For one of the remaining cases, the *pants surface* P, we use techniques of transversality theory to examine the fixed point behavior of boundary-preserving maps of P, and show that P is almost bW.

1. Introduction. Throughout this paper, we will be working in the setting of compact manifolds. Given a map $f\colon X \to X$ of a compact manifold X, we denote the Nielsen number of f by $N(f)$ and let $MF[f]$ be the minimum number of fixed points of all maps homotopic to f. The manifold X is said to be *Wecken* if $MF[f] = N(f)$ for all maps $f\colon X \to X$. Wecken [W] proved that all n-manifolds are Wecken for $n \geq 3$ and Jiang [J] proved that a 2-manifold is Wecken if and only if its euler characteristic is non-negative. The interval is obviously Wecken and it is a classical result that the circle is Wecken.

Now suppose that the manifold X has nonempty boundary ∂X and that f is boundary-preserving, that is, f maps ∂X to itself so f is a map of pairs $f\colon (X, \partial X) \to (X, \partial X)$. We denote the relative Nielsen number by $N_\partial(f)$ and write $MF_\partial[f]$ for the minimum number of fixed points of all maps homotopic to f as maps of pairs. We say that a manifold X with nonempty boundary is *boundary-Wecken*, abbreviated bW, if $MF_\partial[f] = N_\partial(f)$ for all maps $f\colon (X, \partial X) \to (X, \partial X)$. It is obvious that the interval is bW and Schirmer [S] proved that all n-manifolds are bW for $n \geq 4$. The purpose of this paper is to investigate the bW property for boundary-preserving maps of 2-manifolds. We begin, however, with a remark about 3-manifolds. Although all 3-manifolds are Wecken, it is easy to see that not all of them are bW. To construct a simple example, let X be a closed 2-manifold of negative euler characteristic. Let $g: X \to X$ be a map, as in [J], with $MF[g] > N(g)$ and define $f: X \times I \to X \times I$ by $f(x, t) = (g(x), t^2)$ for $x \in X$ and $t \in I$. Then $N_\partial(f) = 2N(g)$ since f has no fixed points on the interior of $X \times I$. On the other hand, a boundary-preserving map of $X \times I$ must take each boundary component to itself, so $MF_\partial[f] = 2MF[g] > N_\partial(f)$. Thus the properties Wecken and bW are not equivalent in the setting of 3-manifolds with boundary.

In contrast to 3-manifolds, it is considerably more difficult to find 2-manifolds with boundary which are not either both Wecken and bW or neither Wecken nor bW. In §2, we will prove that the 2-manifolds with boundary that have non-negative euler characteristic: the disc, annulus and Möbius band, are bW as well as Wecken. In §3, we show that for many surfaces with negative euler chararcteristic, the coincidence of the Wecken and bW properties goes beyond just the absence of these properties. We call a manifold X *totally non-Wecken* if, for any integer m, there is a map $f_m: X \to X$ such that $MF[f_m] - N(f_m) \geq m$. If the manifold X has non-empty boundary, then in the same way we define X to be *totally non-bW* if, for any m, there is a boundary-preserving map f_m with $MF_\partial[f_m] - N_\partial(f_m) \geq m$. It follows from a result of Kelly ([**K2**], Theorem 1.1) that the 2-manifolds with boundary of negative euler characteristic (with possibly a finite number of exceptions) not only fail to have the Wecken property but are in fact totally non-Wecken. In §3, we will show that if $X = S \backslash (D_1 \cup D_2 \cup \cdots \cup D_r)$ is the 2-manifold obtained by removing $r \geq 1$ disjoint open discs D_j from a closed 2-manifold S, then X is totally non-bW if S is *not* in the following list: sphere, projective plane, torus, Klein bottle, connected sum of three projective planes. In addition, the torus minus two or more discs is also totally non-bW.

Thus the possibilities for 2-manifolds with boundary which might behave differently in terms of the Wecken and the bW properties are quite limited. In §4, we carry out a detailed analysis of the fixed point behavior of the homotopy classes of boundary-preserving maps of one such 2-manifold: the sphere with three open discs removed, often called the "pants surface" P. We show that although P is totally non-Wecken, it at least comes very close to the bW property. For maps $f: (P, \partial P) \to (P, \partial P)$, except for a few exceptional cases, we prove that $MF_\partial[f] = N_\partial(f)$. For the remaining cases, we can show that $MF_\partial[f] \leq N_\partial(f) + 1$ and thus P is *almost* bW in the sense that $MF_\partial[f] - N_\partial(f)$ is bounded (by 1) for all f.

The techniques employed in §4 are of independent interest. In many of the proofs, our approach is to use methods of transversality theory to show that a map of the type being considered can be homotoped to one that is in a convenient standard form. It is then possible to describe explicit constructions for further homotoping the map, to one with only the relative Nielsen number of fixed points.

We demonstrate in §4 that the Wecken and bW properties are not identical in the 2-manifold setting, but we do not succeed in characterizing the bW property for all 2-manifolds with boundary. Therefore, in §5 we discuss the problems that remain.

2. Disc, annulus and Möbius band. Throughout the paper, given a boundary-preserving map $f\colon (X, \partial X) \to (X, \partial X)$ of a surface, we will denote the restriction of f to the boundary by $\overline{f}\colon \partial X \to \partial X$.

In this section, we show that the three surfaces with boundary that have non-negative euler characteristic, that is, the ones listed in the title of the section, are bW.

(a) *The disc.* We view the disc D as the unit disc in the complex plane with boundary $\partial D = C$. For a boundary-preserving map $f\colon (D, C) \to (D, C)$, if $\overline{f}$ is of degree $d \neq 1$, then it is homotopic to the map $\varphi_d\colon C \to C$ given by $\varphi_d(z) = z^d$ if $d \geq 0$ and $\varphi_d(z) = \overline{z}^{|d|}$ if $d < 0$. The map φ_d is of degree d and has $|d-1|$ fixed points. Then f is homotopic to a map $g\colon (D, C) \to (D, C)$ whose restriction to C is φ_d and that has no fixed points in the interior of the disc. (See §2 of **[BG]**.) Since $N(\overline{f}) \leq N_\partial(f)$, we have shown that $MF_\partial[f] = N_\partial(f)$ when $d \neq 1$. In the case $d = 1$, it is easy to see that f may be homotoped to a map with $N_\partial(f) = 1$ fixed point.

(b) *The annulus.* Write the boundary of the annulus A as $\partial A = C_0 \cup C_1$ where the C_i are circles. For a map $f\colon (A, \partial A) \to (A, \partial A)$, denote the restriction of f to C_i in the form $f_i\colon C_i \to C_{i^*}$.

LEMMA 2.1. *If $f, g\colon (A, \partial A) \to (A, \partial A)$ are maps such that $\overline{f}, \overline{g}\colon \partial A \to \partial A$ are homotopic, then f and g are homotopic as maps of pairs.*

Proof. Let Σ be the "spindle-shaped" subset of $A \times I$ defined by $\Sigma = (A\times\{0, 1\})\cup(C_0\times I)$. By hypothesis, there is a homotopy between

f_0 and g_0. Since Σ is a strong deformation retract of $A \times I$, there is an extension of that homotopy to a homotopy $H: A \times I \to A$ between f and g. Of course, H might not take $C_1 \times I$ to $C_{1^\#}$. Therefore, we let T be the subset of $A \times I \times I$ defined by

$$T = (A \times I \times \{0\}) \cup (\Sigma \times I) \cup (C_1 \times I \times I).$$

Let $r: A \times I \to A$ be a strong deformation retraction of A onto $C_{1^\#}$. A map from T to A may be defined by letting it be H on $A \times I \times \{0\}$ and on each level of $\Sigma \times I$, and on $C_1 \times I \times I$ it is the composition $r((H|C_1 \times I) \times 1_I)$. By the Homotopy Extension Theorem, this map extends to a map $\Gamma: A \times I \times I \to A$. The restriction of Γ to $A \times I \times \{1\}$ is a homotopy between f and g as maps of pairs. □

THEOREM 2.2. *The annulus is* bW.

Proof. Let $p_0: A = C_0 \times I \to C_0$ be projection and note that $p_0 f$ may be viewed as a homotopy between $p_0 f_0$ and $p_0 f_1$, so choosing orientations of the C_i to agree with a chosen orientation of A, the maps $f_0: C_0 \to C_{0^\#}$ and $f_1: C_1 \to C_{1^\#}$ are of the same degree, call it d and assume for now that $d \neq 1$. If $(0^\#, 1^\#) = (0, 1)$, it follows that $N_\partial(f) \geq N(\overline{f}) = 2|d-1|$. We let $g(z, t) = (\varphi_d(z), t^2)$ for $(z, t) \in C \times I$, for φ_d as in part (a), then g has $2|d-1|$ fixed points for any value of d. The fact that g is homotopic to f as a map of pairs follows from the lemma. If $(0^\#, 1^\#) = (1, 0)$, then $N(\overline{f}) = 0$ so $N_\partial(f) = N(f) = |d-1|$ and in this case we define $g(z, t) = (\varphi_d(z), 1-t)$. If $(0^\#, 1^\#) = (j, j)$, j either 0 or 1, then $N_\partial(f) \geq N(\overline{f}) = N(f_j) = |d-1|$ and letting $g(z, t) = (\varphi_d(z), j)$, we see that g has $N_\partial(f)$ fixed points. Since $\overline{g}$ is homotopic to $\overline{f}$, the lemma completes the argument. The case $d = 1$ is left to the reader. □

(c) *The Möbius band*. We denote the Möbius band by M and picture it as a triangle with vertices v_0, v_1 and v_2, with the edges oriented as $[v_0, v_1]$ $[v_1, v_2]$ and $[v_0, v_2]$. Identify $[v_0, v_1]$ to $[v_1, v_2]$ and call the corresponding embedded curve a. After identification, the edge $[v_0, v_2]$ is a curve we call b. We have given M the structure of a cell complex with 1-skeleton $a \cup b$. In the proof of the lemma that follows, we continue to use the maps $\varphi_d: C \to C$ and we need the obvious maps $a, b: S^1 \to M$. These maps and their homotopy classes will be based.

The key to the proof that the Möbius band is bW is the following result, which establishes for the Möbius band the property that Lemma 2.1 gives us for the annulus.

LEMMA 2.3. *If* $f, g\colon (M, \partial M) \to (M, \partial M)$ *are maps such that* $\overline{f}, \overline{g}\colon \partial M \to \partial M$ *are homotopic, then* f *and* g *are homotopic as maps of pairs.*

Proof. We allow confusion between maps and homotopy classes. Noting that $a\varphi_2 = b$, we have $a^{-1}b = \varphi_2$. By hypothesis, the maps fb and gb are homotopic, so they are of the same degree, call it d. Then $fb = gb = b\varphi_d$. For the maps fa and ga we can find maps $K(f), K(g)\colon S^1 \to S^1$ such that $fa = aK(f)$ and $ga = aK(g)$. Therefore, up to homotopy, we may write

$$\varphi_2\varphi_d = a^{-1}b\varphi_d = a^{-1}fb = K(f)a^{-1}b = K(f)a^{-1}a\varphi_2 = K(f)\varphi_2$$

which implies that $K(f)$ is of degree d. But the same argument works for g as well, so $K(g)$ is also of degree d and we conclude that fa and ga are (based) homotopic. Thus f and g are homotopic on the one-skeleton $a \cup b$ of M. But $\pi_2(M) = 0$ so f and g are in fact homotopic as maps of $(M, \partial M)$. □

THEOREM 2.4. *The Möbius band is* bW.

Proof. For each integer d, we will exhibit a map $f_d\colon (M, \partial M) \to (M, \partial M)$ such that $\overline{f}_d\colon \partial M \to \partial M$ is of degree d and f_d has $N_\partial(f_d)$ fixed points. This will prove the theorem since, given a map $f\colon (M, \partial M) \to (M, \partial M)$ and letting d be the degree of $\overline{f}$, then Lemma 2.3 implies that f is homotopic to that f_d as a map of pairs and it further follows from [S] that $N_\partial(f) = N_\partial(f_d)$. If d is an even integer, we write $d = 2k$. The maps $a, b\colon S^1 \to M$ orient the curves $a(S^1)$ and $b(S^1)$ which we abbreviate as a and b, respectively, and it makes sense to define $\varphi_k\colon a \to b$. We can further retract M onto a in such a way as to identify the restriction of the retraction r to $\partial M = b$ with $\varphi_{-2}\colon b \to a$. We define $f_d = \varphi_k r$ and note that the restriction of f_d to b is φ_d. Since f_d maps M onto ∂M, we see that f_d has $|d-1| = N(\overline{f}_d) = N_\partial(f_d)$ fixed points, as required. When d is odd, we need slightly different constructions of f_d depending on whether d is positive or negative. The case $d = 1$ is easily dealt with by taking f_d to be a fixed point free map. For the case $d = 2k+1 > 1$,

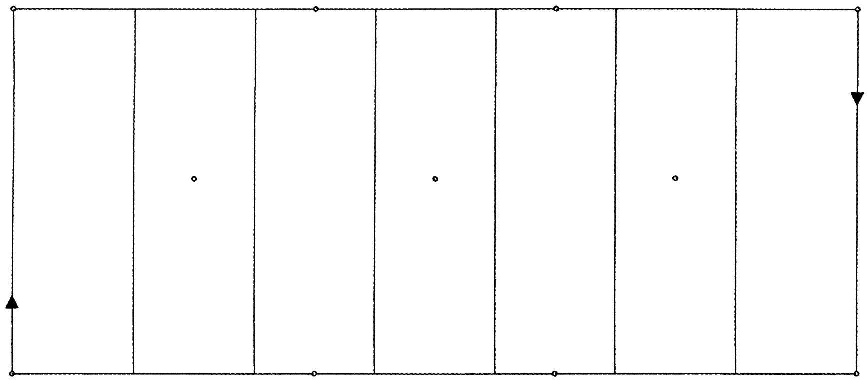

FIGURE 1

we let f_d be the $2k+1$-fold covering map; pictured in Figure 1 for the case $k = 3$.

The left-hand rectangle is stretched over the entire Möbius band and the left-hand edge perturbed slightly so that there are only two fixed points, as indicated in the figure, where those points reappear on the right side in the reversed order. The next rectangle is flipped over as well as stretched over M, so it has only the single fixed point indicated. The third rectangle behaves like the first one: there is a vertical interval of fixed points which can be reduced to two by a perturbation of the interior of the interval. Continuing in this way, we obtain the map $f_d = f_{2k+1}$ with $2k$ fixed points in ∂M and k fixed points in the interior of M. Since both f_d and $\overline{f}_d$ are generically d-to-one maps, they are of degree d and so $N(f_d) = N(\overline{f}_d) = |d-1| = 2k$. It is clear from the construction that when two fixed points lie in the same rectangle, they are Nielsen equivalent, so at most k of the fixed point classes of f_d are represented on ∂M and therefore each of the k fixed points on the interior of M must be a different essential fixed point class of f_d. We conclude that $N_\partial(f_d) = 3k$ and therefore f_d has the required properties in this case. Finally we suppose that $d = 1-2k$ where $k \geq 1$ and we define f_d as a $|1-2k|$-fold cover. The definition is similar to that of the last case, except that each rectangle is reflected about a vertical line segment dividing it in half before it is mapped to M. For the left-most rectangle this reverses the points at the "corners" that were fixed in the previous case and instead we have a fixed point at the center of the interval, as indicated. Now, since the top segment of that rectangle is reversed before mapping to the top of M, it must contain a fixed point. The bottom segment also must contain a fixed point, which is obviously in the same fixed point

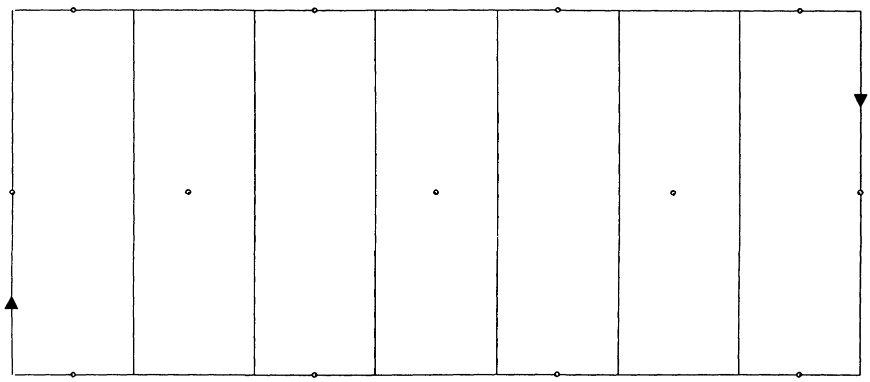

FIGURE 2

class as the one on the top. In the next rectangle the top and bottom segments are still reversed, so we have just a single fixed point, as in the case when d was positive. As Figure 2 shows us in the $k = 4$ case, there are thus $2k$ fixed points on ∂M and another k fixed points in the interior. Since again $N(f_d) = N(\overline{f}_d) = |d - 1| = 2k$ and there are at most k fixed point classes of f_d appearing in ∂M we have $N_\partial(f_d) = 3k$ in this case also. □

3. Totally non-bW surfaces.

THEOREM 3.1. *Let X be the surface obtained in one of the following ways*: (i) *deleting $r \geq 2$ open discs from the torus,* (ii) *deleting $r \geq 1$ open discs from the connected sum of two or more tori,* (iii) *deleting $r \geq 1$ open discs from the connected sum of four or more projective planes. Given an integer $m > \frac{r}{2}$ there exists a map $f\colon (X, \partial X) \to (X, \partial X)$ such that $N_\partial(f) = r$ and $MF_\partial[f] \geq 2m$. Therefore, X is totally non-bW.*

Proof. We first construct a map $f'\colon X \to X$. The surface X can be projected onto the plane as shown in Figure 3 (see next page). In case (i) the broken handle is not present. In the other cases it is present and may be twisted, and there may be additional handles, some of which may be twisted, attached to the dotted region in the figure. Assume for now that we are in case (ii) or (iii). Then $\partial X = C \cup C_1 \cup \cdots \cup C_{r-1}$ where C denotes the "outside" boundary component. The loop α intersects each C_j for $j = 1, \dots, r-1$ in a point x_j. There is a "pants surface" P (disc with two holes) imbedded in X, containing the loops α and β with part of α on the boundary of P, as indicated

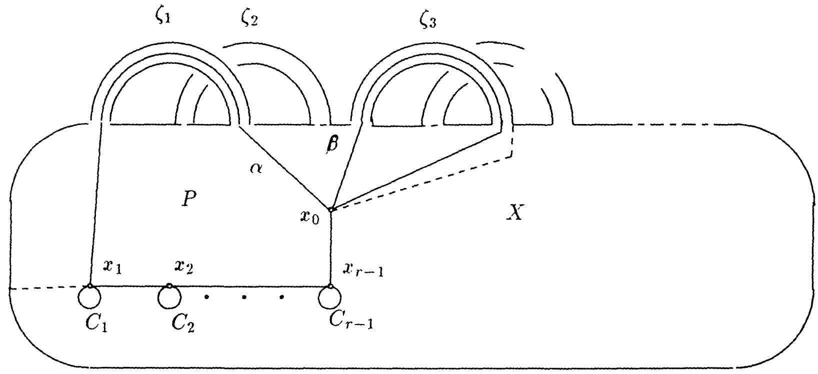

FIGURE 3

by the dashed lines in the figure. Let $\rho: X \to \alpha$ be a retraction such that the loop β is taken to an arc in α and $\rho(C_j) = x_j$. Define a map $\eta: \alpha \to \alpha \cup \beta$ in the following way. Send the arc in α from x_0 to x_1 to itself by the identity map. Then send the rest of α to $\alpha \cup \beta$ so that, as an element of $\pi_1(X, x_0)$, we have α sent to the word $[\beta, \alpha]^m \beta\alpha$, where $[\beta, \alpha]$ denotes the commutator $\beta\alpha\beta^{-1}\alpha^{-1}$. Let $f' = i\eta\rho: X \to X$, where i is the inclusion of $\alpha \cup \beta$ in X. By the commutativity property of the Nielsen number, $N(f') = N(\rho(i\eta)) = 0$ since $\rho(i\eta): \alpha \to \alpha$ is of degree one.

The map $f': X \to X$ is not boundary-preserving because the boundary component C does not go into ∂X, so we must next modify the definition to obtain this property. We note that $\rho|C: C \to X$, the restriction of the retraction to C, is an inessential map. To demonstrate this fact, choose a point x^* on α that lies in the handle through which α passes. Then $(\rho|C)^{-1}(x^*)$ consists of two points on the boundary of the handle, and $\rho|C$ is of opposite degrees at these points, so the degree of $\rho|C$ is zero. We conclude that $f'|C = i\eta\rho|C$ is also inessential. Therefore, there is a homotopy of $f'|C$ to a constant map taking C to a point x_r and thus a homotopy of the restriction of f' to ∂X to a map taking each boundary component to a point x_j, for $j = 1, \ldots, r$, on that component. By the Homotopy Extension Theorem, therefore, we can produce a map $f: (X, \partial X) \to (X, \partial X)$ homotopic to f'. Since $N(f) = N(f') = 0$, it follows that $N_\partial(f) = N(\overline{f}) = r$.

We will show that $MF[f'] = 2m$ and since clearly $MF_\partial[f] \geq MF[f] = MF[f']$, this will complete the proof of Theorem 3.1. The map f' was defined so that its restriction to the pants $f'|P: P \to P$ is the map g_m of Corollary 1.2 of **[K1]** which, according to that result, has the property $MF[g_m] = 2m$. Since P is a retract of X, the inclusion $i: P \to X$ induces a monomorphism of the fundamental groups. The map f' was constructed so that its image is $\alpha \cup \beta$, a subset of P. We clearly have that $i(f'|P) = f'i$, so the hypotheses of Theorem 1.1 of **[K2]** are satisfied, and we conclude that $MF[f'] = MF[f'|P] = 2m$. In case (i), there are only three handles, denoted in Figure 3 by $\zeta_1, \zeta_2, \zeta_3$, and there are $r-2$ curves C_j because the handle ζ_3 determines a boundary component. Since $f'(\beta)$ is contractible, we may still homotope f' to a boundary preserving map and complete the proof as in the other cases. □

4. The pants surface. Let P denote the disc with two holes, known informally as the "pants surface". It was proved in **[K1]** that P is totally non-Wecken: for any integer $m \geq 1$ there is a map $f_m: P \to P$ such that $MF[f_m] - N(f_m) \geq m$. In this section, we will show that with respect to the fixed point theory of boundary-preserving maps, the surface P behaves very differently: $MF_\partial[f] - N_\partial(f) \leq 1$ for any map $f: (P, \partial P) \to (P, \partial P)$.

The components of its boundary, ∂P, are written as C_0, C_1, and C_2. For a map $f: (P, \partial P) \to (P, \partial P)$, we continue to write $f_j: C_j \to C_{j^\#}$ for the restriction of f to each boundary component, just as we did for maps of the annulus.

Assume that P is embedded in the plane so that C_1 and C_2 are contained in the bounded component of the complement of C_0. Choose a clockwise orientation for C_1 and C_2 and a counterclockwise orientation for C_0. Select a base point $x_0 \in C_0$ and arcs ω_j for $j = 1, 2$, each with one endpoint at x_0 and the other at a point x_j in C_j, see Figure 4 on next page.

Define loops σ_j at x_0 by $\sigma_j = \omega_j C_j \omega_j^{-1}$; then we may view $\pi_1(P, x_0)$ as the free group generated by $[\sigma_1]$ and $[\sigma_2]$. Set $[\sigma_0] = [C_0]$ and note that $[\sigma_1][\sigma_2] = [\sigma_0]^{-1}$ in $\pi_1(P, x_0)$.

PROPOSITION 4.1. *Suppose $f: (P, \partial P) \to (P, \partial P)$ is a map such that f_j is essential for all $j = 0, 1, 2$. If $g: (P, \partial P) \to (P, \partial P)$ is homotopic to f as a map from P to itself and $\overline{g}: \partial P \to \partial P$ is homotopic to $\overline{f}$, then g is homotopic to f as a map of pairs.*

Proof. Let $H^{[-1]}: P \times I \to P$ be a homotopy between f and g, the existence of which is given by hypothesis. We will construct homotopies $H^{[j]}$ for $j = 0, 1, 2$ between f and g such that $H^{[j]}$ maps $C_i \times I$ into $C_{i^\#}$ for all $i \leq j$ and therefore $H^{[2]}$ will be the

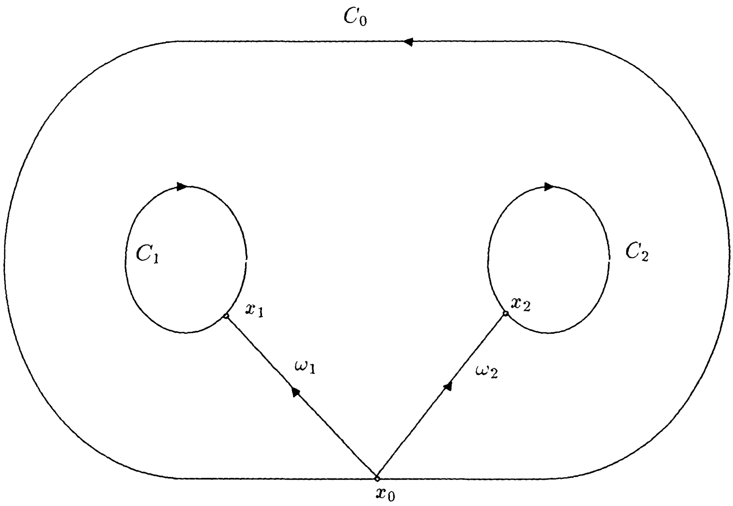

FIGURE 4

required homotopy of pairs. Thus we assume $H^{[j-1]}$ has been constructed and let $\eta: C_j \times I \to P$ be the restriction of $H^{[j-1]}$. Choose an arc a in P that meets ∂P only in its endpoints, one in each component of ∂P other than $C_{j^\#}$. Noting that, by hypothesis, f and g map C_j to $C_{j^\#}$, we make η transverse to $a \operatorname{rel} C_j \times \{0, 1\}$ so that $\eta^{-1}(a)$ is a union of simple closed curves in the interior of $C_j \times I$. Imbed $C_j \times I$ in the plane so that $C_j \times \{0\}$ lies in the bounded component of the complement of $C_j \times \{1\}$. Let K be a component of $\eta^{-1}(a)$; then, by the Schönflies Theorem, the closure of one of the components of the complement of K is a disc D. If D contained $C_j \times \{0\}$, then K and $C_j \times \{0\}$ would bound an annulus and $\eta|K$ and $\eta|(C_j \times \{0\}) = f|C_j = f_j$ would be homotopic. But $\eta(K)$ is contained in the arc a whereas, by hypothesis, f_j is an essential map of C_j onto $C_{j^\#}$, which is freely homotopic, and therefore conjugate in $\pi_1(P, x_0)$, to a nontrivial element of that group. We conclude that D does not contain $C_j \times \{0\}$ but instead lies entirely in $C_j \times I$ and thus K is inessential. Since $\eta^{-1}(a)$ is a union of inessential simple closed curves, we may use an innermost circle argument to homotope η (rel $C_j \times \{0, 1\}$) so that the image is disjoint from a. Furthermore, since $C_{j^\#}$ is a strong deformation retract of $P \backslash a$, we have a homotopy $h: C_j \times I \times I \to P$ between η and a map that takes $C_j \times I$ to $C_{j^\#}$. Let T be the subset of $P \times I \times I$ which is the union of $P \times I \times \{0\}$, $P \times \{0, 1\} \times I$ and all $C_i \times I \times I$ for $i \leq j$. Define a map from T to P to be $H^{[j-1]}$ on $P \times I \times \{0\}$ and $C_i \times I \times \{t\}$ for all t, for $i < j$,

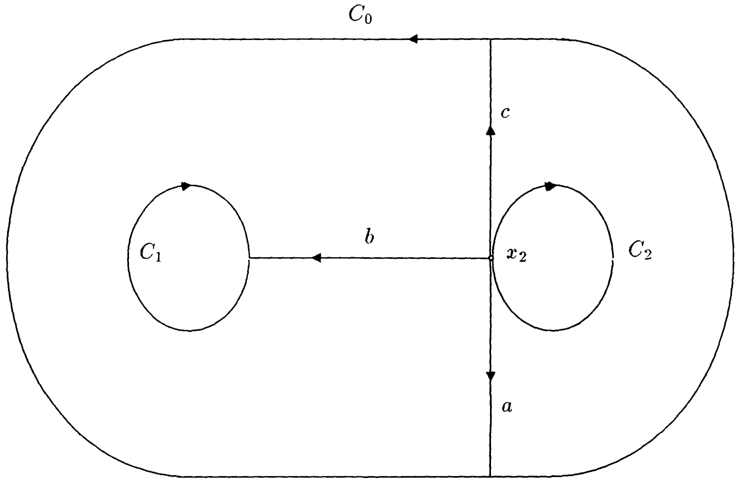

FIGURE 5

to be f and g on $P \times \{0\} \times \{t\}$ and $P \times \{1\} \times \{t\}$, respectively, for all t, and to be h on $C_j \times I \times I$. Extend the map to $\Gamma\colon P \times I \times I \to P$ by the Homotopy Extension Theorem and the required homotopy is the restriction of Γ to $P \times I \times \{1\}$ (compare Lemma 2.1). $\square$

REMARK. Proposition 4.1 is false if the f_j are not essential. Consider maps which are constant on each component of ∂C, constant outside collars of C_0 and C_1 and constant on each parallel circle within the collars. The images of two such maps f and g are illustrated in Figure 5 where $f(P) = a \cup b$, $g(P) = b \cup c$ and both f and g preserve boundary components. These maps are clearly homotopic. Suppose they were homotopic as maps of pairs. From now on we only consider f and g on $a \cup b$. We can assume that f is the identity on $a \cup b$ and g is the identity on b, and $g(a) = c$. Now change g, by a homotopy, so that g is given by $b \mapsto b$, $a \mapsto C_2 a$. Let $H\colon f \simeq g$ on $a \cup b$. The track of x_2 under this homotopy is a loop in C_2. Now the loop once around C_2 is homotopic in P to the loop given by $bC_1^{-1}b^{-1}aC_0^{-1}a^{-1}$ so we can get a homotopy rel x_2 between a modification of the identity given by:

$$a \mapsto (bC_1^{-1}b^{-1}aC_0^{-1}a^{-1})^k a,$$
$$b \mapsto (bC_1^{-1}b^{-1}aC_0^{-1}a^{-1})^k b \quad \text{for some } k$$

and the map g given by $a \mapsto C_2 a$, $b \mapsto b$. Now this homotopy can be assumed to take place in $C_0 \cup a \cup b \cup C_1$ since this space is a strong deformation retract of P. By collapsing $b \cup C_1$ and looking at the maps on a we see that $k = 1$, but by collapsing $a \cup C_0$ and looking at the maps on b we get $k = 0$.

Given a map $f: (P, \partial P) \to (P, \partial P)$, we can homotope f as a map of pairs so that $f(x_j) = x_{j^*}$ for $j = 0, 1, 2$. The maps $f_j: C_j \to C_{j^*}$ are of degrees d_j with respect to the given orientations.

LEMMA 4.2. *Let $f_\pi: \pi_1(P, x_0) \to \pi_1(P, x_0)$ be induced by a map $f: (P, \partial P) \to (P, \partial P)$. Then $f_\pi[\sigma_j]$ is conjugate to $[\sigma_{j^*}]^{d_j}$, for $j = 0, 1, 2$.*

Proof. Since f_j is of degree d_j, then

$$[\omega_{j^*} f(C_j) \omega_{j^*}^{-1}] = [\omega_{j^*} (C_{j^*})^{d_j} \omega_{j^*}^{-1}]$$

in $\pi_1(P, x_0)$, and therefore

$$\begin{aligned} f_\pi[\sigma_j] &= [f(\omega_j C_j \omega_j^{-1})] \\ &= [f(\omega_j)\omega_{j^*}^{-1}][\omega_{j^*} f(C_j)\omega_{j^*}^{-1}][\omega_{j^*} f(\omega_j^{-1})] \\ &= [f(\omega_j)\omega_{j^*}^{-1}][\omega_{j^*} (C_{j^*})^{d_j}\omega_{j^*}^{-1}][\omega_{j^*} f(\omega_j^{-1})] \\ &= [f(\omega_j)\omega_{j^*}^{-1}][\sigma_{j^*}]^{d_j}[\omega_{j^*} f(\omega_j^{-1})]. \end{aligned}$$ □

For a map $f: (P, \partial P) \to (P, \partial P)$, let $\mathrm{Im}_\partial(f)$ denote the number of components of ∂P that contain points of $f(\partial P)$.

LEMMA 4.3. *Let $f: (P, \partial P) \to (P, \partial P)$ be a map such that $\mathrm{Im}_\partial(f) = 3$; then all the maps $f_j: C_j \to C_{j^*}$, for $j = 0, 1, 2$, are of the same degree, d.*

Proof. We can make P into a 2-sphere by attaching discs D_j along the boundary components C_j. The map f then extends to a map $g: S^2 \to S^2$ by using the fact that each D_j is a cone on C_j. After an excision we see that the induced homomorphism $g_*: H_2(D_j, C_j) \to H_2(D_{j^*}, C_{j^*})$ is given by multiplication by $\deg(g)$. The homomorphism extends to a homomorphism of exact sequences of pairs. From this we see that $\deg(g) = \deg(f_j)$. □

LEMMA 4.4. *Let $f: (P, \partial P) \to (P, \partial P)$ be a map such that $\mathrm{Im}_\partial(f) = 3$; then $|d| \leq 1$.*

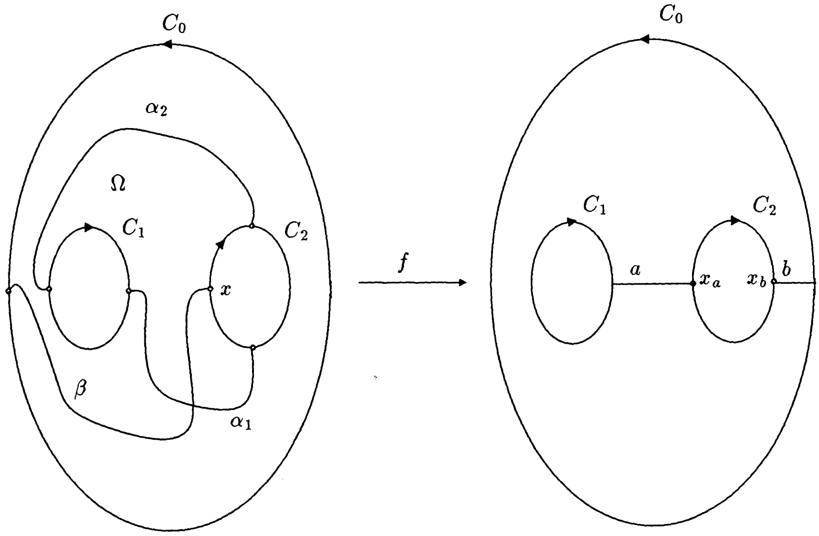

FIGURE 6

Proof. We may assume without loss of generality that f maps each C_j to itself and that each f_j can be identified with a map φ_d from §2. By the preceding lemma, the degree d is the same for all j. Let a and b be arcs connecting C_1 to C_2 and C_2 to C_0, respectively, and make f transverse to $a \cup b$. (See Figure 6.) Now assume that $|d| \geq 2$.

By transversality, $f^{-1}(a)$ contains $|d|$ arcs connecting a point of C_1 to a point of C_2; let α_1 and α_2 be any two of them. Let x_a and x_b be the intersection of C_2 with a and b, respectively. The $|d|$ points of $f_2^{-1}(x_a)$ alternate with the $|d|$ points of $f_2^{-1}(x_b)$. The arcs α_1 and α_2 together with properly chosen arcs of C_1 and C_2 bound a region Ω in P. Let $x \in C_2$ be a point of $f_2^{-1}(x_b)$ that is on the boundary of Ω; then there is an arc β of $f^{-1}(b)$ connecting x to a point of C_0. The arc β must contain points of Ω, yet C_0 is in the unbounded complementary domain of Ω, so the Jordan Curve Theorem implies, since β is contained in P, that β must intersect α_1 or α_2, and that is impossible since a and b are disjoint. We conclude that $|d| \leq 1$. □

LEMMA 4.5. *If* $f\colon (P, \partial P) \to (P, \partial P)$ *is a map with* $\mathrm{Im}_\partial(f) = 3$ *and* $d \neq 0$, *then* $MF_\partial[f] = N_\partial(f)$.

Proof. We claim that the homomorphism $f_\pi\colon \pi_1(P, x_0) \to \pi_1(P, x_{0^\#})$ induced by f is an isomorphism, and furthermore that there is a homeomorphism $h: P \to P$ that induces f_π. Assume first that f maps C_j to itself, for $j = 0, 1, 2$, and that $d = 1$. By Lemma 4.2, we may write $f_\pi[\sigma_1] = \alpha[\sigma_1]\alpha^{-1}$, and $f_\pi[\sigma_2] = \beta[\sigma_2]\beta^{-1}$. We may assume that these expressions have been reduced in the group $\pi_1(P, x_0)$. Certainly we have $f_\pi[\sigma_0]^{-1} = [\sigma_0]^{-1}$. Recalling that $[\sigma_1][\sigma_2] = [\sigma_0]^{-1}$, we see that

$$\alpha[\sigma_1]\alpha^{-1}\beta[\sigma_2]\beta^{-1} = [\sigma_1][\sigma_2].$$

Therefore, the left-hand side must reduce and since we assumed the conjugations were already reduced, we conclude that $\alpha = \beta$. But the only word in $\pi_1(P, x_0)$ that is reduced when conjugated with both generators is the identity, so f_π is the identity isomorphism, which is induced by a homeomorphism, the identity. Now let f retain only the property that $d = 1$, then there is an orientation-preserving homeomorphism θ of P such that θf takes each component of ∂P to itself. By the first part of the proof, $\theta_\pi f_\pi$ is the identity and thus $f_\pi = (\theta_\pi)^{-1} = (\theta^{-1})_\pi$ so f_π is an isomorphism induced by a homeomorphism, $h = \theta^{-1}$. For the case that $d = -1$, we need only choose θ to be orientation-reversing, and this will complete the proof of the claim. The homeomorphism h is homotopic to f because P is a $K(\pi, 1)$. By Lemma 4.2, we see that h must take C_1 and C_2 to the same components of ∂P as f does. Recalling that the f_j are all essential, we see that the hypotheses of Proposition 4.1 are satisfied and therefore h is homotopic to f as a map of pairs. By Theorem 5.1 of **[JG]**, the homeomorphism h is isotopic to a homeomorphism with exactly $N_\partial(h) = N_\partial(f)$ fixed points. □

LEMMA 4.6. *Let $f: (P, \partial P) \to (P, \partial P)$ be a map with* $\mathrm{Im}_\partial(f) = 2$. *Of the maps $f_j: C_j \to C_{j^\#}$, the sum of the degrees of the two that map to the same component of ∂P is zero and the remaining map is inessential.*

Proof. Let $C_{i^\#}$ be the component of ∂P to which one boundary component, C_i, is mapped by f. Then if we attach discs D_i to the domain and $D_{i^\#}$ to the range we have a map of an annulus to an annulus and the first part follows—project the image annulus to a circle. If as in the proof of 4.3 we extend to a map g of a 2-sphere then we can see that g has degree zero, since g is not onto. The result now follows by computing the degree in two ways—look at inverse images of the two discs which are in the image of g. □

Suppose that $f\colon (P, \partial P) \to (P, \partial P)$ is a map. When we want to be specific as to where boundary components go, we will say that f is of *type* $(0^\#, 1^\#, 2^\#)$ to mean that $f_j\colon C_j \to C_{j^\#}$ for $j = 0, 1, 2$. Call f *boundary inessential* if f is null homotopic on each boundary component.

Now suppose $\mathrm{Im}_\partial(f) = 2$ and f is not boundary inessential. Suppose $\mathrm{im}(f) \cap C_0 = \varnothing$, and $f(C_0) = C_1$; then up to numbering of the components of ∂P there are three cases to be considered given by $(1, 1, 2)$, $(1, 2, 1)$, and $(1, 2, 2)$.

THEOREM 4.7. *If $f\colon (P, \partial P) \to (P, \partial P)$ is a map such that $\mathrm{Im}_\partial(f) = 2$ and f is not boundary inessential, then $MF_\partial[f] = N_\partial(f)$.*

Proof. Case $(1, 1, 2)$. We will find a homotopy of f to a map which has no fixed points on the interior of P. Let a be an arc intersecting C_1 and C_2 at x_1 and x_2, respectively, and otherwise disjoint from ∂P. Let $Z = C_1 \cup a \cup C_2$; then there is a deformation retraction $r\colon P \to Z$ and we note that rf is homotopic to f as a map of pairs. By the Homotopy Extension Theorem we may assume that f_1 can be identified with the map φ_d, that we introduced in §2, with $f(x_1) = x_1$. Furthermore, by the preceding lemma, we may assume that f_2 is the constant map at x_2. Choose b_i in C_i for $i = 1, 2$ so that $b_i \neq x_i$ and choose b_3 in the interior of a. We now invoke the theory of transverse cw complexes of **[BRS]**, Chapter 7. After a homotopy we can assume that $\mathrm{im}(f) = Z$ and f is transverse to Z; in particular this means $Y = f^{-1}\{b_1 \cup b_2 \cup b_3\}$ is a 1-manifold, and inverse images of the 1-cells in Z form a trivialised tubular neighbourhood of Y. The regions between the components of the tubular neighbourhood get mapped to $\{x_1, x_2\}$. After a further homotopy rel ∂P, which will eliminate innermost circles, we can assume that Y appears as in Figure 7 (see next page), except that the loop h is not in Y.

The arc a is now partitioned into intervals each of which is mapped into one of the following: C_1, C_2, a, x_1, x_2. Interior fixed points can now only occur in a. After a homotopy we can assume that neighbouring circles of Y are not both mapped to b_3—they would represent a "fold" along a. Consider one of the subintervals of a which is now mapped to a, it will lie between subintervals which are mapped to $\{x_1, x_2\}$. Change the partition of a so that these three subintervals are now counted as one and labelled as a if the map preserves orientation on this amalgamated interval and a^{-1} otherwise.

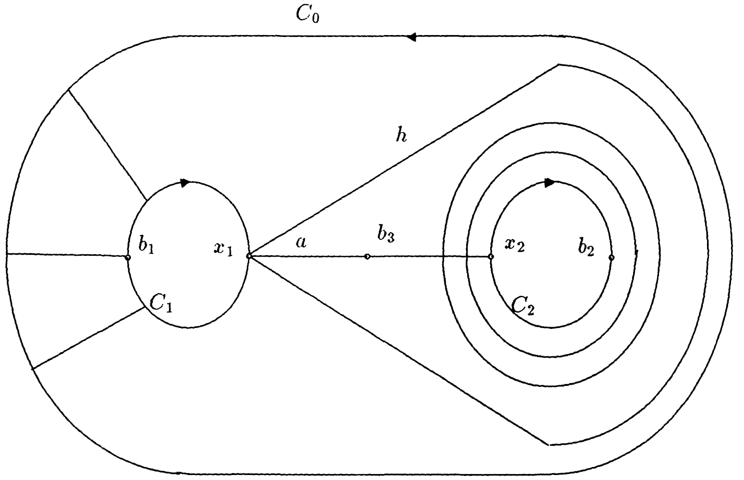

FIGURE 7

Similarly label the remaining intervals by C_i or C_i^{-1} for $i = 1, 2$. Reading from x_1 to x_2 along a we can assume that we get a word of the form:

$$C_1^{n_1} a C_2^{n_2} a^{-1} C_1^{n_3} a C_2^{n_4} a^{-1} \cdots C_1^{n_r} a.$$

The word begins with $C_1^{n_1}$ because $f(x_1) = x_1$, but it may be that $n_1 = 0$. Otherwise we may assume all $n_i \neq 0$. Since $f(x_2) = x_2$, the word could end with $C_2^{n_{r+1}}$. However since f is constant on C_2, we may deform the map so that the final C_2 term vanishes.

Let h be a loop in P based at x_1 and lying in the region outside the regular neighbourhood, so that $f(h) = x_1$. Over each triple of intervals of a that corresponds to an expression $aC_2^k a^{-1}$, we deform that loop based at x_1 to the loop h^k. We still call our map f, though its image is now $Z \cup h$ and its effect on a is now represented by the word:

$$C_1^{n_1} h^{n_2} C_1^{n_3} h^{n_4} \cdots C_1^{n_r} a.$$

The map f has only the fixed point x_1 on h since $f(h) = x_1$ and now the only interval of a that is mapped to a is the one containing x_2, which is stretched over a and thus has a fixed point only at x_2. Thus f has no fixed points on the interior of P. If $d \neq 1$, then f has exactly $N(\overline{f})$ fixed points on ∂P, so certainly $MF_\partial[f] = N_\partial(f)$.

If $d = 1$ then f_1 has x_1 as fixed point. We must modify f so that it has no fixed points on C_1. To do this note that C_1 has a collar neighbourhood N which is mapped to C_1. Change f on N, keeping the outer boundary fixed, and so that f on C_1 is a degree one map

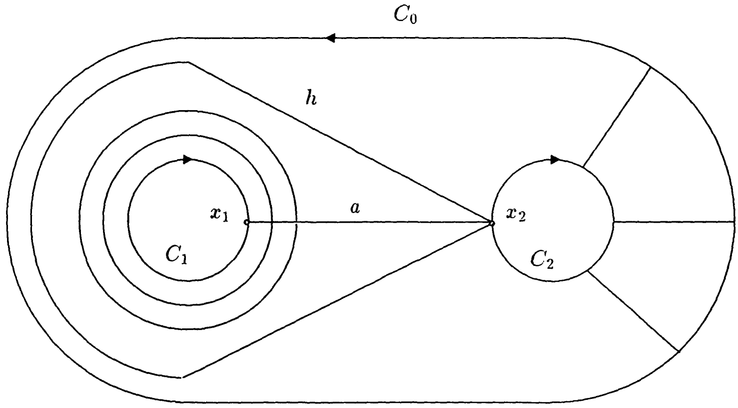

FIGURE 8

without fixed points. There is now only one fixed point in P, namely x_2.

Case (1, 2, 1). In this case Y is pictured in Figure 8 and the image of a is given by a word of the form

$$a^{-1}C_1^{n_1}aC_2^{n_2}a^{-1}C_1^{n_3}\cdots a^{-1}C_1^{n_{r-2}}aC_2^{n_{r-1}}a^{-1}C_1^{n_r}$$

which does not have to begin with a C_2-term because f is constant on C_1. For h the loop in Figure 8, we here replace $a^{-1}C_1^k a$ by h^k to obtain

$$h^{n_1}C_2^{n_2}h^{n_3}\cdots h^{n_{r-2}}C_2^{n_{r-1}}a^{-1}C_1^{n_r}.$$

There are no fixed points on ∂P and a single fixed point on a corresponding to the final a^{-1} in the word. The map f must have at least one fixed point because its Lefschetz number is nonzero.

Case (1, 2, 2). There will be no interior fixed points in this case. The picture for Y is as shown in Figure 9 (see next page). This time we can assume $f(a) = x_2$. □

THEOREM 4.8. *If $f\colon (P, \partial P) \to (P, \partial P)$ is a boundary inessential map then $MF_\partial[f] = N_\partial(f)$.*

Proof. We use the cw decomposition shown in Figure 4. We can assume that $f(\partial P) \subset \{x_0, x_1, x_2\}$. First consider the case $\mathrm{Im}_\partial(f) = 3$. Up to numbering of the components, there are three cases: $(0, 1, 2)$, $(0, 2, 1)$ and $(1, 2, 0)$. In the case $(0, 1, 2)$, we will show that f is homotopic as a map of pairs to a map which has no fixed points on

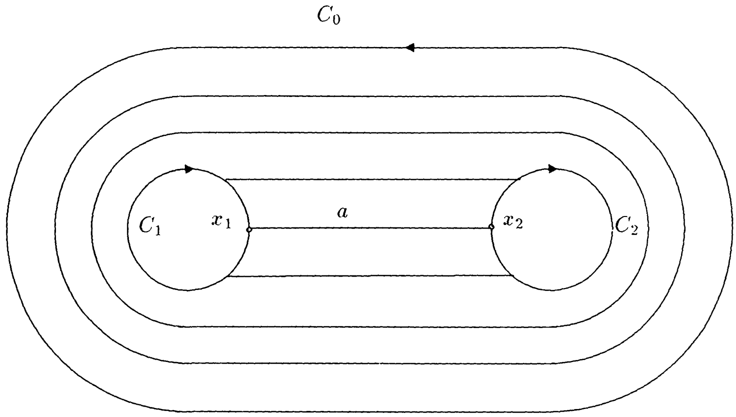

FIGURE 9

the interior of P. Since $\omega_1 \cup \omega_2 \cup C_1 \cup C_2$ is a strong deformation retract of P we can assume that $\mathrm{im}(f) \subset (\omega_1 \cup \omega_2 \cup C_1 \cup C_2)$.

We now make f transverse to the cw complex $\omega_1 \cup \omega_2 \cup C_1 \cup C_2$. The corresponding picture is shown in Figure 10. Any (innermost) outer circle parallel to C_0 could be eliminated by a homotopy which replaces it by a pair of circles—one around C_1 and one around C_2. The curves h_1, h_2 are chosen to lie in a region outside the circles which maps to x_0. As in the proof of 4.7 the fixed points of f must lie on $\omega_1 \cup \omega_2$. The effect of f on ω_1 is given by a word in ω_1, ω_2, C_1, C_2. Any subword of the form $\omega_1 C_1^n \omega_1^{-1}$ can be replaced by h_1^n. It is now easy to see that after this replacement the word represents a map which has no fixed point on the interior of ω_1. Similarly we may use h_2 to eliminate the fixed points on ω_2. The case $(0, 2, 1)$ can be treated similarly. For the case $(1, 2, 0)$ after replacement there will still be an interior fixed point in ω_1 coming from the beginning of the word. This fixed point cannot be eliminated, since the Lefschetz number of f is nonzero.

Now suppose that $\mathrm{Im}_\partial(f) = 2$. Cases $(1, 1, 2)$ and $(1, 2, 1)$ can be proved as $(0, 1, 2)$ and $(1, 2, 0)$ were above, except that for the $(1, 1, 2)$ case we need to switch the labels C_0 and C_1 in Figure 4. The fixed point in the $(1, 2, 1)$ case is again in ω_1 and again the Lefschetz number is nonzero. For the $(1, 2, 2)$ case there will be no fixed points in $\mathrm{int}(P)$—switch C_0 and C_2 in Figure 4. Finally suppose $\mathrm{Im}_\partial(f) = 1$; we can suppose f is of type $(0, 0, 0)$ and a similar argument demonstrates that f can be homotoped to have no interior fixed points in P. □

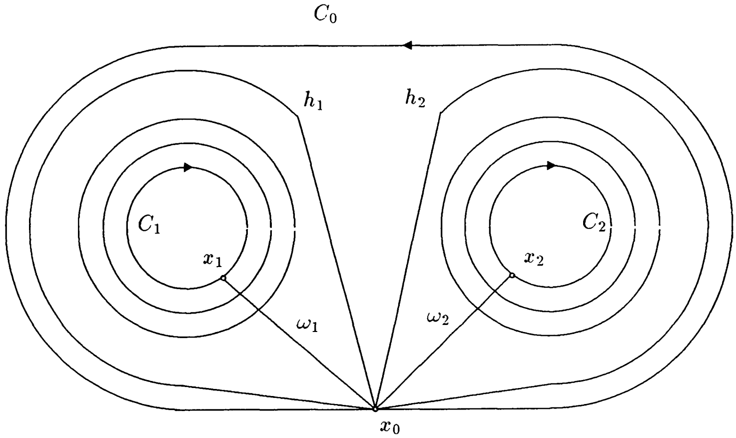

FIGURE 10

THEOREM 4.9. *If* $f\colon (P, \partial P) \to (P, \partial P)$ *is a map such that* $\mathrm{Im}_\partial(f) = 3$, *then* $MF_\partial[f] = N_\partial(f)$.

Proof. This is immediate from 4.5 and 4.8. □

For $f\colon (P, \partial P) \to (P, \partial P)$ with $\mathrm{Im}_\partial(f) = 1$, we will assume that ∂P is mapped to the component C_2, with $f_j\colon C_j \to C_2$ of degree d_j with respect to the orientations above. We may in fact assume that $f_j = \varphi_{d_j}$, as in §2, and that $f(x_j) = x_{j^\#}$. Of the three maps f_j it cannot happen that one is essential while the others are inessential since after attaching two discs to the domain of f we would get a null homotopy in P of the essential map. For two of the f_j essential, it will be sufficient to consider the following cases:

(i) all f_j essential, (ii) f_0 and f_2 essential, f_1 inessential, (iii) f_0 and f_1 essential, f_2 inessential.

THEOREM 4.10. *In each of cases* (i) *and* (ii) $MF_\partial[f] = N_\partial(f)$. *In case* (iii) $MF_\partial[f] \le N_\partial(f) + 1$.

Proof. We have the arc a, the points b_j for $j = 1, 2, 3$ and let $Z = C_1 \cup a \cup C_2$ as in Theorem 4.7. We still assume f maps P to Z and is transverse to Z, and let $Y = f^{-1}(b_1 \cup b_2 \cup b_3)$. We will assume that innermost circles in Y have been removed wherever possible. In each of cases (i) and (ii) we will deform f rel the boundary so that there are no fixed points on $\mathrm{int}(P)$ and we can conclude that $MF_\partial[f] = N(f_2) = N(\overline{f}) = N_\partial(f)$. Furthermore, in the cases where it may be that $d_2 = 1$, we modify f on C_2 so it has no fixed points there, as in the proof of 4.7.

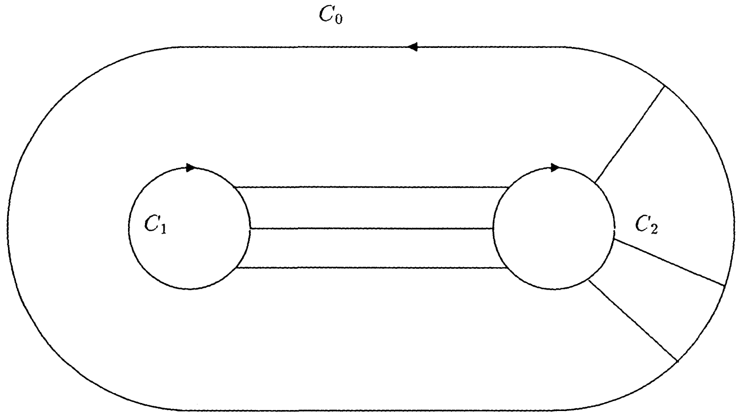

FIGURE 11

Case (i). After removing innermost circles, Y has the form of Figure 11 with no circles in Y, and f now maps all of P to C_2, so there are certainly no fixed points in $\operatorname{int}(P)$.

Case (ii). The manifold Y is pictured in Figure 8. As in the proof of 4.7, we represent the image of a by a word. In the present case, it has the form

$$a^{-1}C_1^{n_1}aC_2^{n_2}a^{-1}C_1^{n_3}a\cdots C_2^{n_{r-2}}a^{-1}C_1^{n_{r-1}}aC_2^{n_r}.$$

We note that the word does not start with a C_2-term because f_1 is inessential in this case. Again as in the proof of 4.7, we deform each loop of the form $a^{-1}C_1^k a$ to h^k and obtain the word

$$h^{n_1}C_2^{n_2}h^{n_3}\cdots C_2^{n_{r-2}}h^{n_{r-1}}C_2^{n_r}.$$

Thus the only possible fixed point on a is at x_2.

Case (iii). We see the form of Y in Figure 7. The word describing f on a is this time

$$C_2^{n_1}a^{-1}C_1^{n_2}aC_2^{n_3}a^{-1}C_1^{n_4}\cdots aC_2^{n_{r-1}}a^{-1}C_1^{n_r}a$$

which does not end in a C_2-term because f_2 is constant. Similar to the argument in case (ii) of this theorem, we replace terms of the form $aC_2^k a^{-1}$ by h^k to obtain the word

$$C_2^{n_1}a^{-1}C_1^{n_2}h^{n_3}C_1^{n_4}\cdots h^{n_{r-1}}C_1^{n_r}a.$$

Recall that the subinterval, at the end of the word, which is mapped to a is made up of three subintervals. The first goes to x_1, the second to a and the third to x_2. There is thus a collar region around C_2 which maps to x_2. We can change f on the last two subintervals so that the x_2 region disappears and the only fixed point corresponding to a in the word is now x_2. However, there must be a fixed point x^* on the interval of a that is mapped to a^{-1}. Since f_2 is the constant map, we see that every map f of Case (iii) is homotopic as a map of pairs to a map with two fixed points, x_2 and x^*. We will show that x_2 and x^* are in the same fixed point class. Let a_- be the sub-arc of a from x_1 to x^* and a_+ be the sub-arc from x^* to x_2. Let ω be the path in Z from x^* to x_2 defined as follows:

$$\omega = a_-^{-1} a C_2^k a^{-1} C_1^{n_2} a C_2^{n_3} a^{-1} \cdots a^{-1} C_1^{n_r} a$$

where

$$k = d(n_2 + n_4 + \cdots + n_r),$$

the n_j are determined by f, as earlier in the proof of this case, and d is the degree of f_1. Keeping in mind that therefore $f(C_1^{n_j}) = C_2^{dn_j}$ whereas $f(C_2^{n_j}) = x_2$ and $f(a_-^{-1}) = a_+ C_2^{-n_1}$ we have

$$f(\omega) = a_+ C_2^{-n_1} C_2^k f(a).$$

Substituting the first word of this case for $f(a)$, we see that $f(\omega)$ is homotopic to ω rel the endpoints, so the fixed points are in the same class. Of course $N(\overline{f}) = 1$ and therefore $N_\partial(f) = 1$. Since our construction gave a map with two fixed points, we can only claim that $MF_\partial[f] \leq 2$ and therefore in this case that $MF_\partial[f] \leq N_\partial(f) + 1$. □

To summarize the results of this section

THEOREM 4.11. *If $f\colon (P, \partial P) \to (P, \partial P)$ is a boundary preserving map of the pants surface, then $MF_\partial[f] \leq N_\partial(f) + 1$ and therefore P is almost bW.*

5. Conclusion. The obvious question remaining from the preceding section is whether or not the pants surface P is bW. We think not, in fact:

Conjecture 1. The disc with two or more discs removed is not bW.

If our conjecture is correct, then P exhibits a new behavior in Nielsen-Wecken fixed point theory: the difference between the minimum number of fixed points in a homotopy class and the Nielsen number is bounded, but it is not zero. A good candidate for a counterexample to the bW property for P comes from case (iii) of Theorem 4.10. The simplest case corresponds to the word $a^{-1}C_1a$. It is easy to construct similar examples on a disc with more than two holes.

Conjecture 2. The disc with three or more discs removed is almost bW.

If the disc with three discs removed is almost bW but not bW, it would be interesting to know whether the bound on $MF_\partial[f] - N_\partial(f)$ is one, as for the pants, or whether the bound must be greater for this, in some sense more complicated, surface. Then one could ask the corresponding question for all these surfaces.

The 2-manifolds with boundary that we have not yet discussed are the Möbius band with one or more discs removed, the torus with one disc removed, the Klein bottle with one or more discs removed and the surfaces obtained by removing discs from the connected sum of three copies of the projective plane. Are these surfaces bW, almost bW or totally non-bW?

References

[BG] R. Brown and R. Greene, *An interior fixed point property of the disc*, Amer. Math. Monthly, to appear.

[BRS] S. Buoncristiano, C. Rourke and B. Sanderson, *A Geometric Approach to Homotopy Theory*, London Mathematical Society Lecture Notes Series 18 (Cambridge University Press, 1976).

[J] B. Jiang, *Fixed points and braids*, II, Math. Ann., **272** (1985), 249–256.

[JG] B. Jiang and J. Guo, *Fixed points of surface diffeomorphisms*, Pacific J. Math., to appear.

[K1] M. Kelly, *Minimizing the number of fixed points for self-maps of compact surfaces*, Pacific J. Math., **126** (1987), 81–123.

[K2] ——, *Minimizing the cardinality of the fixed point set for self-maps of surfaces with boundary*, Michigan Math. J., **39** (1992), 201–217.

[S] H. Schirmer, *A relative Nielsen number*, Pacific J. Math., **122** (1986), 459–473.

[W] F. Wecken, *Fixpunktklassen*, III, Math. Ann., **118** (1942), 544–577.

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