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ON A NON-LINEAR EQUATION RELATED TO THE GEOMETRY OF THE DIFFEOMORPHISM GROUP

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Let M be a compact boundaryless Riemannian manifold. We derive the equations on M which characterize asymptotic vectors on $\mathrm{Diff}_{\mathrm{vol}}(M)$. We classify those M's whose volume-preserving diffeomorphism groups admit asymptotic vectors which are represented by *harmonic* vector fields on M. We then show that these harmonic solutions can be used to construct other (typically non-harmonic) solutions.

Introduction. In this paper, we examine the extrinsic geometry of $\mathrm{Diff}_{\mathrm{vol}}(M)$ as a submanifold of the full diffeomorphism group $\mathrm{Diff}(M)$. We derive and study the equations on M which characterize asymptotic vectors on $\mathrm{Diff}_{\mathrm{vol}}(M)$. These equations on M constitute, a priori, a second order pde system.

The contents of the paper are as follows. Section 1 reviews the weak Riemannian geometry and the 'Lie group'-like structure of $\mathrm{Diff}(M)$ and $\mathrm{Diff}_{\mathrm{vol}}(M)$. Section 2 recalls the Levi-Cività connection on $\mathrm{Diff}(M)$. Section 3 discusses the induced connection on $\mathrm{Diff}_{\mathrm{vol}}(M)$ and derives the equations for asymptotic vectors. In §4, we show that the second order pde system described in §3 is equivalent to a *single* first order equation in the compact boundaryless case. We classify those Riemannian manifolds whose volume-preserving diffeomorphism groups admit asymptotic vectors which are represented by *harmonic* vector fields on M. These harmonic solutions can be used to construct other (non-harmonic) solutions. In particular, we show that any 2-dimensional manifold carries metrics such that the corresponding volume-preserving diffeomorphism group admits asymptotic (but possibly non-harmonic!) vectors. Section 5 discusses the system of equations derived in the previous section, in the setting of noncompact boundaryless 2-dimensional manifolds.

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1. The groups $\mathrm{Diff}(M)$ **and** $\mathrm{Diff}_{\mathrm{vol}}(M)$ **as weak Riemannian manifolds.** Let (M, g) be a C^∞ m-dimensional compact boundaryless manifold. The Riemannian density is denoted by μ and its local expression is given by $\mu(x) = \sqrt{g(x)}\, d^m x$, where $x \in M$, $g(x) = \det(g_{ij}(x))$, and $d^m x$ is the usual Lebesgue measure on $\mathbb{R}^m$. For vector fields X, Y on M, $\nabla_X Y$ denotes the covariant derivative with respect to the Levi-Cività connection of g.

The diffeomorphisms and vector fields used throughout this paper are all of class H^s. We assume throughout this paper that $s > \frac{m}{2} + 1$, which guarantees that all objects are at least C^1. We will denote by $\mathrm{Diff}(M)$ the group of all H^s diffeomorphisms on M, by $\mathscr{X}(M)$ the vector space of all H^s vector fields on M, by

$$\mathrm{Diff}_{\mathrm{vol}}(M) = \{\eta \in \mathrm{Diff}(M) | \eta^* \mu = \mu\}$$

the subgroup of all volume-preserving H^s diffeomorphisms of M, and by

$$\mathscr{X}_{\mathrm{div}}(M) = \{X \in \mathscr{X}(M) | \operatorname{div} X = 0\}$$

the subspace of divergence-free H^s vector fields on M.

The groups $\mathrm{Diff}(M)$ and $\mathrm{Diff}_{\mathrm{vol}}(M)$ are topological groups, C^∞ Hilbert manifolds, and only right translation $R_\eta(\varphi) := \varphi \circ \eta$ is C^∞. They are not Hilbert Lie groups since left translation and inversion are only C^0. $\mathrm{Diff}_{\mathrm{vol}}(M)$ is a closed Hilbert submanifold of $\mathrm{Diff}(M)$. The formal Lie algebras of $\mathrm{Diff}(M)$ and $\mathrm{Diff}_{\mathrm{vol}}(M)$ are $\mathscr{X}(M)$ and $\mathscr{X}_{\mathrm{div}}(M)$ respectively; see **[EM]** and **[MEF]** for more details.

Any tangent vector X_η to $\mathrm{Diff}(M)$ at η is of the form $X \circ \eta$ for some $X \in \mathscr{X}(M)$. For a given $X \in \mathscr{X}(M)$, let X^R denote the right-invariant vector field on $\mathrm{Diff}(M)$ whose value at the identity is X, i.e., $X^R(\eta) = X \circ \eta$. In other words, to get X^R, we use the *same* X at every η. Lie differentiation is functorial on right-invariant vector fields:

$$(1.1) \qquad (\mathscr{L}_{X^R} Y^R)(\eta) = (\mathscr{L}_X Y) \circ \eta = [X, Y]_R \circ \eta,$$

where X, $Y \in \mathscr{X}(M)$, $\eta \in \mathrm{Diff}(M)$, and $\mathscr{L}_X Y = [X, Y]_R$ denotes the usual Jacobi-Lie bracket of vector fields on M; namely, $([X, Y]_R)^i = X^j Y^i_{,j} - Y^j X^i_{,j}$. This bracket is the *right* Lie algebra bracket of $\mathscr{X}(M)$ thought of as the formal Lie algebra of $\mathrm{Diff}(M)$, hence the subscript R.

For $X_\eta, Y_\eta \in T_\eta \mathrm{Diff}(M)$, define the inner product

$$\langle X_\eta, Y_\eta \rangle = \int_M g_{\eta(x)}(X_\eta(x), Y_\eta(x))\mu(x). \tag{1.2}$$

The choice of this non-right-invariant (for a justification of this choice, see the discussion just before Proposition 2) metric makes $\mathrm{Diff}(M)$ into a weak Riemannian manifold. The induced metric on $\mathrm{Diff}_{\mathrm{vol}}(M)$, however, is right-invariant, as can be seen through the use of $\eta^*\mu = \mu$ and a change-of-variables argument.

2. The Levi-Cività connection on $\mathrm{Diff}(M)$. The classical proof of the existence and uniqueness of the Levi-Cività connection on a finite dimensional Riemannian manifold does *not* generalize to infinite dimensional Hilbert manifolds endowed with a weak metric. Only the uniqueness part of the proof stays unchanged. The existence of such a connection is proved in **[EM]**, inspired by an outline in **[Ar]**. Below we shall sketch an alternative approach which avoids the machinery of *connectors* used in **[EM]**.

A. We state for later use a general fact; for the details of the proof one can consult, for example, **[MRR]**. Let G be an arbitrary (possibly infinite dimensional) C^1 manifold which is also a topological group with a C^1 right translation; for example, G can be $\mathrm{Diff}(M)$ or $\mathrm{Diff}_{\mathrm{vol}}(M)$. If Y is a given vector field on G and $g \in G$, denote by $[Y_g]^R$ the right-invariant vector field on G whose value at g is $Y(g)$ (also denoted Y_g). That is,

$$[Y_g]^R(h) = (T_e R_h \circ T_g R_{g^{-1}})(Y_g), \tag{2.1}$$

where $g, h \in G$, R_k denotes right translation by k on G, and $T_s R_k$ is its derivative at $s \in G$.

PROPOSITION 1. *Assume that all vector fields below are of class* C^1 *on* G.

(i) *There exists a unique connection* ∇^c *on* G *such that* $\nabla^c Z = 0$ *for all right-invariant vector fields* Z *on* G. *In fact, for any* $X, Y \in \mathscr{X}(G)$,

$$(\nabla^c_X Y)(g) = (\mathscr{L}_X Y)(g) - (\mathscr{L}_X [Y_g]^R)(g). \tag{2.2}$$

(ii) *Any connection* ∇ *on* G *is completely determined by its values on right-invariant vector fields in the following manner:*

$$(\nabla_X Y)(g) = (\nabla_{[X_g]^R}[Y_g]^R)(g) + (\nabla^c_X Y)(g). \tag{2.3}$$

B. The basic step consists of a functorial relationship between the geodesics on Diff(M) and those on M. For $Z_\eta = Z \circ \eta \in T_\eta \operatorname{Diff}(M)$, we denote by $\underline{\exp}(tZ_\eta)$ the geodesic of the weak Riemannian metric (1.2) with initial velocity Z_η. Ebin and Marsden [EM] proved the local existence of such geodesics by first establishing a rather general theorem about vector fields. They then deduced that

$$\underline{\exp}(tZ_\eta) = (\exp(tZ)) \circ \eta, \tag{2.4}$$

where $[\exp(tZ)](y) := \exp(tZ(y))$ is the geodesic on M through y with initial velocity $Z(y)$.

C. Using (2.4), one directly calculates the values of the Riemannian connection on right-invariant vector fields, and finds that: for $X, Y \in \mathscr{X}(M)$, $\eta \in \operatorname{Diff}(M)$, one has

$$(\underline{\nabla}_{X^R} Y^R)(\eta) = (\nabla_X Y) \circ \eta, \tag{2.5}$$

where ∇ and $\underline{\nabla}$ are the Levi-Cività connections on M and Diff(M) respectively. Note that the torsion-freeness of $\underline{\nabla}$ follows, in view of (2.5), directly from that of ∇. Its metric compatibility can also be ascertained in a similar manner. The values of the connection $\underline{\nabla}$ on arbitrary vector fields of Diff(M) now follows from Proposition 1.

Formula (2.5) implies a functorial relationship between the curvature of M and that of Diff(M), made specific by a statement in the proof of the following proposition. This functorial/natural relationship is an asset because we are using Diff(M) as an ambient space to study the geometry of $\operatorname{Diff}_{\mathrm{vol}}(M)$ and, to this end, the more accessible the geometrical information on the ambient space the better.

PROPOSITION 2. (i) *If M is flat, then so is* Diff(M).

(ii) *If M has non-negative (respectively non-positive) sectional curvatures, then so has* Diff(M).

Proof. (i) By (1.1) and (2.5), together with the tensorial nature of curvatures, we see that the curvature tensors $\underline{R}$ of Diff(M) and R of M are related by

$$\underline{R}_{X_\eta Y_\eta} Z_\eta = (R_{XY} Z) \circ \eta. \tag{2.6}$$

Formula (2.6) shows that if $R_{XY}Z = 0$ for all X, Y, Z, then Diff(M) is flat.

(ii) Let $X_\eta = X \circ \eta$, $Y_\eta = Y \circ \eta \in T_\eta \operatorname{Diff}(M)$ be any two linearly independent tangent vectors at η, set

$$A_{XY}^2 = g(X, X) g(Y, Y) - (g(X, Y))^2$$

and

$$\underline{A}^2_{X_\eta Y_\eta} = \langle X_\eta , X_\eta \rangle \langle Y_\eta , Y_\eta \rangle - \langle X_\eta , Y_\eta \rangle)^2 .$$

Recall that linear independence in $T_\eta \operatorname{Diff}(M)$ is over $\mathbb{R}$, *not* over the functions on M. For example, if $f \colon M \mapsto \mathbb{R}$ is a non-constant function, then $X \circ \eta$ and $(fX) \circ \eta$ are linearly independent. This shows that A^2_{XY} can be identically zero on M while $\underline{A}^2_{X_\eta Y_\eta}$ is strictly positive by the Cauchy-Schwarz inequality.

By the definition of the sectional curvature $\underline{K}$, we have

$$\begin{aligned} \underline{A}^2_{X_\eta Y_\eta} \underline{K}_{X_\eta Y_\eta} &:= \langle \underline{R}_{X_\eta Y_\eta} Y_\eta , X_\eta \rangle = \langle (R_{XY} Y) \circ \eta , X \circ \eta \rangle \\ &= \int_M (K_{XY})_{\eta(x)} (A^2_{XY})_{\eta(x)} \mu(x) . \end{aligned} \tag{2.7}$$

Therefore, a constant sign of the sectional curvature on M implies the same for $\underline{K}$ with the possibility of additional vanishing due to A^2_{XY}, for instance when X, Y are collinear over the functions on M but linearly independent over $\mathbb{R}$. For this reason, we also see that sharpening the hypothesis from non-negative to positive does *not* produce a corresponding change in the conclusion. □

For example, when S^2 is given the usual metric, $\operatorname{Diff}(S^2)$ has non-negative sectional curvature.

If the sign of the sectional curvature of M is non-constant, then that of $\operatorname{Diff}(M)$ cannot, in general, be controlled, as the following example of $\operatorname{Diff}(\mathbb{T}^2)$ shows. Here, $\mathbb{T}^2$ is the torus of revolution obtained by rotating, about the z-axis, the circle of radius r in the xz-plane with center at $x = R > r$. Therefore, a parametric representation of this torus is (with $0 \le \theta \le 2\pi$, $0 \le \varphi \le 2\pi$)

$$(\theta , \varphi) \mapsto ((R + r\cos\varphi)\cos\theta , (R + r\cos\varphi)\sin\theta , r\sin\varphi) , \tag{2.8}$$

and its Gaussian curvature is given by $K = \cos\varphi / [r(R + r\cos\varphi)]$. One finds that

$$A^2_{XY} = (X^1 Y^2 - X^2 Y^1)^2 r^2 (R + r\cos\varphi)^2$$

and, since $K_{XY} = K$, formula (2.7) becomes

$$\underline{A}^2_{X_\eta Y_\eta} \underline{K}_{X_\eta Y_\eta} = \int_{\mathbb{T}^2} ((X^1 Y^2 - X^2 Y^1)^2 ar\cos\varphi)_{\eta(x)} \mu(x) , \tag{2.9}$$

where we have abbreviated the quantity $(R + r\cos\varphi)$ by a.

Let $p(\varphi)$ be an axially symmetric 2π-periodic function on $\mathbb{T}^2$ and let $X = p\frac{\partial}{\partial\theta}$, $Y = \frac{\partial}{\partial\varphi}$. Then (2.9) becomes

$$\underline{A}^2_{X_\eta Y_\eta}\underline{K}_{X_\eta Y_\eta} = \int_{\mathbb{T}^2}(p^2 ar\cos\varphi)_{\eta(x)}\mu(x)\,. \tag{2.10}$$

Note that for each choice of $p(\varphi)$, the corresponding 2-plane in $T_\eta\,\mathrm{Diff}(M)$ has

$$\begin{aligned}\underline{A}^2_{X_\eta Y_\eta} &= \int_{\mathbb{T}^2}(p^2a^2)_{\eta(x)}\mu(x)\int_{\mathbb{T}^2} r^2\mu(x)\\ &= r^2 S\int_{\mathbb{T}^2}(p^2a^2)_{\eta(x)}\mu(x)\,,\end{aligned} \tag{2.11}$$

where $S = 4\pi^2 rR$ is the total surface area of $\mathbb{T}^2$. Therefore (2.10) and (2.11) give

$$\underline{K}_{X_\eta Y_\eta} = \frac{\int_{\mathbb{T}^2}(p^2 a\cos\varphi)_{\eta(x)}\mu(x)}{rS\int_{\mathbb{T}^2}(p^2a^2)_{\eta(x)}\mu(x)}\,. \tag{2.12}$$

Let us draw some conclusions from (2.12). If $\varphi = \pm\frac{\pi}{2}$, we have $\cos\varphi = 0$ and $a = R$. From this we see that if we let $p(\varphi)$ be a bump function which is supported and peaks inside a small neighborhood on the immediate left (resp. right) of $\frac{\pi}{2}$ and is zero outside, $\underline{K}_{X_\eta Y_\eta}$ can be made to take on *any* small positive (resp. negative) values. Secondly, from $a = R + r\cos\varphi$, we deduce

$$|\cos\varphi| \le 1 \le 1 + \frac{r(1+\cos\varphi)}{R-r} = \frac{a}{R-r}\,,$$

so (2.12) implies that $|\underline{K}_{X_\eta Y_\eta}| \le 1/[r(R-r)S]$. Thus, our uncountable family of 2-planes indexed by $p(\varphi)$ is not large enough to tell us whether $|\underline{K}_{X_\eta Y_\eta}|$ can take on arbitrarily large values. Lastly, if $X_\eta = (p\frac{\partial}{\partial\theta})\circ\eta$ and $Z_\eta = \frac{\partial}{\partial\theta}\circ\eta$, then $\underline{A}^2_{X_\eta Z_\eta} > 0$ by the strict Cauchy-Schwarz inequality whereas $\underline{A}^2_{X_\eta Z_\eta}\underline{K}_{X_\eta Z_\eta} = 0$ by (2.9). So $\underline{K}_{X_\eta Z_\eta} = 0$. We have proved the following:

PROPOSITION 3. *Let $\mathbb{T}^2$ be the torus of revolution with radii r and R, $R > r > 0$. For each $\eta \in \mathrm{Diff}(\mathbb{T}^2)$, there exists an uncountable family of 2-planes in $T_\eta\,\mathrm{Diff}(\mathbb{T}^2)$, each indexed by an axially symmetric 2π-periodic continuous function $p = p(\varphi)$, such that the corresponding sectional curvatures of* $\mathrm{Diff}(\mathbb{T}^2)$:

(i) *are uniformly bounded above by $1/[r(R-r)S]$, where $S = 4\pi^2 rR$ is the total surface area of $\mathbb{T}^2$;*

(ii) *take on positive, zero, and negative values;*

(iii) *take on values of arbitrarily small magnitude, so that a positive uniform lower bound* (*on* $|\underline{K}|$ *is not possible.*

3. The connection and the second fundamental form of $\mathrm{Diff}_{\mathrm{vol}}(M)$**.** The metric $\langle\ ,\ \rangle$ on $\mathrm{Diff}(M)$, given by (1.2), induces a metric on $\mathrm{Diff}_{\mathrm{vol}}(M)$ which we denote by the same symbol. As remarked before, this restricted metric is right-invariant. In fact, more is true: given $\eta \in \mathrm{Diff}_{\mathrm{vol}}(M)$, we have

$$\langle X \circ \eta, Y \circ \eta \rangle = \int_M g(X, Y)(x)\mu(x) \tag{3.1}$$

for any vector fields (not necessarily divergence-free) $X, Y \in \mathscr{X}(M)$, as a change-of-variables argument shows; note that the right-hand side is independent of η. Put another way, (3.1) says that whenever the diffeomorphism η is *volume-preserving*, the map $X \mapsto X \circ \eta$ is an isometry from $\mathscr{X}(M)$ onto $T_\eta \mathrm{Diff}(M)$. Now, from Hodge theory one gets (see e.g. **[EM]** or **[MEF]**) the following $\langle\ ,\ \rangle$-orthogonal direct sum decomposition (of closed spaces) on a compact Riemannian manifold:

$$\mathscr{X}(M) = \mathscr{X}_{\mathrm{div}}(M) \oplus \mathrm{Grad}(M), \tag{3.2}$$

where all vector fields are of class H^s, $s > \frac{m}{2} + 1$, and Grad denotes gradient vector fields (relative to g) of H^{s+1} functions on M. Using the aforementioned isometry, this induces an $\langle\ ,\ \rangle$-orthogonal splitting

$$T_\eta \mathrm{Diff}(M) = T_\eta \mathrm{Diff}_{\mathrm{vol}}(M) \oplus (T_\eta \mathrm{Diff}_{\mathrm{vol}}(M))^\perp, \tag{3.3}$$

where

$$(T_\eta \mathrm{Diff}_{\mathrm{vol}}(M))^\perp = \mathrm{Grad} \circ \eta. \tag{3.4}$$

We shall denote by

$$\underline{P}_\eta \colon T_\eta \mathrm{Diff}(M) \to T_\eta \mathrm{Diff}_{\mathrm{vol}}(M)$$

the orthogonal projection given by (3.3) and, if η is the identity, we let $\underline{P}_\eta$ be the projection $\underline{P} \colon \mathscr{X}(M) \to \mathscr{X}_{\mathrm{div}}(M)$ implied by the decomposition in (3.2). Note that on account of the isometry, one has $\underline{P}_\eta(X \circ \eta) = (\underline{P}X) \circ \eta$.

As in §2, the metric $\langle\ ,\ \rangle$ on $\mathrm{Diff}_{\mathrm{vol}}(M)$ induces the Levi-Cività connection; its uniqueness can be proved as in the finite dimensional case, whereas its existence needs a separate argument. This is easier than in §2 due to the splitting (3.3). Put

$$(3.5) \qquad (\underline{\nabla}^v_U V)(\eta) := \underline{P}_\eta(\underline{\nabla}_U V),$$

where the U, V on the left-hand side of (3.5) denote arbitrary vector fields on $\mathrm{Diff}_{\mathrm{vol}}(M)$, while those on the right-hand side denote their (non-unique) extensions to a neighbourhood of $\mathrm{Diff}(M)$ containing $\mathrm{Diff}_{\mathrm{vol}}(M)$. One can verify that (3.5) defines indeed the covariant derivative of a torsion-free affine connection which is compatible with the metric $\langle\ ,\ \rangle$. By the uniqueness of such a connection, it must equal the Levi-Cività connection defined by $\langle\ ,\ \rangle$.

The explicit calculation of $\underline{\nabla}^v_U V$, or for that matter the projection operator $\underline{P}$, involves the solution of a Poisson equation (obtained by taking the divergence of (3.2)) on M, and can be done in terms of Green's functions.

Next, let us consider the decomposition

$$(3.6) \qquad (\underline{\nabla}_U V)(\eta) = \underline{P}_\eta[(\underline{\nabla}_U V)(\eta)] + [(\underline{\nabla}_U V)(\eta)]^\perp$$

according to (3.3), where U, V denote arbitrary vector fields on $\mathrm{Diff}(M)$. It can be checked, as in the finite dimensional case, that the quantity $[(\underline{\nabla}_U V)(\eta)]^\perp$ depends (bilinearly) on U_η and V_η—the values of the vector fields U and V at the point η — and is hence tensorial. It defines the second fundamental form $\underline{S}$ of $(\mathrm{Diff}_{\mathrm{vol}}(M), \langle\ ,\ \rangle)$ in $(\mathrm{Diff}(M), \langle\ ,\ \rangle)$ and its symmetry follows from the torsion-freeness of $\underline{\nabla}$. Exploiting the tensorial nature of $\underline{S}$ and using (2.5), we arrive at the following formula: for tangent *vectors* (rather than vector fields) $X_\eta = X \circ \eta$, $Y_\eta = Y \circ \eta$ on $\mathrm{Diff}_{\mathrm{vol}}(M)$,

$$(3.7) \qquad \underline{S}_\eta(X_\eta, Y_\eta) := [(\underline{\nabla}_{X^R} Y^R)(\eta)]^\perp = (\nabla_X Y - \underline{P}(\nabla_X Y)) \circ \eta .$$

The explicit computation of $\underline{S}_\eta$ can again be done in terms of Green's functions.

4. Asymptotic vectors of $\mathrm{Diff}_{\mathrm{vol}}(M)$ **in** $\mathrm{Diff}(M)$**.** In this section we shall characterize all asymptotic vectors $X_\eta \in T_\eta \mathrm{Diff}_{\mathrm{vol}}(M)$, i.e. all vectors X_η satisfying $\underline{S}_\eta(X_\eta, X_\eta) = 0$, and look for manifolds M for which such vectors exist on $\mathrm{Diff}_{\mathrm{vol}}(M)$. If $X_\eta = X \circ \eta$ with $X \in \mathscr{X}_{\mathrm{div}}(M)$, then (3.7) says that $\underline{S}_\eta(X_\eta, X_\eta) = (\nabla_X X - \underline{P}(\nabla_X X))\circ\eta$. Therefore $X_\eta = X \circ \eta$ is an asymptotic vector if and only if $\operatorname{div} X = 0$ and $\nabla_X X = \underline{P}(\nabla_X X)$, which is equivalent to the system

$$(4.1) \qquad \operatorname{div} X = 0 \quad \text{and} \quad \operatorname{div}(\nabla_X X) = 0 .$$

Let us give a motiviation for studying asymptotic vectors of $\mathrm{Diff}_{\mathrm{vol}}(M)$. Take any $X_\eta \in T_\eta \mathrm{Diff}_{\mathrm{vol}}(M)$ and view it as the time t_0 velocity of some curve η_t in $\mathrm{Diff}_{\mathrm{vol}}(M)$ which happens to pass through the point η at time t_0. Let $V_t := X_t \circ \eta_t$ be the velocity vector field of this curve; then $V_{t_0} = X_\eta$. Considering η_t as a curve in $\mathrm{Diff}(M)$ instead of $\mathrm{Diff}_{\mathrm{vol}}(M)$, the ambient acceleration vector field $\underline{\nabla}_{V_t} V_t$ makes sense. If we project this quantity tangent to $\mathrm{Diff}_{\mathrm{vol}}(M)$ and set the result to zero, we get the condition which characterizes η_t as a geodesic of $\mathrm{Diff}_{\mathrm{vol}}(M)$. The corresponding pde on M is then (see **[Ar]** and **[EM]**) the Euler equation for an incompressible fluid:

$$\frac{\partial X_t}{\partial t} + \nabla_{X_t} X_t = \mathrm{grad}\, p_t ,$$

where X_t is divergence-free and p_t, a time-dependent scalar potential (the negative of the pressure in physics), is part of the "unknowns." On the other hand, if we project the ambient acceleration $\langle\ ,\ \rangle$- orthogonal to $\mathrm{Diff}_{\mathrm{vol}}(M)$, evaluate at time t_0, and set the result to zero, we get $\underline{S}_\eta(X_\eta , X_\eta) = 0$ (essentially because $\underline{S}$ is tensorial). Thus our focus on asymptotic vectors in this paper is naturally complementary to the question studied by Arnold, Ebin, and Marsden.

THEOREM 4. *For a compact boundaryless Riemannian m-manifold* (M , g), $X \circ \eta$ *is an asymptotic vector of* $(\mathrm{Diff}_{\mathrm{vol}}(M), \langle\ ,\ \rangle)$ *in* $(\mathrm{Diff}(M), \langle\ ,\ \rangle)$ *if and only if* X *satisfies the equation*

$$\|\tfrac{1}{2}\mathscr{L}_X g\|^2 - \|\tfrac{1}{2}\, dX^\flat\|^2 + \mathrm{Ric}(X , X) = 0, \tag{4.2}$$

where the tensor norms and our conventions on curvature tensors are clarified below. Also, the vector field X *is of class* H^s, *with* $s > \frac{m}{2} + 1$.

REMARK. On any submanifold, asymptotic vectors are characterized by a *first order* differential equation on *that* submanifold. For example, here on $\mathrm{Diff}_{\mathrm{vol}}(M)$, the first order equation in question is $[(\underline{\nabla}_{X^R} X^R)(\eta)]^\perp = 0$. Note, however, that its restatement on the underlying M is $\mathrm{div}(\nabla_X X) = 0$ (together with $\mathrm{div}\, X = 0$), which is a priori of second order *on* M. Nevertheless, Theorem 4 tells us that the divergence-freeness of X reduces this second order equation to a first order one, *on* M.

Proof. We begin by stating our convention for the Ricci tensor; it will be used in the computations below. Our curvature tensor R is defined as $R_{XY}Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X , Y]} Z$. To avoid clutter, we have left out the subscript R on the Lie-bracket. In components, we have $R_{XY}Z = (R^i_{jkl}X^kY^lZ^j)\partial_i$. The Ricci tensor is symmetric and has components $\mathrm{Ric}_{jl} = R^i_{jil}$; that is $\mathrm{Ric}(Y, Z) = \mathrm{trace}(X \mapsto R_{XY}Z)$ where, on the right-hand side, the trace is that of the linear operator obtained from $R_{XY}Z$ by keeping Y and Z fixed.

Next we recall a Bochner formula (see, for example, [L] or [**BY**]):

$$\|\nabla X^\flat\|^2 - (\Delta X^\flat, X^\flat) + \mathrm{Ric}(X, X) = -\tfrac{1}{2}\Delta[\|X\|^2]. \tag{4.3}$$

Here, $\nabla X^\flat$ is regarded as a covariant 2-tensor, not as a 2-form; hence its point-wise norm-squared is taken in the tensor sense, namely $(\nabla_j X_i)(\nabla^j X^i) = X_{i|j}X^{i|j}$. Also: $(\ ,\ )$ is the inner product on differential forms; $\Delta X^\flat := (\delta_2 d_1 + d_0\delta_1)X^\flat$ and $\Delta = \delta_1 d_0 = -\,\mathrm{div} \circ \mathrm{grad}$ on functions. A calculation in components re-expresses (4.3) as

$$\begin{aligned}\|\tfrac{1}{2}\mathscr{L}_X g\|^2 - \|\tfrac{1}{2}\,dX^\flat\|^2 + \mathrm{Ric}(X, X)\\ = \mathrm{div}(\nabla_X X) - \mathrm{div}[(\mathrm{div}\,X)X] + (\mathrm{div}\,X)^2,\end{aligned} \tag{4.4}$$

where *all* norms are taken in the tensor sense. In other words, even though the *curl*

$$\begin{aligned} dX^\flat &= X_{j,i}\,dx^i \wedge dx^j = \tfrac{1}{2}(X_{j,i} - X_{i,j})dx^i \wedge dx^j \\ &= \sum_{i<j}(X_{j,i} - X_{i,j})dx^i \wedge dx^j \end{aligned}$$

is a 2-form, we choose to rewrite and work with it here as the 2-tensor $(X_{j,i} - X_{i,j})dx^i \otimes dx^j$. Note that $\|dX^\flat\|^2_{\mathrm{tensor}} = 2\|dX^\flat\|^2_{\mathrm{form}}$.

From (4.4), it is clear that (4.1) implies (4.2). The hypothesis that M is compact and boundaryless enters only in establishing the converse. Indeed, suppose (4.2) holds, then (4.4) becomes

$$\mathrm{div}(\nabla_X X) - \mathrm{div}[(\mathrm{div}\,X)X] + (\mathrm{div}\,X)^2 = 0 \tag{4.5}$$

which, upon integrating over the boundaryless M and using the fact that in such case exact divergences integrate to zero (this requires compactness unless one uses function spaces with appropriate decay conditions), yields $\int_M (\mathrm{div}\,X)^2\mu = 0$; hence $\mathrm{div}\,X = 0$ because our X here is at least C^1. Plugging this conclusion back into (4.5) gives $\mathrm{div}(\nabla_X X) = 0$. □

DEFINITION. A vector field X is said to be *asymptotic* if it satisfies the first order non-linear pde (4.2). It is said to be *harmonic* if $\delta X^\flat = 0$ (equivalently, $\mathrm{div}\,X = 0$) and $dX^\flat = 0$; equivalently, since M is boundaryless, if $\Delta X^\flat = 0$.

PROPOSITION 5. *Let X be a harmonic vector field on a compact boundaryless Riemannian manifold (M, g), then X is asymptotic if and only if $\|X\|^2$ is constant on each connected component of M.*

Proof. We use here a slightly re-expressed version of Bochner's formula (4.3), namely:

$$(4.6)\quad \|\tfrac{1}{2}\mathscr{L}_X g\|^2 + \|\tfrac{1}{2}dX^\flat\|^2 - (\Delta X^\flat, X^\flat) + \mathrm{Ric}(X, X) = -\tfrac{1}{2}\Delta[\|X\|^2].$$

(Note that the sign here between the first two terms is a plus, whereas that in (4.4) is a minus.)

By (4.2), a harmonic vector field is asymptotic if and only if

$$\|\tfrac{1}{2}\mathscr{L}_X g\|^2 + \mathrm{Ric}(X, X) = 0.$$

In view of (4.6), this is equivalent to $\Delta(\|X\|^2) = 0$, and to $\|X\|^2 =$ constant on each connected component if M is compact and boundaryless. □

Therefore, in order to classify compact boundaryless manifolds (M, g) which admit a harmonic asymptotic vector field, it suffices to characterize those which carry a harmonic 1-form with constant norm.

We begin by enumerating some consequences of having a closed 1-form $Y^\flat$ on an n-dimensional Riemannian manifold (N, h) which is not necessarily compact. The reason for such a degree of generality is that this will be applied, for example, to the universal Riemannian cover of our compact boundaryless manifold (M, g).

PROPOSITION 6. *Let Y be an H^s, $s > \frac{n}{2} + 1$, vector field on an n-dimensional Riemannian manifold (N, h) such that the corresponding 1-form $Y^\flat$ is closed. Then,*

(A) *The following two properties are equivalent:*

(i) $\|Y\|^2 = h(Y, Y) =$ constant *on each connected component of N,*

(ii) $\nabla_Y Y = 0$, *where ∇ is the Levi-Cività connection of h.*

(B) *The distribution $Y^\perp$, defined as $Y^\perp(x) := \{v \in T_xN \mid h(v, Y(x)) = 0\}$, is involutive, hence integrable, and the leaves foliate N. If Y is nowhere zero, then each leaf is the one-to-one immersion into N of an $(n-1)$-dimensional connected manifold. If, in addition to the above, N is boundaryless and $H^1(N, \mathbb{R}) = 0$, then each leaf is actually a connected component of some level set of a submersion $f\colon N \to \mathbb{R}$,* *and hence is a closed* $(n-1)$*-dimensional connected submanifold of* N.

(C) *If* Y *has nonzero constant length, which without loss of generality may be taken to be* 1, *then (in view of parts* (A) *and* (B)) *its integral curves are unit speed geodesics which intersect orthogonally the leaves of* $Y^{\perp}$. *Let* φ_t *be the time* t *map of the flow of* Y. *Whenever* $\tau > 0$ *is such that* φ_τ *is defined at and near* x, *then for all* $v \in Y^{\perp}(x)$, *we have:*

$$h(\varphi_{t*}v\,,\,Y[\varphi_t(x)]) = 0\,, \qquad 0 \le t \le \tau\,. \tag{4.7}$$

A similar statement would hold if $\tau < 0$. *In other words, the initial orthogonality between* v *and* $Y(x)$ *is preserved by the flow* φ_t.

Proof. (A) Since $Y^\flat$ is closed, we have $Y_{i,j} = Y_{j,i}$ and hence $Y_{i|j} = Y_{j|i}$ because the connection ∇ on N is torsion-free. Using this and the metric-compatibility of ∇, we observe that

$$\begin{aligned} d(\|Y\|^2) &= \nabla(Y_iY^i) = 2Y_{i|j}Y^i\,dx^j = 2Y_{j|i}Y^i\,dx^j \\ &= 2(Y^k_{\ |i}Y^i)h_{kj}\,dx^j = 2(\nabla_Y Y)^\flat\,. \end{aligned}$$

(B) Let U, V be vector fields on N with values in $Y^{\perp}$. Using the fact that ∇ is torsion-free and h-compatible, and that U, V are orthogonal to Y, we can show that $\mathscr{L}_U V$ again takes values in $Y^{\perp}$. Indeed, a calculation gives

$$h(\mathscr{L}_U V\,,\,Y) = \tfrac{1}{2}(Y_{i|j} - Y_{j|i})(U^iV^j - U^jV^i)\,,$$

which vanishes because $Y_{i|j} = Y_{j|i}$ (i.e. $Y^\flat$ is closed). The first half of conclusion (B) thus follows from Frobenius' theorem.

Next suppose $H^1(N, \mathbb{R}) = 0$. Then the closed 1-form $Y^\flat$ is equal to df for some globally defined H^{s+1} function $f\colon N \to \mathbb{R}$. This f is a submersion because Y is nowhere zero. So, on a boundaryless N, its level sets are closed $(n-1)$-dimensional submanifolds of N. Finally, the tangent spaces to these submanifolds are given by the kernels of df, which are simply the hyperplanes $Y^{\perp}$.

(C) Given $v \in Y^{\perp}(x)$, let Q be the leaf which passes through x. There exists a curve $s \mapsto q_s$, $0 \le s \le \xi$ in Q which passes through the point x with initial velocity v. This curve and the 1-parameter family of maps $\{\varphi_t : 0 \le t \le \tau\}$ now define a "rectangle" of curves in the following sense. For each fixed s, one gets a unit speed geodesic $\varphi_t(q_s)$, $0 \le t \le \tau$ which is issued orthogonal to the leaf Q. At each point $\varphi_t(x)$ on the "base geodesic," there emanates a transversal curve $s \mapsto \varphi_t(q_s)$, $0 \le s \le \xi$, with initial velocity vector $\varphi_{t*}v$. Conclusion (4.7) now follows from the formula for the first variation of arc length. □

We are now ready to deduce the following structure theorem which, in view of Proposition 5 and its ensuing remark, completely characterizes those compact boundaryless Riemannian manifolds that admit *harmonic* asymptotic vector fields. This theorem is related to some classical properties of Riemannian foliations (see, for example, [**Mo**]).

THEOREM 7. *Let (M, g) be a compact connected boundaryless m-dimensional Riemannian manifold, and let (N, h) denote its universal Riemannian cover. The following two conditions are equivalent (and, the theorem remains valid if, in its statement, we delete the sentences which come after the semicolons, followed by obvious deletions in the proof):*

(i) *(M, g) admits a nonzero H^s, $s > \frac{m}{2} + 1$, vector field X of constant norm such that $X^\flat$ is closed; and X is divergence-free (i.e., $X^\flat$ is co-closed).*

(ii) *There exists an isometry $\Phi: (\mathbb{R} \times L, dt \otimes dt + h_t^L) \to (N, h)$, where L is a closed boundaryless $(m-1)$-dimensional submanifold of N, h_t^L is a Riemannian metric on the slice $\{t\} \times L$, and the (globally defined) vector field $\Phi_* \frac{\partial}{\partial t}$ is the lift of some globally defined H^s vector field X on M; and μ_t^L—the volume form of h_t^L on $\{t\} \times L$—is independent of t when regarded as an $(m-1)$-form on L.*

Proof. (i) ⇒ (ii). Normalizing by a constant if necessary, we may assume without loss of generality that $\|X\| = 1$. By definition, (N, h) is locally isometric to (M, g); hence X lifts to a vector field Y on N such that: $\|Y\| = 1$, $dY^\flat = 0$, and $\operatorname{div} Y = 0$. Furthermore, since X is complete (because M is compact and boundaryless), so is Y. Since N is connected and simply-connected, its first cohomology is zero; hence $Y^\flat = df$ for some globally defined H^{s+1} function $f: N \to \mathbb{R}$. This f is a submersion because Y is nowhere zero. Let $x \in N$ and $\{\varphi_t : t \in \mathbb{R}\}$ be the flow of the complete vector field Y; one checks that

$$f(\varphi_t(x)) = t + f(x) \quad \text{for all } t \in \mathbb{R}. \tag{4.8}$$

Thus f maps N onto $\mathbb{R}$.

Let $L := f^{-1}(0)$. Since N is boundaryless and f is a submersion, L is a closed boundaryless $(m-1)$-dimensional submanifold of N.

From (4.8), we have $\varphi_t(L) = f^{-1}(t)$, so it is natural to consider the map $\Phi\colon \mathbb{R}\times L \to N$ given by

$$\Phi(t, x) := \varphi_t(x) \quad \text{for all } t\in\mathbb{R},\ x\in N,$$

which is well-defined (because Y is complete) and differentiable. It is surjective because any $z \in N$ can be expressed as $z = \varphi_{f(z)}(x)$, where $x := \varphi_{[-f(z)]}(z)$ lies in L by virtue of (4.8). It is injective because distinct integral curves do not intersect. Therefore Φ is a diffeomorphism.

For each $t\in\mathbb{R}$, the closed $(m-1)$-dimensional submanifold $\varphi_t(L)$ is diffeomorphic to L and is everywhere orthogonal to the unit vector field Y. This orthogonality follows if we first observe that the tangent bundle of L lies in the distribution $Y^{\perp}$, and then use (4.7). Such a geometrical picture is equivalent to the following statement:

$$\Phi^*h = dt\otimes dt + h_t^L, \tag{4.9}$$

where h_t^L is the Riemannian metric on $\{t\}\times L$ given by

$$h_t^L(u, v) := h(\varphi_{t*}u, \varphi_{t*}v). \tag{4.10}$$

On the left-hand side of (4.10), u, v denote tangent vectors of $\{t\}\times L$; but on the right-hand side, they are regarded as tangent vectors of L before having φ_{t*} applied to them. Endowing $\mathbb{R}\times L$ with the metric Φ^*h makes Φ an isometry.

Lastly, let us choose local coordinates $(t, x^1, \dots, x^{m-1})$ on $\mathbb{R}\times L$, write

$$\mu_t^L = k(t, x^1, \dots, x^{m-1})dx^1\wedge\cdots\wedge dx^{m-1}, \tag{4.11}$$

note that $\Phi_*\frac{\partial}{\partial t} = Y$, (hence, since Φ is an isometry,) $\Phi^*\operatorname{div} Y = \operatorname{div}\frac{\partial}{\partial t}$, and (from $\mathscr{L} = i\circ d + d\circ i$) $\mathscr{L}_{\frac{\partial}{\partial t}}\, dt = 0 = \mathscr{L}_{\frac{\partial}{\partial t}}\, dx^j$. Computing $\mathscr{L}_{\frac{\partial}{\partial t}}(dt\wedge\mu_t^L)$ with this information gives:

$$\frac{1}{k}\frac{\partial k}{\partial t} = \operatorname{div}\frac{\partial}{\partial t} = \Phi^*\operatorname{div} Y. \tag{4.12}$$

Thus $\operatorname{div} Y = 0$ gives $\frac{\partial k}{\partial t} = 0$ and completes the proof of (i) $\to$ (ii).

(ii) $\Rightarrow$ (i). From the product structure (4.9) of the metric on $\mathbb{R}\times L$, we see that the globally defined vector field $\frac{\partial}{\partial t}$ has norm 1, and $(\frac{\partial}{\partial t})^\flat = dt$, hence the latter is a closed 1-form; also, $\operatorname{div}\frac{\partial}{\partial t} = 0$ by (4.11), (4.12), and the hypothesis that μ_t^L is independent of t. The same three properties hold for $Y := \Phi_*\frac{\partial}{\partial t}$ because Φ is an isometry. Note, however, that one only needs a local isometry to preserve these three properties. Thus the same is true for the vector field $X := \pi_* Y$ (which incidentally is well-defined *by hypothesis*) because the covering projection $\pi: (N, h) \to (M, g)$ is a local isometry. □

When Theorem 7 is applied to 2-dimensional compact boundaryless surfaces (M, g), more specific information can be obtained. Indeed, in that case the submanifold L is 1-dimensional; hence the product structure described in Theorem 7 implies, among other things, that the universal cover is flat. Consequently (M, g) is also flat and, in view of the Gauss-Bonnet theorem, must be of genus 1. In other words, only a flat torus or a flat Klein bottle can possibly satisfy criterion (ii) of Theorem 7; an inspection shows that they indeed do. We thus have:

COROLLARY 8. *Let (M, g) be a compact boundaryless connected 2-dimensional Riemannian manifold. Then (M, g) admits nonzero harmonic asymptotic vector fields if and only if it is a flat torus or a flat Klein bottle.*

In some special situations, asymptotic (but typically non-harmonic) vector fields can be obtained in the following way,

LEMMA 9. *Let Ω be a "domain" in the compact boundaryless m-dimensional Riemannian manifold (M, g). Suppose*

(1) X is an H^s, $s > \frac{m}{2} + 1$, vector field defined on Ω satisfying $\operatorname{div} X = 0$ and $\nabla_X X = 0$ (note: this is admittedly a stronger requirement than $\operatorname{div}(\nabla_X X) = 0$, but then it only needs to be satisfied on Ω),

(2) f is an H^s function supported in Ω, and

(3) in Ω, f is constant along each integral curve of X;
then the vector field which is fX on Ω and zero elsewhere is asymptotic on (M, g).

This is an immediate consequence of (4.1) and the two formulas below:

$$\operatorname{div}(fX) = f \operatorname{div} X + \mathscr{L}_X f$$

and

$$\nabla_{fX} fX = f^2 \nabla_X X + (\mathscr{L}_X f) fX .$$

PROPOSITION 10. *Let (M, g) be a compact boundaryless m-dimensional Riemannian manifold.*

(i) *If (M, g) admits a nonzero harmonic asymptotic vector field,* *then it also admits an infinite-dimensional (over* $\mathbb{R}$) *set of nonharmonic asymptotic vector fields.*

(ii) *If* (M, g) *is 2-dimensional, then by "slightly" modifying* g *to some other metric* $\tilde{g}$ *if necessary, we can always find asymptotic vector fields on* $(M, \tilde{g})$.

Proof. (i) By Proposition 5 and Theorem 7, we see from the structure of the universal cover that in a small neighbourhood Ω of M, we can find coordinates $(t, x^1, \dots, x^{m-1})$ such that the metric g takes the form

$$dt \otimes dt + h_{ij}(t, x^1, \dots, x^{m-1}) dx^i \otimes dx^j \qquad (1 \le i, j \le m-1),$$

and that $\sqrt{\det(h_{ij})}\, d^{m-1}x$ is independent of t. Thus, the vector field $X := \frac{\partial}{\partial t}$, defined on Ω, satisfies $\operatorname{div} X = 0$ (see (4.12)) and $\nabla_X X = 0$ (by part (A) of Proposition 6). Now let f be any H^s function supported in Ω and depending only on the variables $x^1, \dots, x^{m-1}$. The conclusion follows immediately from Lemma 9.

(ii) Work on any open subset U of M for which the following procedure makes sense; for example, U can be any geodesic disc of (M, g), or it can be the domain of a local chart. We shall see that once a workable U is chosen, all the modifications of the metric g can be localized in U.

In U, take a simple closed curve and generate a tubular neighbourhood Ω (still in U) which is diffeomorphic, through some map Φ, to a "vertical" cylinder $(S^1 \times I, d\theta \otimes d\theta + dz \otimes dz)$ in Euclidean $\mathbb{R}^3$. Equip Ω with the product metric $\Phi^*(d\theta \otimes d\theta + dz \otimes dz)$, which we then prolongate to a Riemannian metric h on M.

Next we use a standard procedure to confine the changes on g to within the neighbourhood U. Construct a smooth function $\psi: M \to \mathbb{R}$ with the following properties: $0 \le \psi \le 1$; $\psi = 1$ on Ω; $\psi = 0$ on $M \setminus U$. The "slightly" modified metric $\tilde{g}$ is then defined to be $\psi h + (1 - \psi) g$. It is still Riemannian because the space of all Riemannian metrics is an open convex cone in the space of metrics. Denote its Levi-Cività connection by $\widetilde{\nabla}$.

Since Φ is an isometry between $(\Omega, \tilde{g})$ and $(S^1 \times I, d\theta \otimes d\theta + dz \otimes dz)$, one can readily verify that the vector field $X := \Phi_*^{-1} \frac{\partial}{\partial \theta}$, defined on Ω, satisfies $\widetilde{\nabla} \cdot X = 0$ and $\widetilde{\nabla}_X X = 0$. Now let f be the pullback under Φ of any H^s function supported in $S^1 \times I$ and depending only on the variable z. We are thus again in a position to apply Lemma 9. $\square$

REMARKS. (1) We hasten to point out a subtlety in the above construction. It is imperative that Ω be given a geometry which is isometric to a cyclinder $(S^1 \times I, d\theta \otimes d\theta + dz \otimes dz)$ in $\mathbb{R}^3$ and not to an annulus $(S^1 \times I, r^2 d\theta \otimes d\theta + dr \otimes dr)$ in $\mathbb{R}^2$, even though both are flat. On the practical level, this means that in deforming the given surface, one must take care to create a 'vertical' cylindrical band and not a 'horizontal' annular plateau. More discussion on this issue is given in **[BR]**.

(2) By Corollary 8, the torus of revolution (which is geometrically different from the flat torus) does not admit any nonzero harmonic asymptotic vector fields. Proposition 10, however, assures us that if we flatten one of its two equators into a vertical equatorial band, then asymptotic (though typically non-harmonic) vector fields do exist on the slightly deformed surface.

5. The non-compact two-dimensional case. Although the geometrical interpretation in terms of $\mathrm{Diff}_{\mathrm{vol}}(M)$ fails, the system (4.1), namely, $\operatorname{div} X = 0$ and $\operatorname{div}(\nabla_X X) = 0$, still makes sense in the non-compact boundaryless case. Here we give some examples which seemed to us geometrically interesting, restricting ourselves to 2-dimensional surfaces without boundary.

On a 2-manifold (M, g), divergence-free vector fields can be represented by "stream function"—which are in general only locally defined unless $H^1(M, \mathbb{R})$ vanishes—as follows. Let M be orientable (otherwise, work with its orientable cover) and let $*$ denote the Hodge star operator. The condition $\operatorname{div} X = 0$ is, up to a sign, the same as $*d * X^\flat = 0$. Hence *locally* we have $*X^\flat = df$ for some stream function f; in other words,

$$X = -(*df)^\# \tag{5.1}$$

solves the $\operatorname{div} X = 0$ equation on the domain of f. Here, X is H^s with $s > \frac{m}{2} + 1 = 2$, and f is H^{s+1}. It can be checked that geometrically, (5.1) says that X is the gradient of f rotated "clockwise" (relative to the chosen orientation) by 90 degrees in each tangent plane of M. Also, the demand that X is to be a globally defined vector field means that its collection of local stream functions must differ only by constants on their domain overlaps.

Next, we plug (5.1) into $\operatorname{div}(\nabla_X X) = 0$ to obtain a pde that f must satisfy. Such a calculation is detailed in **[BR]**, and the result is the following degenerate Monge-Ampère equation:

$$\det(f_{|i|j}) = \tfrac{1}{2} K \det(g_{ij}) \|df\|^2, \tag{5.2}$$

where K is the Gaussian curvature function of the metric g (our convention being that the round 2-sphere of radius a has $K = +1/a^2$), $\|df\|^2 = f_{,i}f_{,j}g^{ij}$, and $f_{|i|j} = f_{,i,j} - f_{,k}\Gamma^k{}_{ij}$.

For the sake of simplicity, we shall restrict our search to globally defined solutions of (5.2), even if $H^1(M, \mathbb{R}) \neq 0$; that is, we are solving $\operatorname{div}(\nabla_X X) = 0$ for those X such that $*X^\flat$ is globally exact.

EXAMPLE 1. The Euclidean plane. In this case, (5.2) reduces to $f_{,x,x}f_{,y,y} - (f_{,x,y})^2 = 0$, which says that the graph $\{(x, y, z)|z = f(x, y), (x, y) \in \mathbb{R}^2\}$ is a complete, boundaryless, flat, injectively immersed surface in $\mathbb{R}^3$. These are then precisely the so-called generalized cylinders (see, for example, [S] Chapter 5) which hover over the xy-plane. Consequently, it is not hard to see that the solutions $f = f(x, y)$ are of the form $\varphi(ax + by + c)$, where a, b, c are arbitrary constants and φ is any H^{s+1}, $s > 2$, function of one variable.

EXAMPLE 2. Non-compact boundaryless surfaces of revolution. Let us generate such surfaces by revolving, around the z-axis, a parametric curve $(a(t), 0, b(t))$, $0 \leq t < \tau$ (τ could $= +\infty$), given in the xz-plane. The parametrization is chosen such that the speed of the curve is always 1; that is, $(a')^2 + (b')^2 = 1$, where the prime denotes differentiation with respect to t. Hence the metric on the resulting surface has the form $g = a^2(t)d\theta \otimes d\theta + dt \otimes dt$, $0 \leq \theta \leq 2\pi$, $0 \leq t < \tau$, and the Gaussian curvature K is calculated to be $-a''/a$. We shall *assume* that the curve begins on the z-axis, namely

$$a(0) = 0. \tag{5.3}$$

For a technical reason that will become evident later, let us *suppose* that it heads off in the positive x direction and never turns back:

$$a'(t) > 0, \qquad 0 \leq t < \tau. \tag{5.4}$$

Note that (5.4) is automatic if $a'(0) > 0$ and $K \leq 0$.

For an axially symmetric function $f = f(t)$, the Monge-Ampère equation (5.2) becomes, after a straightforward calculation,

$$(f'aa')f'' = \left(\frac{-a''}{2a}\right)(a^2)(f')^2. \tag{5.5}$$

Note that it remains regular at $t = 0$, the only time when a is 0. Let us look for those solutions which lack critical points. Since f is axially-symmetric and at least C^2, and (5.5) is invariant under $f \to -f$, this is equivalent to the *assumption* that

$$f'(t) > 0, \qquad 0 \leq t < \tau. \tag{5.6}$$

Conditions (5.4) and (5.6) simplifies (5.5) to $f''/f' = \frac{-1}{2}(a''/a')$, which can then be integrated twice (the second integration needs (5.6) again) to give

$$f(t) = f(0) + C\int_0^t \frac{du}{\sqrt{a'(u)}}, \qquad C > 0. \tag{5.7}$$

Here, C and $f(0)$ are the two arbitrary constants of integration.

We would like to mention here that the technical conditions (5.3), (5.4), and (5.6) are highlighted not for fastidious reasons. As is demonstrated in **[BR]**, one can gather a good deal of information about the solution f by studying its interplay with these technical conditions.

EXAMPLE 3. The hyperbolic plane H^2. We can write the metric in normal coordinates as $g = (\sinh^2 t)d\theta \otimes d\theta + dt \otimes dt$ (see, for example, **[GHL]** p. 119). Then the computations in Example 2 are applicable, and we find that the axially symmetric the monotonically increasing solutions $f = f(t)$ are given by

$$f(t) = f(0) + C\int_0^t \frac{du}{\sqrt{\cosh(u)}}, \qquad C > 0. \tag{5.8}$$

From this, one sees that given any $x_0 \in H^2$, the function $h(x) := f(\mathrm{dist}(x_0, x))$ is a solution.

On the other hand, taking the Poincaré upper half-plane model, with $g = (1/y^2)(dx \otimes dx + dy \otimes dy)$, $y > 0$, one can check that (**[P]**) all solutions of the type $f = f(y)$ with $\frac{df}{dy}$ never 0 are of the form

$$f(y) = a\sqrt{y} + b, \qquad a > 0. \tag{5.9}$$

Geometrically, (5.9) can be viewed as $\frac{1}{\sqrt{h}}$, where h is a horofunction of H^2.

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