

VOLUMES OF TUBULAR NEIGHBOURHOODS OF REAL ALGEBRAIC VARIETIES

RICHARD WONGKEW

This paper concerns the following problem:

Let V be an algebraic variety in n -dimensional euclidean space. For each pair of positive numbers ρ and R find an upper bound on the volume of the set of points that are within a distance of ρ from V and within a distance R from a fixed point p_0 . Obtain this upper bound so that it is independent of the choice of p_0 . In particular, does there exist a universal n th-degree polynomial, say $P_n(\cdot, \cdot, \cdot)$ which automatically provides an upper bound upon entering ρ , R and the degree of V ?

Introduction. Among the people who have worked on this problem are Demmel, Renegar, Ocneanu and myself. The one common factor which led us to this problem was our familiarity with Smale's Bulletin article [S]. In it Smale obtained an upper bound for the case where V is a complex hypersurface. His method however lacks rigor and possesses a serious flaw. This flaw involves incorrectly applying Fubini's Theorem and ignoring the special points of V where the "relative curvature" is larger than $\Omega(\frac{1}{\rho})$. Since then Renegar who was a thesis student of Smale has obtained a correct solution to this special case where V is a complex algebraic hypersurface. Later on, Demmel using some elementary results from integral geometry extended Renegar's results to all complex algebraic varieties and all tubes of length less than 1. See Theorem 4.1 of Demmel [D]. Crucial to Demmel's result is an estimate of the r -dimensional measure of the portion of an r -dimensional algebraic variety contained in an n -ball S with radius R . More specifically Demmel used the following fact: There exists a constant c depending only on n , so that if $\text{Vol}_r(\cdot)$ is the r -dimensional measure of r -manifolds, then

$$\text{Vol}_r(V \cap S) \leq c \cdot \text{degree}(V) \cdot R^r.$$

For the methods used in proving this result one can see Wongkew ([W], pp. 9–10) or Demmel ([D], p.19). This fact generalizes Proposition 6.3 in Renegar [R].

Independently of Renegar, Adrian Ocneanu obtained the most general result of all. Unfortunately, Ocneanu did not publish his result and only a handwritten rough draft is in circulation. Ocneanu gave a talk on his result and the methods he used in achieving it. See [O]. In this talk he gave the following estimate:

$$\text{Vol}_n[T_\rho(V \cap S)] \leq \sum_{k=m}^n 2mO_n C_k^n (d\rho)^k R^{n-k}.$$

Here C_k^n is the number of ways to choose k things from n , d is the algebraic degree of V , m is the codimension of V and O_n is the volume of the unit n -ball. Crucial to Adrian's proof of this inequality was his ability to express the infinitesimal volume form of a manifold in terms of a sum of symmetric functions of its principal curvatures. Looking through the standard references on geometry and measure theory I have only been able to find such formulas for convex bodies in E^n .

1. Using purely topological and algebraic arguments I will demonstrate the existence of a solution to our problem. More precisely, I will prove the following.

MAIN THEOREM. *Let m be the codimension of a real algebraic variety V whose defining polynomials are all bounded in degree by d . And let S be an arbitrary n -ball in E^n with radius R . There exist constants $\{c_j\}$ which depend only on n , so that for all positive ρ the following is true:*

$$\text{Vol}_n[T_\rho(V \cap S)] \leq \sum_{j=m}^n c_j d^j \rho^j R^{n-j}.$$

A variation of this theorem can be derived from a result due to Weyl [We]. However, in this variation, the elements of $\{c_j\}$ depend on integral invariants of $V \cap S$ as well as the degree of V ; furthermore, the size of ρ is bounded.

In this paper we will use $|\cdot|$ to signify either the length (norm) of a vector (operator) or the number of elements in a set. The meaning will be clear from the context.

Proof of Main Theorem. From this point onwards the symbols ρ and R will represent non-negative real numbers and V will represent a real algebraic variety of degree d and codimension m in E^n . Let (ρ, R, V) be a randomly chosen triple. Assume that S is a randomly

chosen n -ball with radius R . We will apply induction on n . So assume $n = 1$. The Fundamental Theorem of Algebra tells us that V is a finite set with $|V| \leq d$. Thus for $n = 1$ we have

$$\text{Vol}_n[T_\rho(V \cap S)] \leq 2^n \cdot (d\rho)^n.$$

Thus our main theorem is obviously true for $n = 1$. So assume our theorem is true for algebraic varieties that are defined by $n - 1$ or less variables. To carry out our inductive argument, we will need some results from Differential Topology and Algebraic Geometry. More specifically we will need the following:

THEOREM (M). ¹ *Let V be a real algebraic variety in n -dimensional euclidean space. If each of the defining polynomial f for V has degree bounded above by d , then the sum of the Betti numbers of V is bounded above by the following n th degree polynomial in d :*

$$O(d) = d(2d - 1)^{n-1}.$$

THEOREM (S). ² *For any topological space X , the 0th Homology group for X is a free group. The rank of this free group equals the number of non-empty components of X .*

By a change of coordinates of E^n we may assume that S is centered at the origin. Thus S is the standard R -ball, $\{|x| \leq R\}$. And it is clear that for any pair of positive numbers ρ and R , we can cover the ball S with regular cubes ³ $\{V_k\}$ which satisfy the following five properties:

(1) There exists a family of hyperplanes $G_\rho^R = \{H_\alpha\}$ so that each V_k is in the closure of a component of $S \sim \bigcup H_\alpha$; and each hyperplane H_α is normal to e_j for some j .

(2) $V_k \cap V_j \neq \emptyset$ implies $k = j$ or $V_k \cap V_j$ is a regular $n - 1$ dimensional cube in R^n .

(3) The diameter of each V_k is less than $\rho\sqrt{n}/n$.

(4) If $H \in G_\rho^R$ then $V \cap H$ is a real algebraic variety which satisfies the following:

(a) dimension of $V \cap H = n - m - 1 = \dim(V) - 1$.

(b) the algebraic degree ⁴ of $V \cap H = d = \text{degree of } V$.

¹ See Theorem 2 of Milnor [M], p. 275.

² For Theorem [S] see Corollary 8 of Spanier [Sp], p. 175.

³ Here I am using the term regular cube to refer to an n -dimensional rectangle; i.e. all the edges are perpendicular.

⁴ By the algebraic degree of a variety we mean the degree of the defining polynomials for that variety.

(5) For each j , the number of hyperplanes in G_ρ^R with normal vector e_j is bounded above by $2nR/\rho$.

Assume $\{V_k\}$ and G_ρ^R are cubes and hyperplanes which satisfy (1) through (5) above; then the following inequalities are true:

$$(1) \quad \text{Vol}_n[T_\rho(V \cap S)] \leq \left\{ \sum_{H \in G_\rho^R} \text{Vol}_n[T_{\rho+\rho/n}(V \cap H \cap S)] \right\} + E(\rho, V, R).$$

Here $E(\rho, V, R)$ is equal to the volume of $T_\rho(S_\rho^R)$ and S_ρ^R is defined thus:

DEFINITION. For each $x \in V$ let $h(x)$ be the distance from x to $\bigcup[V \cap H: H \in G_\rho^R]$. Then S_ρ^R is the set of points $x \in V \cap S$ where $h(x) \geq \rho/\sqrt{n}$.

From our induction hypothesis, there exist $n-1$ constants $\{c_j\}$, so that for all hyperplanes $H \in G_\rho^R$ we have

$$(2) \quad \text{Vol}_{n-1}[T_\rho(V \cap H \cap S)] \leq \sum_m^{n-1} c_j d^j \rho^j R^{n-1-j}.$$

This implies that

$$(3) \quad \text{Vol}_n[T_\rho(V \cap H \cap S)] \leq \sum_m^{n-1} c_j d^j \rho^{j+1} R^{n-(1+j)}.$$

Now for each i , $1 \leq i \leq n$, let G_i be the set of all hyperplanes in G_ρ^R with normal vector e_i . From properties (5) and (1) we have for each i ,

$$|G_i| \leq \frac{2nR}{\rho} \quad \text{and} \quad G_\rho^R = \bigcup(G_i: 1 \leq i \leq n).$$

This plus the inequalities above imply that for each i ,

$$(4) \quad \sum_{H \in G_i} \text{Vol}_n[T_\rho(V \cap H \cap S)] \leq \sum_{j=m}^{n-1} \frac{2nR}{\rho} c_j d^j \rho^{j+1} R^{n-(j+1)} \\ = \sum_{j=m}^{n-1} 2nc_j d^j \rho^j R^{n-j}.$$

Since i only takes values between 1 and n , we know that for some i , the following is true:

$$(5) \quad \sum_{H \in G_\rho^R} \text{Vol}_n[T_\rho(V \cap H \cap S)] \leq n \sum_{H \in G_i} \text{Vol}_n[T_\rho(V \cap H \cap S)].$$

Combining this with (4) we get

$$(6) \quad \sum_{H \in G_\rho^R} \text{Vol}_n[T_\rho(V \cap H \cap S)] \leq \sum_{j=m}^{n-1} 2n^2 c_j d^j \rho^j R^{n-j}.$$

Thus upon replacing ρ with $\rho + \rho/n$ in (6) and plugging the result into (1) we get

$$(7) \quad \text{Vol}_n[T_\rho(V \cap S)] \leq \sum_{j=m}^{n-1} 2n^2 c_j \left(1 + \frac{1}{n}\right)^j d^j \rho^j R^{n-j} + \text{Vol}_n[T_\rho(V \cap S_\rho^R)].$$

Now for each $1 \leq j \leq n - 1$, let us redefine c_j to be equal to $2n^2 c_j (\frac{1+n}{n})^j$. Then we can rewrite (7) as

$$(8) \quad \text{Vol}_n[T_\rho(V \cap S)] \leq \sum_{j=m}^{n-1} c_j d^j \rho^j R^{n-j} + \text{Vol}_n[T_\rho(V \cap S_\rho^R)].$$

Clearly, we are finished as soon as we demonstrate the existence of a constant c_n independent of ρ , R , and V so that

$$(9) \quad \text{Vol}_n[T_\rho(V \cap S_\rho^R)] \leq c_n (d\rho)^n.$$

Our proof of (9) follows from the following:

Claim. If O_n is the volume of the unit ball in E^n and we set $c_n = 3^n O_n$ then (9) is true for all R , ρ , V and d .

To prove our claim we will need the following lemma:

LEMMA (C). *Let d be the degree of the defining polynomials for an algebraic variety V in n -dimensional euclidean space. Then the number of components of V is bounded above by*

$$d(2d - 1)^{n-1}.$$

Proof. By definition the 0th Betti number of V is the rank of the 0th Homology group of V . And from Theorem (S) we know that the number of connected components of V is equal to the 0th Betti number of V . Thus in a rather crude fashion, it follows that the sum of the Betti numbers V is an upper bound for the number of bounded components of V . From the works of Milnor [M], Thom [T] and Oleinik [OI] we know this sum of Betti numbers is bounded

above by an n th degree polynomial in d . From Theorem (M) we may assume this polynomial is

$$d(2d - 1)^{n-1}.$$

Thus we have completed the proof of our lemma.

Proof of Claim. Let $\{U_k\}$ be the set of components of $E^n \sim \bigcup\{H \in G_\rho^R\}$ which contains S_ρ^R . Now $\{U_k\}$ is defined as a cover of S_ρ^R ; and the elements of S_ρ^R are bounded components of V . From Lemma (C), the number of components of V is bounded above by

$$O(d) = d(2d - 1)^{n-1}.$$

Thus the following inequality is true:

$$(10) \quad |\{U_k\}| = \text{cardinality of } \{U_k\} \leq d(2d - 1)^{n-1}.$$

But because $\{U_k\}$ covers S_ρ^R we get

$$T_\rho(S_\rho^R) \subset T_\rho \left[\bigcup U_k \right].$$

Thus

$$(11) \quad \text{Vol}_n T_\rho(S_\rho^R) \leq \text{Vol}_n T_\rho \left[\bigcup U_k \right].$$

From the sub-additivity of Lebesgue measure we know that

$$(12) \quad \text{Vol}_n T_\rho \left[\bigcup U_k \right] \leq \sum_k \text{Vol}_n T_\rho[U_k].$$

Now each U_k is the result of a translation of the n -cube, $[0, \frac{\rho}{n}]^n$. Thus upon combining this fact with (10) and (12) we get

$$(13) \quad \text{Vol}_n T_\rho \left[\bigcup U_k \right] \leq d(2d - 1)^{n-1} \text{Vol}_n \left[T_\rho \left(\left[0, \frac{\rho}{n} \right]^n \right) \right].$$

Now the center of the n -cube, $[0, \frac{\rho}{n}]^n$ is the point $p_0 = \frac{\rho}{2n} \sum_{k=1}^n e_k$. And every point of the n -cube, $[0, \frac{\rho}{n}]^n$ is within a distance of $\frac{\rho}{2}$ from p_0 . In other words, the following relation holds:

$$\left[T_\rho \left(\left[0, \frac{\rho}{n} \right]^n \right) \right] \subset \left(x : |x - p_0| \leq \frac{3\rho}{2} \right).$$

This implies that for each U_k ,

$$(14) \quad \text{Vol}_n [T_\rho(U_k)] < \text{Vol}_n \left[|p - p_0| \leq \frac{3\rho}{2} \right] = O_n \cdot \left(\frac{3\rho}{2} \right)^n.$$

Upon combining (11) and (13) with our previous inequality we obtain that

$$(15) \quad \text{Vol}_n T_\rho(S_\rho^R) < (d)(2d-1)^{n-1} \cdot O_n \left(\frac{3}{2}\rho \right)^n.$$

From this inequality and the fact that

$$(d)(2d-1)^{n-1} \cdot O_n \left(\frac{3}{2}\rho \right)^n < 3^n O_n(d\rho)^n$$

we see that we have proven our claim; and thus the proof of our Main Theorem is complete.

Postscript. It is a little unsettling to use the sum of the Betti numbers as the upper bound on the number of components. It is only the 0th Betti number of V that one needs for the proof. One would hope that we could get a much sharper bound for this. Thus I shall end with some of the questions which bothered me throughout the paper:

1. Are there sharp estimates for the 0th Betti number of V in terms of d and n ?
2. What are the best constants for the formula in our main theorem?
3. Is there a formula for expressing the infinitesimal volume form of an arbitrary singular (non-singular) submanifold in terms of a sum of symmetric functions of its principal curvatures?

REFERENCES

- [A] L. Auslander and R. E. Mackenzie, *Introduction to Differentiable Manifolds*, McGraw-Hill Series in Higher Math., 1963.
- [BT] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, New York, 1982.
- [D] J. Demmel, *The probability that a numerical analysis problem is difficult*, Courant Institute, New York University, preprint 1987.
- [DFN] B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, *Modern Geometry—Methods and Applications. Part II. The Geometry and Topology of Manifolds*, Springer-Verlag, New York, 1985.
- [H] H. Hotelling, *Tubes and spheres in n -spaces, and a class of statistical problems*, Amer. J. Math., **61** (1939), 440–446.
- [HY] J. Hocking and G. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961.
- [K] E. Kostlan, *Complexity Theory of Numerical Linear Algebra*, preprint M.S.R.I. 1985.
- [M] J. Milnor *On the Betti numbers of real varieties*, Proc. Amer. Math. Soc., **15** (1964), 275–280.
- [O] A. Ocneanu, *On the Volumes of Tubular Neighbourhoods of Algebraic Varieties*, presentation 1984, MSRI, Berkeley.

- [O1] O. A. Oleĭnik, *Estimates of the Betti numbers of real algebraic hypersurfaces*, Mat. Sb. (N.S.), **28** (70) (1951), 635–640 (Russian).
- [R] J. Renegar, *On the Efficiency of Newton's Method in Approximating all Zeros of a System of Complex Polynomials*, preprint M.S.R.I. 1985.
- [S] S. Smale, *The fundamental theorem of algebra and complexity theory*, Bull. Amer. Math. Soc. (N.S.), **4** (1981), 1–36.
- [Sp] E. H. Spanier, *Algebraic Topology*, McGraw-Hill Series in Higher Mathematics, 1966.
- [T] R. Thom, *Sur l'Homologie des Variétés Algébriques Reelles in A Symposium in Honor of Marston Morse*, Princeton University Press, Stuart S. Cairnes (editor) 1965.
- [Wa] B. L. Van Der Waerden, *Modern Algebra*, vol. 2, Ungar, New York 1950.
- [We] H. Weyl, *On the volume of tubes*, Amer. J. Math., **61** (1939), 461–472.
- [Wo] R. Wongkew, *Partitioning for Strong Convergence of Newton's Method*, Mathematics Dept. UC Berkeley, preprint 1985.

Received December 9, 1989 and in revised form November 25, 1991.

UNIVERSITY OF THE VIRGIN ISLANDS
ST. THOMAS, VI 00802

AND

WORLD INSTITUTE FOR SCIENCE
KINGSTON
JAMAICA, WEST INDIES