PACIFIC JOURNAL OF MATHEMATICS
Vol. 159, No. 1, 1993

PSEUDO REGULAR ELEMENTS AND THE AUXILIARY MULTIPLICATION THEY INDUCE

B. E. JOHNSON

An element f of a commutative Banach algebra is *pseudo regular* if there is a constant M with $\|abf\| \le M\|af\|\,\|bf\|$ $(a, b \in \mathfrak{A})$. In many cases pseudo regularity implies formally stronger conditions such as relative invertibility; that is, f is invertible in some subalgebra of $\mathfrak{A}$. In this paper we describe some algebraic methods which can be used to establish results of this kind. Given a pseudo regular element f of $\mathfrak{A}$, $af \circ bf = abf$ extends by continuity to a multiplication $\circ$, called the *auxiliary multiplication*, on J, the closed ideal generated by f. This leads to the fundamental inequality $\|\phi\|_{J^*} \le M|\phi(f)|$ where ϕ is a multiplicative linear functional on $\mathfrak{A}$. As applications of these ideas we identify the pseudo regular elements of the algebra $C^{(n)}[0, 1]$ as being the elements such that $f, f', \ldots, f^{(n)}$ have no common zeros and the pseudo regular elements of the group algebra of a locally compact abelian group as being the relatively invertible elements. Similar constructions can be made when f is an element of an $\mathfrak{A}$ module $\mathfrak{X}$ though the structure is less rich in this case.

1. Introduction. The idea of pseudo regularity was introduced by Arens in [**1**] where he first proved results of the kind in this paper. The definition of auxiliary multiplication is given in §2. The product $j \circ k$ is, heuristically, jk/f and this is made precise in Proposition 2.4 and Corollary 2.5. The fundamental inequality is proved in §3 where the equivalence of pseudo regularity and relative invertibility also appears (Corollary 3.6). The two remaining sections deal with pseudo regularity of an element f of an $\mathfrak{A}$ module $\mathfrak{X}$ (§4) and of a system (§5). In §5 we show that a system $F = (f_1, \ldots, f_n) \in \mathfrak{A}^n$ where $\mathfrak{A}$ is a uniform algebra is pseudo regular if and only if $(0, \ldots, 0)$ is an isolated point of $\{(\phi(f_1), \ldots, \phi(f_n)),\ \phi \in \partial\mathfrak{A}\} \cup \{(0, \ldots, 0)\}$ where $\partial\mathfrak{A}$ is the Šilov boundary of $\mathfrak{A}$. This extends a result in [**1**] for the case $n = 1$.

The author is indebted to Professor Arens for letting him have a copy of an early version of [**1**].

2. The auxiliary multiplication generated by a pseudo regular element. Let $\mathfrak{A}$ be a commutative Banach algebra and let $f \in \mathfrak{A}$. We define a product $\circ$ on $\mathfrak{A}f$ by $af \circ bf = abf$ $(a, b \in \mathfrak{A})$. This is well defined because if $af = 0$ then $abf = 0$. Then $\circ$ is jointly continuous if and only if f is pseudo regular. Also it is jointly continuous if and only if it has a continuous extension, which we also denote by $\circ$, to the closed ideal $(\mathfrak{A}f)^-$ generated by f. In fact if $f \notin (\mathfrak{A}f)^-$ it is convenient to replace $(\mathfrak{A}f)^-$ by $J = (\mathfrak{A}^1 f)^-$, where $\mathfrak{A}^1$ is the algebra obtained by adjoining an identity to $\mathfrak{A}$ so that J is the smallest closed ideal in $\mathfrak{A}$ containing f. This replacement is possible because we have Proposition 2.1 below. Of course if $\mathfrak{A}$ has an identity or even an approximate identity, not necessarily bounded, then $J = (\mathfrak{A}f)^-$ and this maneuver is unnecessary.

PROPOSITION 2.1. *Let $f \in \mathfrak{A}$. Then f is pseudo regular in $\mathfrak{A}$ if and only if it is pseudo regular in $\mathfrak{A}^1$.*

Proof. If $f \in \mathfrak{A}$ is pseudo regular in $\mathfrak{A}^1$ then it is obviously pseudo regular in $\mathfrak{A}$. If f is pseudo regular in $\mathfrak{A}$ then $\circ$ is a continuous bilinear map $(\mathfrak{A}f)^- \times (\mathfrak{A}f)^- \to \mathfrak{A}$. If $f \in (\mathfrak{A}f)^-$ then $\circ$ is a continuous bilinear map $(\mathfrak{A}^1 f)^- \times (\mathfrak{A}^1 f)^- \to \mathfrak{A}$ so f is pseudo regular in $\mathfrak{A}^1$ provided the two definitions of $\circ$ on $\mathfrak{A}^1 f$ agree. Suppose $\{a_n\}$ is a sequence in $\mathfrak{A}$ with $a_n f \to f$. Then for $a, b \in \mathfrak{A}$ and $\lambda, \mu \in \mathbb{C}$

$$[(a + \lambda 1)a_m f] \circ [(b + \mu 1)a_n f] = (a + \lambda 1)(b + \mu 1)a_m a_n f.$$

Taking the limit first on n then on m gives the required consistency condition. If $f \notin (\mathfrak{A}f)^-$ then $(\mathfrak{A}^1 f)^- = (\mathfrak{A}f)^- + \mathbb{C}f$ and $(j + \lambda f) \circ (k + \mu f) = j \circ k + \lambda k + \mu j + \lambda\mu f$ is a continuous bilinear map $\mathfrak{A}^1(f)^- \times (\mathfrak{A}^1 f)^- \to \mathfrak{A}$ which extends

$$(a + \lambda 1)f \circ (b + \mu 1)f \to (a + \lambda 1)(b + \mu 1)f \qquad (a, b \in \mathfrak{A},\ j, k \in (\mathfrak{A}f)^-,\ \lambda, \mu \in \mathbb{C})$$

so again f is pseudo regular.

It is easy to see that $\circ$ is commutative and associative on $\mathfrak{A}^1 f$ and hence on J if it extends. Thus we have

PROPOSITION 2.2. *The element f of the commutative Banach algebra $\mathfrak{A}$ is pseudo regular if and only if the auxiliary multiplication*

$$af \circ bf = abf$$

on $\mathfrak{A}^1 f$ is continuous and so has a continuous extension to $J = (\mathfrak{A}^1 f)^-$. If f is pseudo regular then J° is a Banach equivalent algebra and f is the identity element of J°.

Here J° denotes the Banach space J with multiplication $\circ$. Saying it is a Banach equivalent algebra means that it is a Banach algebra under some equivalent norm—that is, that multiplication is continuous so that $\|j \circ k\| \le M\|j\|\,\|k\|$ $(j, k \in J)$ for some constant M but M is not necessarily 1. The only part of the proposition we have to prove is that f is the identity and this follows from $af \circ f = af \circ 1f = af$ $(a \in \mathfrak{A}^1)$ so $j \circ f = j$ $(j \in J = (\mathfrak{A}^1 f)^-)$.

PROPOSITION 2.3. *If f is pseudo regular then for all j, k and l in J,*

(i) $(j \circ k)f = jk$,
(ii) $(jk) \circ l = (j \circ k)l = j \circ (kl) = j(k \circ l)$.

Proof. This is obvious for $j, k, l \in \mathfrak{A}^1 f$ and follows for more general values by continuity.

The auxiliary product $j \circ k$ behaves formally like jk/f. For semisimple algebras we get a more precise statement.

PROPOSITION 2.4. *Let $\mathfrak{A}$ be a commutative semisimple Banach algebra and let $f \in \mathfrak{A}$. Then f is pseudo regular if and only if for all j, k in $J_0 = (\mathfrak{A}f)^-$ the function g on $\widehat{\mathfrak{A}}$ defined for $\phi \in \widehat{\mathfrak{A}}$ by*

$$\begin{aligned} g(\phi) &= \phi(j)\phi(k)\phi(f)^{-1} \quad \textit{if } \phi(f) \neq 0, \\ &= 0 \quad \textit{if } \phi(f) = 0, \end{aligned}$$

is the Gelfand transform $\hat{a}$ of some element a of $\mathfrak{A}$.

Proof. If f is pseudo regular and $\phi(f) \neq 0$ then by Proposition 2.3 (i) if $j, k \in J_0$, then $\phi(j \circ k)\phi(f) = \phi(j)\phi(k)$ so $g(\phi) = \phi(j \circ k)$. If $\phi(f) = 0$ then $\phi(j) = 0$ for all j in J_0 and so $g(\phi) = \phi(j \circ k)$ because both sides of the equation are zero.

For the converse, fix k and consider the map T of J_0 into $\mathfrak{A}$ defined by $(Tj)^\wedge = g$ where g is as above. This is well defined because $\mathfrak{A}$ is semisimple. If $j_n \to 0$ in J_0 and $(Tj_n) \to b$ in $\mathfrak{A}$ then, for $\phi \in \widehat{\mathfrak{A}}$ with $\phi(f) \neq 0$, $\phi(Tj_n) = \phi(j_n)\phi(k)/\phi(f) \to 0$ so $\phi(b) = 0$. If $\phi(f) = 0$ then $\phi(Tj_n) = 0$ so $\phi(b) = 0$. Thus $b = 0$ and T is continuous by the closed graph theorem. Now consider the bilinear map $B\colon J_0 \times J_0 \to \mathfrak{A}$ defined by $B(j, k)^\wedge = g$ as above. We have shown that it is continuous in the first variable and by symmetry it is continuous in the second variable. Thus it is separately and hence jointly continuous. However if $j = af$, $k = bf$ then $B(j, k)$ is just

abf so the auxiliary multiplication is continuous and f is pseudo regular.

COROLLARY 2.5. *If $f \in \mathfrak{A}$ and $\hat{f}^{-1}(0)$ is nowhere dense in $\widehat{\mathfrak{A}}$ then f is pseudo regular if and only if for all j, k in $J_0 = (\mathfrak{A}f)^-$ the function $g(\phi) = \phi(j)\phi(k)/\phi(f)$ on $\widehat{\mathfrak{A}}\backslash\hat{f}^{-1}(0)$ is the restriction of the Gelfand transform of an element of $\mathfrak{A}$.*

Proof. The only change necessary in the proof of Proposition 2.4 is that $b = 0$ because $\phi(b) = 0$ for all ϕ in $\widehat{\mathfrak{A}}$ with $\phi(f) \neq 0$, that is, on a dense subset of $\widehat{\mathfrak{A}}$, and hence everywhere.

As an application of Corollary 2.5 we are able to characterise the pseudo regular elements of $C^n[0, 1]$.

PROPOSITION 2.6. *Let $\mathfrak{A} = C^n[0, 1]$ and $f \in \mathfrak{A}\backslash\{0\}$. Then f is pseudo regular if and only if the functions $f, f', \dots, f^{(n)}$ have no common zero.*

Proof. Suppose $f, f', \dots, f^{(n)}$ have no common zero. Put $Z = f^{-1}(0)$. Let $c \in Z$ and let l be the least value of m with $f^{(m)}(c) \neq 0$. Then $0 < l \leq n$. By considering $f(t) - f^{(l)}(c)(t-c)^l/l! = o(t-c)^l$ as $t \to c$ we see that c is an isolated point of Z and so Z is finite and in particular, nowhere dense.

By Corollary 2.5 we need to prove that for each j and k in J, the function jk/f, which is clearly a C^n function on $[0, 1]\backslash Z$, has a C^n extension to $[0, 1]$. We do this by showing that for $0 \leq m \leq n$, $(jk/f)^{(m)}$ has a continuous extension to $[0, 1]$ and using

LEMMA 2.7. *Let F be a C^n function on $[0, 1]\backslash Z$ such that, for $0 \leq m \leq n$, $F^{(m)}$ has a continuous extension to $[0, 1]$. Then F has a C^n extension to $[0, 1]$.*

Proof. We prove the result by induction on n. It is trivial for $n = 0$. If it is true for $n = k$ and F satisfies the hypothesis with $n = k+1$ then F' has a C^k extension G to $[0, 1]$. Let $c \in [0, 1]\backslash Z$ and put $H(t) = F(c) + \int_c^t G(s)\,ds$. Then $H \in C^{k+1}[0, 1]$ and $F - H$ has a zero derivative on $[0, 1]\backslash Z$. Thus $F - H$ is constant on the components of $[0, 1]\backslash Z$. As however it has a continuous extension to $[0, 1]$ and is zero at c we see $F = H$ on $[0, 1]\backslash Z$ so H is a C^{k+1} extension of F.

Returning to the proof of the proposition, let $c \in Z$. If $j \in J$ then $j(c) = \cdots = j^{(l-1)}(c) = 0$ because this is true for $j \in \mathfrak{A}f$ and $j \mapsto j^{(m)}(c)$ is a continuous functional on $C^n[0, 1]$ for $0 \leq m \leq n$. By elementary calculus $f^{m+1}(jk/f)^{(m)}$ is a linear combination of terms of the form $j^{(p)}k^{(q)}f^{(r_1)}\cdots f^{(r_m)}$ where $p, q, r_1, \ldots, r_m \geq 0$ and $p + q + r_1 + \cdots + r_m = m$. We write the Taylor series for these functions as

$$j^{(p)} = \sum_{i=s}^{m} \alpha_i i(i-1)\cdots(i-p+1)(x-c)^{i-p} + A_p(x-c)^{m-p},$$

$$k^{(p)} = \sum_{i=s}^{m} \beta_i i(i-1)\cdots(i-p+1)(x-c)^{i-p} + B_p(x-c)^{m-p},$$

$$f^{(p)} = \sum_{i=s}^{m} \gamma_i i(i-1)\cdots(i-p+1)(x-c)^{i-p} + C_p(x-c)^{m-p},$$

where $s = \max\{l, p\}$. We refer to the A_p, B_p and C_p as *remainders.* If we multiply $m+2$ functions of the same form as j, k and f and treat the remainders as independent unknowns then the coefficient of $(x-c)^k$ in the product for $k < (m+1)l + m = lm + l + m$ does not involve the remainders and for $k = lm + l + m$ it is a first degree polynomial in the remainders. This applies to

$$\begin{aligned}&[(x-c)^p j^{(p)}][(x-c)^q k^{(q)}][(x-c)^{r_1} f^{(r_1)}]\cdots[(x-c)^{r_m} f^{(r_m)}]\\ &\quad = (x-c)^m j^{(p)} k^{(q)} f^{(r_1)} \cdots f^{(r_m)}\end{aligned}$$

and shows that the coefficient of $(x-c)^k$ for $k < lm + l$ for each term in $f^{m+1}(jk/f)^{(m)}$, and hence in $f^{m+1}(jk/f)^{(m)}$ itself, does not contain the remainders and hence can be calculated by examining the case in which the remainders are all zero. Accordingly we put

$$\tilde{j} = \sum_{i=l}^{m} \alpha_i (x-c)^i, \quad \tilde{k} = \sum_{i=l}^{m} \beta_i (x-c)^i \quad \text{and}$$

$$\tilde{f} = \sum_{i=l}^{m} \gamma_i (x-c)^i \quad \text{where } \gamma_l \neq 0.$$

Then $\tilde{j}\tilde{k}/\tilde{f}$ is a rational function with a zero of order at least l at c so that $\tilde{f}^{m+1}[\tilde{j}\tilde{k}/\tilde{f}]^{(m)}$ is a rational function with a zero of order at least $l(m+1) = lm + l$ at c. This shows that for $k < lm + l$, the coefficient of $(x-c)^k$ in $f^{m+1}(jk/f)^{(m)}$ is zero so that

$$(jk/f)^{(m)} = (x-c)^{lm+m} Q/f^{m+1}$$

where Q is a polynomial in $(x-c)$ and the remainders. As

$$(x-c)^{lm+m}/f^{m+1} = [(x-c)^l/f]^{m+1} \to \gamma_l^{-m-1} \quad \text{as } x \to c$$

and the remainders tend to zero we see that $\lim_{x\to c}[jk/f]^{(m)}$ exists.

For the converse suppose that Z_n, the set of common zeros of $f, f', \ldots, f^{(n)}$ is not empty. As f is not zero, $Z_n \neq [0, 1]$ and hence ∂Z_n is not empty. Let $c \in \partial Z_n$. Then c is not an interior point of Z so either there is $d \in (c, 1]$ with $Z \cap (c, d] = \varnothing$ or $d \in [0, c)$ with $Z \cap [d, c) = \varnothing$. We consider the first case; the second is similar. We shall find a function j in J such that if $g = j^2/f$ then c is a point of accumulation of the zeros of $g^{(k)}$ $(k = 0, 1, \ldots, n)$ in $(c, d]$ and yet $g(t) \neq o(t-c)^n$ as $t \to c$. If g had a C^n extension G to $[0, 1]$ then the first property of g in the last sequence implies that $G(c) = G'(c) = \cdots = G^{(n)}(c)$ from which it follows that $g(t) = o(t-c)^n$ as $t \to c$.

As $c \in Z_n$ and $f \in C^n$ we have $f(t) = o(t-c)^n$ as $t \to c$. Put $t_m = c + 2^{-m}(d-c)$ and $\sigma_m = [f(t_m)(t_m - c)^n]^{1/2}$. Then $\sigma_m = o(2^{-mn})$ as $m \to \infty$ so there is a positive sequence $\{\tau_m\}$ with $\tau_m \to \infty$ and $\sigma_m\tau_m = o(2^{-mn})$ as $m \to \infty$. There is a C^n function ρ on $\mathbb{R}$ with support in $[-1, +1]$ and $\rho(0) = 1$. Put $K = \sup_t |\rho^{(n)}(t)|$ and $\rho_m(t) = \sigma_m\tau_m\rho(2^{-m-2}(d-c)(t-t_m))$. Then $\rho_m \in C^n[0, 1]$, $\rho_m(t_m) = \sigma_m\tau_m$, the support of ρ_m lies in $[c + \frac{3}{4}2^{-m}(d-c),\ c + \frac{5}{4}2^{-m}(d-c)]$ and, for all t, $|\rho_m^{(n)}(t)| \leq K\sigma_m\tau_m 2^{nm}4^n/(d-c)^n \to 0$ as $m \to \infty$. Thus the ρ_m have disjoint support and $j = \sum_m \rho_m$ converges in $C^m[0, 1]$. We have $\rho_m \in C^n[0, 1]f$ because $\rho_m = (\rho_m/f)f$ so $j \in J$. Also $j(t_m)^2/f(t_m) = \tau_m^2(t_m - c)^n$ with $\tau_m \to \infty$ as $m \to \infty$ so $g = j^2/f$ is not $o(t-c)^n$ as $t \to c$. However g is identically zero on the intervals between the supports of the ρ_m and so we see that g does not have a C^n extension to $[0, 1]$.

3. Pseudo regularity and relative invertibility. Recall that $\mathfrak{A}$ is a commutative Banach algebra.

DEFINITION 3.1. We say that an element f of $\mathfrak{A}$ is *relatively invertible* if there is an element g of $\mathfrak{A}$ with $gf^2 = f$.

If f is relatively invertible then $gf \in J_0 = (\mathfrak{A}f)^-$, $f \in (\mathfrak{A}f)^-$ and $(gf)j = j$ for all $j \in \mathfrak{A}f$ and hence for all j in J_0. Thus gf is the identity of J_0 and g^2f is the inverse of f in J_0. If $f \in \mathfrak{A}$ is relatively invertible then it is relatively invertible in $\mathfrak{A}^1$. If it is relatively invertible in $\mathfrak{A}^1$ so that $gf^2 = f$ with $g \in \mathfrak{A}^1$ then $gf \in \mathfrak{A}$ is the identity in $J = (\mathfrak{A}^1 f)^-$ so $(gfg)f^2 = f$ with $gfg \in \mathfrak{A}$ and

f is relatively invertible in $\mathfrak{A}$. We have seen that if f is relatively invertible then it is invertible in some subalgebra of $\mathfrak{A}$. The converse is trivial. It is easy to see that a relatively invertible element of $\mathfrak{A}$ is pseudo regular and, using Šilov's idempotent theorem, that if $\mathfrak{A}$ is semisimple then an element f of $\mathfrak{A}$ is relatively invertible if and only if $Z = \hat{f}^{-1}(0)$ is open and closed in $\widehat{\mathfrak{A}}$ (Arens [**1**; Theorem 3.2]). We are interested in this section in situations in which all pseudo regular elements are relatively invertible.

PROPOSITION 3.2. *If f is a pseudo regular element of a Banach algebra $\mathfrak{A}$ and $f \in [(\mathfrak{A}^1 f)^2]^-$ then f is relatively invertible.*

Note that $(\mathfrak{A}^1 f)^2$ is the linear span of products of elements from $\mathfrak{A}^1 f$. The condition $f \in [(\mathfrak{A}^1 f)^2]^-$ is equivalent to J^2 being dense in J.

Proof. Let M be the constant with $\|abf\| \leq M\|af\|\,\|bf\|$ $(a, b \in \mathfrak{A})$. Let $j_1, \dots, j_n, k_1, \dots, k_n \in J$ with $\|f - \sum j_i k_i\| < (2M)^{-1}$. Put $g = \sum j_i \circ k_i$. Then for all l in J, $l \circ (f - \sum j_i k_i) = l - l\sum j_i \circ k_i = l - lg$ so $\|(1-g)l\| \leq M\|l\|/2M = \|l\|/2$. Thus $\|(1-g)^n l\| \leq (1/2)^n\|l\|$ and hence $e = g + (1-g)g + (1-g)^2 g + \cdots$ converges in J. For $l \in J$, $(1-g)^n gl = (1-g)^n l - (1-g)^{n+1} l$ so $el = l$ and e is the identity for J. Then $e \circ e$ is the inverse for f in J because $f(e \circ e) = f \circ (ee) = f \circ e = e$. Thus f is relatively invertible.

COROLLARY 3.2. *If every closed ideal in $\mathfrak{A}$ is the kernel of its hull then every pseudo regular element of $\mathfrak{A}$ is relatively invertible.*

The hull of an ideal is the set of maximal modular ideals which contain it. The kernel of a set of ideals is their intersection (see Rickart [**3**, p. 78]).

Proof. A multiplicative linear functional which is zero on $(\mathfrak{A}^1 f)^2$ is zero on $\mathfrak{A}^1 f$ because $\phi(af)^2 = \phi((af)^2)$. Thus $[(\mathfrak{A}^1 f)^2]^-$ and $(\mathfrak{A}^1 f)^-$ have the same hull and, given the hypothesis, are the same.

Corollary 3.2 applies to the algebra $C_0(X)$ where X is a locally compact topological space [**1**, Theorem 2.3] and to the algebra $L^1(G)$ of a compact abelian topological group [**1**, Theorem 3.3]. More generally

COROLLARY 3.3. *Let $\mathfrak{A}$ be a commutative Banach algebra with an approximate identity (not necessarily bounded) and which is generated* *by its minimal idempotents. Then an element f of $\mathfrak{A}$ is pseudo regular if and only if it is relatively invertible and hence a finite linear combination of minimal idempotents.*

Proof. Let S be the set of finite linear combinations of minimal idempotents in $\mathfrak{A}$. Then $\mathfrak{A}$ has an approximate identity $\{e_\alpha\}$ of elements of S. If J is a closed ideal in $\mathfrak{A}$ let E be the set of minimal idempotents in J and let E' be the others. Then $\widetilde{J}$, the kernel of the hull of J, consists of those elements a of $\mathfrak{A}$ with $ae = 0$ for all e in E'. Let $a \in \widetilde{J}$. Then $e_\alpha a \in S$ and $e_\alpha a e = 0$ for all $e \in E'$ so $e_\alpha a$ is in the linear span of E. Thus $e_\alpha a \in J$ and hence $a \in J$, showing that $J = \widetilde{J}$.

THEOREM 3.4. *Let f be a pseudo regular element of $\mathfrak{A}$.*

(a) *If ϕ is a non zero multiplicative linear functional on J then $\phi(f) \neq 0$ and $\phi(f)^{-1}\phi$ is a multiplicative linear functional on J°.*

(b) *If ϕ is a multiplicative linear functional on J° then $\phi(f^2)\phi$ is a multiplicative linear functional on J.*

Proof. If $\phi(f) = 0$ then $\phi(af)^2 = \phi(a^2f^2) = \phi(a^2f)\phi(f) = 0$ for all a in $\mathfrak{A}^1$ so $\phi = 0$. Thus if $\phi \neq 0$ then $\phi(f) \neq 0$. By Proposition 2.3 we have $\phi(j \circ k)\phi(f) = \phi(jk) = \phi(j)\phi(k)$ $(j, k \in J)$ so $\phi(f)^{-1}\phi$ is a multiplicative linear functional on J°.

For (b) we have, because f is the identity in J°,

$$\phi(jk) = \phi(jk \circ f) = \phi(j \circ kf) = \phi(j)\phi(kf).$$

This gives $\phi(kf) = \phi(k)\phi(f^2)$ and so $\phi(jk) = \phi(j)\phi(k)\phi(f^2)$ $(j, k \in J)$ and shows $\phi(f^2)\phi$ is a multiplicative linear functional on J.

Note that in (b) if $\phi(f^2) = 0$ then of course $\phi(f^2)\phi$ is zero even though ϕ is not. Because $\phi(jk) = \phi(j)\phi(k)\phi(f^2)$ this happens if and only if ϕ is zero on J^2. Thus if J^2 is dense in J we have in (b) that if ϕ is non zero then so is $\phi(f^2)\phi$. Without some such additional hypothesis this is not true. For example if $\mathfrak{A} = C^1[0, 1]$ and $f(t) = t$ $(0 \leq t \leq 1)$ then $J = \{a : a \in \mathfrak{A},\ a(0) = 0\}$, $J^2 \subseteq \{a : a \in \mathfrak{A},\ a(0) = a'(0) = 0\}$ and f is pseudo regular (Proposition 2.5). We have $(af \circ bf)'(0) = (abf)'(0) = a(0)b(0) = (af)'(0)(bf)'(0)$ for all a and b in $\mathfrak{A}$ so $(j \circ k)'(0) = j'(0)k'(0)$ $(j, k \in J)$ and $\phi(j) = j'(0)$ is a non zero multiplicative linear functional on J with $\phi(f^2) = 0$.

COROLLARY 3.5. *Let f be a pseudo regular element of $\mathfrak{A}$.*

(a) *If ϕ is a multiplicative linear functional on J then*

$$\|\phi\| \leq M|\phi(f)|.$$

(b) *If there is* $K > 0$ *such that* $\{\phi\colon \phi \in \widehat{J},\ \|\phi\| \geq K\}$ *is dense in* $\widehat{J}$ *then* f *is invertible in* J.

Proof. For a Banach equivalent algebra the standard proof that in a Banach algebra, if ϕ is a multiplicative linear functional, then $|\phi(a)| \leq \|a\|$, gives $|\phi(a)| \leq M\|a\|$. Thus if ϕ is a non zero multiplicative linear functional on J then, by 3.4, $\phi(f)^{-1}\phi$ is a multiplicative linear functional on J° and so $\|\phi(f)^{-1}\phi\| \leq M$ and (a) follows. For (b), (a) implies that $|\hat{f}(\phi)| \geq KM^{-1}$ on a dense subset of $\widehat{J}$ and hence everywhere. Thus $\widehat{J}$ is compact, J has an identity and f is invertible.

For the next result note that if f is any element of $\mathfrak{A}$ and $\phi \in \widehat{\mathfrak{A}}$ with $\phi(f) \neq 0$ then the restriction of ϕ to J is an element of $\widehat{J}$ and this gives a homeomorphism of $\{\phi\colon \phi \in \widehat{\mathfrak{A}},\ \phi(f) \neq 0\}$ onto $\widehat{J}$.

THEOREM 3.6. *Let* $\mathfrak{A}$ *be a commutative regular semisimple Banach algebra. Suppose that there is* $L > 0$ *such that for each* $\phi \in \widehat{\mathfrak{A}}$ *and each neighbourhood* N *of* ϕ *there is* $a \in \mathfrak{A}$ *with* $\|a\| \leq L$, $\phi(a) = 1$ *and* $\psi(a) = 0$ *for* $\psi \in \widehat{\mathfrak{A}}\backslash N$. *Then an element* f *of* $\mathfrak{A}$ *is pseudo regular if and only if it is relatively invertible.*

Note that it would be enough to have the condition satisfied for all ϕ in a dense subset of $\widehat{\mathfrak{A}}$.

Proof. Suppose that f is pseudo regular. We will show that for all $\phi \in \widehat{J}$, $\|\phi\|_{J^*} \geq L^{-1}$ (where $\|\phi\|_{J^*}$ is the norm of ϕ as an element of J^*). Corollary 3.5 then completes the proof. Identify $\widehat{J}$ with the open subset $\{\phi\colon \phi \in \widehat{\mathfrak{A}},\ \phi(f) \neq 0\}$ of $\widehat{\mathfrak{A}}$. Let $\phi \in \widehat{J}$ and let N be a neighbourhood of ϕ with $\overline{N} \subseteq \widehat{J}$. Take $a \in \mathfrak{A}$ as in the hypothesis and $j \in J$ with $\hat{j} = 1$ on N [3, Theorems 2.7.2 and 2.7.12]. Then $a = aj \in J$ and $\|\phi\|_{J^*} \geq |\phi(a)|\,\|a\|^{-1} \geq L^{-1}$.

COROLLARY 3.7. *Let* G *be a locally compact abelian group. Then an element* f *of* $L^1(G)$ *is pseudo regular if and only if it is relatively invertible.*

Proof. This is essentially the proof that $L^1(G)$ is regular. If $c \in L^2(G)$ is such that $\hat{c}$ is the characteristic function of a relatively compact symmetric neighbourhood N of the identity, 1, in $\widehat{G}$ with Haar measure λ then $c^2 \in L^1(G)$, $\|c^2\| = \|c\|_2^2 = \|\hat{c}\|_2^2 = \lambda$ and $(c^2)^{\wedge} = \hat{c} * \hat{c}$ so that $(c^2)^{\wedge}(1) = \lambda$ and the support of $(c^2)^{\wedge}$ lies in N^2. Thus by taking a as $\lambda^{-1}\phi c^2$ for a suitable choice of N the hypotheses of Theorem 3.6 are satisfied with $L = 1$.

Because of the severe restrictions on idempotents in $L^1(G)$ [**4**, Theorem 3.1.3], Corollary 3.7 implies that $L^1(G)$ may have relatively few pseudo regular elements. If G has no compact open subgroups then 0 is the only pseudo regular element of $L^1(G)$.

The above results can be adapted to give a proof of [**1**, Theorem 2.3].

COROLLARY 3.8. *Let* $\mathfrak{A}$ *be a uniform algebra and let* $f \in \mathfrak{A}$. *Then* f *is pseudo regular if and only if* 0 *is not in the closure of* $Y = \{\phi(f)\,;\, \phi \in \partial\mathfrak{A},\ \phi(f) \neq 0\}$.

Proof. This is Theorem 5.2 with $n = 1$.

4. Pseudo regularity in modules. Let $\mathfrak{A}$ be a commutative Banach algebra and $\mathfrak{X}$ a left Banach $\mathfrak{A}$ module which we convert to a bimodule by writing $xa = ax$ $(a \in \mathfrak{A}, x \in \mathfrak{X})$.

DEFINITION 4.1. An element f of $\mathfrak{X}$ is *pseudo regular* if there is a positive real number M such that $\|abf\| \leq M\|af\|\,\|bf\|$ $(a, b \in \mathfrak{A})$.

This is a direct extension of the definition in [**1**, §6] for the case $A = L^1(G)$, $\mathfrak{X} = L^2(G)$. Much of what we have said in earlier sections carries over to this case. In particular f is pseudo regular if and only if $af \circ bf = abf$ extends to a continuous bilinear map from $X \times X$ into X where $X = (\mathfrak{A}^1 f)^-$. However the structure is less rich if there is no "ordinary" multiplication on $\mathfrak{X}$ to interact with $\circ$ as there is for $\mathfrak{X} = L^2(G)$ when G is compact. We summarise these results.

THEOREM 4.2. *Let* f *be a pseudo regular element of* $\mathfrak{X}$ *and put* $X = (\mathfrak{A}^1 f)^-$. *Then* X° *is a commutative Banach equivalent algebra. We have*

$$a(x \circ y) = (ax) \circ y = x \circ ay \qquad (x, y \in X,\ a \in \mathfrak{A}^1).$$

f *is the identity of* X°. *We have* $X = (\mathfrak{A}f)^-$ *if* $\mathfrak{A}$ *has a bounded approximate identity and* $\mathfrak{X}$ *is an essential* $\mathfrak{A}$ *module. The map* $T\colon a \to af$ *is a continuous algebra homomorphism of* $\mathfrak{A}^1$ *onto a dense subalgebra of* X° *with* $\|T\| \leq \|f\|$ *so that* T^* *maps* $(X^\circ)^\frown$ *injectively and homeomorphically into* $\widehat{\mathfrak{A}}^1$. *If* $f \in (\mathfrak{A}f)^-$ *then* T^* *maps* $(X^\circ)^\frown$ *into* $\widehat{\mathfrak{A}}$.

COROLLARY 4.3 (*see* [**1**, §6]). *Let* G *be a locally compact group,* $\mathfrak{A} = L^1(G)$ *and* $\mathfrak{X} = L^p(G)$ $(1 < p < \infty)$ *or* $C_0(G)$. *For* $a \in \mathfrak{A}$, $x \in \mathfrak{X}$, ax *is the convolution of* a *and* x. *If* G *is not compact then* 0 *is the only pseudo regular element of* $\mathfrak{X}$. *If* G *is compact then* $f \in \mathfrak{X}$ *is pseudo regular if and only if it is a finite linear combination of characters of* G.

Proof. As $\mathfrak{A}$ has a bounded approximate identity and $\mathfrak{X}$ is an essential $\mathfrak{A}$ module, $f \in (\mathfrak{A}f)^-$ for all $f \in \mathfrak{X}$. Suppose that f is a non zero pseudo regular element of $\mathfrak{X}$ and $\phi \in (X^\circ)^\wedge$ ($(X^\circ)^\wedge$ is non void because X° has a unit). Then there is an extension g of ϕ to an element of $\mathfrak{X}^*$. If we identify g with an L^q function for the conjugate index q if $f \in L^p(G)$, or with a measure if $f \in C_0(G)$ then $\tilde{f} * g \in C_0(G)$ where $\tilde{f}(s) = f(-s)$ and we have

$$(T^*\phi)(a) = \phi(af) = \iint a(t) f(s-t) g(s)\, dt\, ds$$
$$= \int a(t)(\tilde{f} * g)(t)\, dt,$$

the interchange being justified because $\iint |a(t)|\,|f(s-t)|\,|g(s)|\, dt\, ds < \infty$. As $T^*\phi \in \widehat{\mathfrak{A}}$ there is a character χ on $L^1(G)$ with

$$(T^*\phi)(a) = \int a(t)\chi(t)\, dt.$$

Thus $\chi(t) = \tilde{f} * g$ almost everywhere. As $\tilde{f} * g \in C_0(G)$ and $|\chi(t)| = 1$ for all $t \in G$ this implies that G is compact.

If G is compact we have $\mathfrak{X} \subseteq \mathfrak{A}$ and

$$(x \circ y) f = xy \qquad (x, y \in X)$$

because this holds trivially for $x, y \in \mathfrak{A}f$. Thus, just as in Theorem 3.4, if $\phi \in \widehat{\mathfrak{A}}$ and $\phi(f) \neq 0$ then $\phi^{-1}(f)\phi \in (X^\circ)^\wedge$ and so, as in 3.5, $\|\phi\|_{X^*} \leq M|\phi(f)|$. If $\chi \in L^\infty(G)$ is the character corresponding to ϕ then $\chi f = \phi(f)\chi \in X$ and $|\phi(f)| = |\phi(\chi f)| \leq \|\phi\|_{X^*} \|\chi f\|_X = \|\phi\|_{X^*} |\phi(f)|$. This shows that $\|\phi\|_{X^*} \geq 1$ and so $|\phi(f)| \geq M^{-1}$. Since $\hat{f} \in C_0(\widehat{G})$ this implies that $\phi(f) \neq 0$ for only a finite number of values of ϕ and so f is a finite linear combination of characters.

Conversely if G is compact and $f \in \mathfrak{X}$ is a finite linear combination of characters then X is finite dimensional so $\circ$ is continuous.

Note that even if G is not compact then any trigonometric polynomial f in $\mathfrak{X}$ is pseudo regular. This shows that for example $\mathfrak{X} = C_b(G)$ has non zero pseudo regular elements.

5. Pseudo regular systems. Let $\mathfrak{A}$ be a commutative Banach algebra and denote the direct sum of n copies of $\mathfrak{A}$ with $\|(a_1, \dots, a_n)\| = \mathrm{Max}\{\|a_1\|, \dots, \|a_n\|\}$ by $\mathfrak{A}^n$. Then $\mathfrak{A}^n$ is also a commutative Banach algebra and is an $\mathfrak{A}$ module with the action $a(b_1, \dots, b_n) = (ab_1, \dots, ab_n)$. Thus Definition 4.1 applies with $\mathfrak{A}^n = \mathfrak{X}$. Pseudo regular elements of $\mathfrak{A}^n$ are called *pseudo regular systems*. This would be identical with [**1**, §4] if we had taken $\|(a_1, \dots, a_n)\| = \sum \|a_i\|$. As this is an equivalent norm to the one we are using, the above definition of pseudo regular system agrees with that given by Arens.

Using the product in $\mathfrak{A}^n$, if $F \in \mathfrak{A}^n$ is a pseudo regular system then

$$(x \circ y)F = xy \quad (x, y(\mathfrak{A}F)^-).$$

From this it follows that if Φ is a multiplicative linear functional on $\mathfrak{A}^n$ with $\Phi(F) \neq 0$ then $\Phi(F)^{-1}\Phi|X$ is a multiplicative linear functional on X° and so $\|\Phi\|_{X^*} \leq M|\Phi(F)|$. This of course also holds trivially if $\Phi(F) = 0$. However if $\Phi \neq 0$ then Φ is of the form $\Phi(a) = \phi(a_j)$ for some j with $1 \leq j \leq n$ and some $\phi \in \widehat{\mathfrak{A}}$ so $\|\Phi\|_{X^*} \leq M|\Phi(f_i)|$. For $x \in X$ and $\phi \in \widehat{\mathfrak{A}}$ we put $\phi(x) = (\phi(x_1), \dots, \phi(x_n)) \in \mathbb{C}^n$ and denote the norm of this map (where $\mathbb{C}^n$ has the Max norm) by $\|\phi\|_n$. Thus we have shown

LEMMA 5.1. *If F is a pseudo regular element of $\mathfrak{A}^n$ and $\phi \in \widehat{\mathfrak{A}}$ then*

$$\|\phi\|_n \leq M\|\phi(F)\|.$$

THEOREM 5.2. *Let $\mathfrak{A}$ be a uniform algebra and let $F \in \mathfrak{A}^n$. Then F is a pseudo regular system if and only if $(0, \dots, 0) \notin (W(F))^-$ where $W = \{\phi\colon \phi \in \partial\mathfrak{A},\ \|\phi(F)\| > 0\}$.*

Proof. If the criterion is satisfied then, just as for $n = 1$, F is a regular system in $C(Y)$ where $Y = \{\phi\colon \phi \in \partial\mathfrak{A},\ \phi(F) \neq (0, \dots, 0)\}$ and so is a pseudo regular system [**1**, §4].

For the converse if F is pseudo regular, $\varepsilon > 0$, ϕ is a weak peak point of $\mathfrak{A}$ with $\phi(F) \neq (0, \dots, 0)$ and N is an open neighbourhood of ϕ with $\|\psi(F) - \phi(F)\| < \varepsilon$ for all $\psi \in N$ then there is $a \in \mathfrak{A}$ with $\phi(a) = 1 = \|a\|$ and $|\psi(a)| < 1$ for all ψ in $\widehat{\mathfrak{A}}\backslash N$. As $\widehat{\mathfrak{A}}\backslash N$ is compact there is $P < 1$ with $|\psi(a)| \leq P$ for all ψ in $\widehat{\mathfrak{A}}\backslash N$ and replacing a by a^m for a sufficiently large value of m we will have $P < \varepsilon$. Then $aF \in X$, $\|af\| \leq \varepsilon\|F\| + \|\phi(F)\| + \varepsilon$ and $\phi(aF) = \phi(F)$ so that $\|\phi\|_n \geq \|\phi(F)\| \ (\varepsilon\|F\| + \|\phi(F)\| + \varepsilon)^{-1}$ and taking the limit as $\varepsilon \to 0^+$ we get $\|\phi\|_n \geq 1$ so that by the lemma, $\|\phi(F)\| \geq M^{-1}$.

Since the weak peak points are dense in $\partial\mathfrak{A}$ and W is open in $\partial\mathfrak{A}$, the weak peak points in W are dense in W and $\|\phi(F)\| \geq M^{-1}$ for all $\phi \in W$, which proves the result.

The question of whether Corollary 3.7 applies to systems based on the algebra $L^1(G)$ seems an interesting question in harmonic analysis.

DEFINITION 5.3. A system F in $\mathfrak{A}^n$ is *relatively regular* if it is a regular system in $(e\mathfrak{A})^n$ for some idempotent e in $\mathfrak{A}$.

A relatively regular system is of course pseudo regular [**1**, §4]. This can be seen by taking $g_1, \dots, g_n \in e\mathfrak{A}$ with $\sum f_i g_i = e$ so that, for $a \in \mathfrak{A}$, $\|ae\| = \|\sum af_i g_i\| \leq \|aF\| \sum \|g_i\|$ and so $\|abF\| = \|aebF\| \leq \|ae\|\,\|bF\| \leq M\|aF\|\,\|bF\|$ with $M = \sum \|g_i\|$.

THEOREM 5.4. *Let G be a compact abelian group. Then a system F in $\mathfrak{A} = L^1(G)^n$ is pseudo regular if and only if it is relatively regular.*

Proof. The proof is similar to Theorem 5.2. If $\phi \in \widehat{\mathfrak{A}}$ with $\phi(F) \neq (0, \dots, 0)$ then we take a to be the minimal idempotent with $\phi(a) = 1$ and we get $\|\phi(F)\| \geq M^{-1}$ as before. By the Riemann-Lebesgue lemma this implies that $\|\phi(F)\| = 0$ for all but a finite number of elements ϕ in $\widehat{\mathfrak{A}}$. Thus X is finite dimensional so $\circ$ is continuous and F is pseudo regular.

REFERENCES

[1] R. F. Arens, *Pseudo regular elements in a normed ring*, Pacific J. Math., **154** (1992), 1–16.

[2] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol II, Springer, Berlin, 1960.

[3] C. E. Rickart, *General Theory of Banach algebras*, Van Nostrand, Princeton, 1960.

[4] W. Rudin, *Fourier Analysis on Groups*, Interscience, New York, 1962.

Received September 9, 1991.

THE UNIVERSITY
NEWCASTLE UPON TYNE
ENGLAND NE1 7RU