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AN APPLICATION OF HOMOGENIZATION THEORY TO HARMONIC ANALYSIS ON SOLVABLE LIE GROUPS OF POLYNOMIAL GROWTH

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Let Q be a connected solvable Lie group of polynomial growth. Let also $E_1, \dots, E_p$ be left invariant vector fields on G that satisfy Hörmander's condition and denote by $L = -(E_1^2 + \cdots + E_p^2)$ the associated sub-Laplacian and by $S(x, t)$ the ball which is centered at $x \in Q$ and it is of radius $t > 0$ with respect to the control distance associated to those vector fields. The goal of this article is to prove the following Harnack inequality: there is a constant $c > 0$ such that $|E_i u(x)| \leq ct^{-1}u(x), x \in Q$, $t \geq 1$, $1 \leq i \leq p$, for all $u \geq 0$ such that $Lu = 0$ in $S(x, t)$. This inequality is proved by adapting some ideas from the theory of homogenization.

0. Introduction. Let Q be a connected solvable Lie group which we assume to be of polynomial growth; i.e., if dg is a left invariant Haar measure on Q and V a compact neighborhood of the identity element e of Q, then there are constants $c, d > 0$ such that

$$dg - \text{measure}(V^n) \leq cn^d, \qquad n \in \mathbb{N}.$$

Notice that the connected nilpotent Lie groups are also solvable and of polynomial growth (cf. [**5**], [**6**]).

Let us identify the Lie algebra $\mathfrak{q}$ of Q with the left invariant vector fields on Q and consider $E_1, \dots, E_p \in \mathfrak{q}$ that satisfy Hörmander's condition; i.e., together with their successive Lie brackets $[E_{i_1}, [E_{i_2}, [\dots[E_{i_{s-1}}, E_{i_s}]\dots]]]$, they generate $\mathfrak{q}$. To these vector fields there is associated, in a canonical way, a left invariant distance $d_E(\cdot, \cdot)$ on G, called control distance. This distance has the property that (cf. [**15**]) if $S_E(x, t) = \{y \in G, d_E(x, y) < t\}$, $x \in G$, $t > 0$ then there is $c \in \mathbb{N}$ such that

$$S_E(e, n) \subseteq V^{cn}, \quad V^n \subseteq S_E(e, cn), \qquad n \in \mathbb{N}. \tag{0.1}$$

According to a classical theorem of L. Hörmander [7] the operator

$$L = -(E_1^2 + \cdots + E_p^2)$$

is hypoelliptic.

The goal of this paper is to prove the following result:

THEOREM 1. *Let Q, $E_1, \dots, E_p$ and L be as above. Then there is $c > 0$ such that*

$$|E_i u(x)| \leq ct^{-1} u(x), \qquad x \in Q, t \geq 1, \tag{0.2}$$

for all $u \geq 0$ such that $Lu = 0$ in $S_E(x, t)$, $1 \leq i \leq p$.

This is a result of technical nature, but a very useful one, when one tries to generalise the "real variable theory" to Q (cf. [9], [10]). For instance, it can be used to obtain estimates for the Poisson kernel and the Green function. Another immediate consequence of Theorem 1 is that every positive harmonic function in Q (i.e. every $u \geq 0$, $u \in C^\infty(Q)$ such that $Lu = 0$ in Q) is constant (cf. [13]).

When Q is also nilpotent then Theorem 1 is a particular case of a more general result of N. Th. Varopoulos [14], namely for all integers $l \geq 0$ there is $c_l > 0$ such that

$$|E_{i_1} \cdots E_{i_l} u(x)| \leq c_l t^{-l} u(x), \qquad x \in Q, t \geq 1, \tag{0.3}$$

for all $u \geq 0$ such that $Lu = 0$, in $S_E(x, t)$.

As we shall see, for $l \geq 2$, (0.3) is not true for general, not necessarily nilpotent, solvable Lie groups.

(0.3) is also true for $0 < t < 1$ (cf. N. Th. Varopoulos [14]), but this is a local result and the Lie group structure does not play any role in proving it.

The main contribution of this article is the observation that the operator L can be viewed as a second order differential operator with quasiperiodic coefficients on the nil-shadow Q_N of Q, which is a nilpotent Lie group (cf. [6]). Once we adopt this point of view, proving Theorem 1 becomes a matter of generalizing results, already known for second order uniformly elliptic differential operators with periodic coefficients (cf. [1], [2]). Indeed, in that context, Theorem 1 has already been proved by M. Avellaneda and F. H. Lin [1].

More precisely, let (we use the summation convention for repeated indices)

$$L_1 = -\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j}$$

be a uniformly elliptic operator in $\mathbb{R}^n$ and assume that its coefficients $a_{ij}(x)$ are periodic (i.e. $a_{ij}(x+z) = a_{ij}(x)$, $x \in \mathbb{R}^n$, $z \in \mathbb{Z}^n$) and Hölder continuous (i.e. there is $\alpha \in (0, 1)$ and $M > 0$ such that $\|a_{ij}(x)\|_{C^\alpha(\mathbb{R}^n)} \leq M$).

Also let

$$L_\varepsilon = -\frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_j}, \qquad 0 < \varepsilon \leq 1,$$

and denote by $B(x, t)$ the Euclidean ball of radius $t > 0$ centered at $x \in \mathbb{R}^n$.

We observe that $L_\varepsilon u_\varepsilon = 0$ in $B(0, 1)$ if and only if $Lu = 0$ in $B(0, t)$, where $u(x) = u_\varepsilon(\varepsilon x)$, $t = \varepsilon^{-1}$. Hence, proving that there is $c > 0$ such that

$$|\nabla u(0)| \le ct^{-1}u(0), \qquad t \ge 1,$$

for all $u \ge 0$ such that $Lu = 0$ in $B(0, t)$, is equivalent to proving that there is a constant $c > 0$, independent of $\varepsilon \in (0, 1)$ such that

$$|\nabla u_\varepsilon(0)| \le cu_\varepsilon(0), \qquad 0 < \varepsilon \le 1,$$

for all $u_\varepsilon \ge 0$ such that $L_\varepsilon u_\varepsilon = 0$ in $B(0, 1)$. This follows from the following result of M. Avellaneda and F. H. Lin [**1**], using Moser's Harnack inequality (cf. [**9**]).

THEOREM 0.1 (*cf. M. Avellaneda and F. H. Lin* [**1**]). *Let* L_ε, $0 < \varepsilon \le 1$, *be as above,* $f \in L^{n+\delta}(B(0, 1))$, $\delta > 0$, *and* $g \in C^{1,\nu}(\partial B(0, 1))$, $0 < \nu \le 1$. *Then there is a constant* $c > 0$ *depending only on* $\alpha, M, n, \nu, \delta$ *and independent of* ε *such that*

$$(0.4)\quad \begin{aligned} &[u_\varepsilon]_{C^{0,1}(B(0,1))} \\ &\quad \le c([g]_{C^{1,\nu}(\partial B(0,1))} + \|f\|_{L^{n+\delta}(B(0,1))}), \qquad 0 < \varepsilon \le 1, \end{aligned}$$

for all u_ε *satisfying*

$$L_\varepsilon u_\varepsilon = f \text{ in } B(0, 1), \quad u_\varepsilon = g \text{ on } \partial B(0, 1), \qquad 0 < \varepsilon \le 1.$$

Notice that although we do not have any, uniform with respect to ε, control of the Hölder continuity of the coefficients of the operators L_ε, the above result gives a uniform with respect to the ε estimate for $[u_\varepsilon]_{C^{0,1}}$. This is due to the fact that there is an elliptic operator with constant coefficients

$$L_0 = -\frac{\partial}{\partial x_i} q_{ij} \frac{\partial}{\partial x_j}$$

called the homogenized operator, which has the property that if

$$L_0 u_0 = f \text{ in } B(0, 1), \quad u_0 = g \text{ on } \partial B(0, 1), \qquad 0 < \varepsilon \le 1,$$

then

$$u_\varepsilon \to u_0, \quad \varepsilon \to 0$$

uniformly on the compact subsets of $B(0, 1)$.

The coefficients q_{ij} of the homogenized operator L_0 are given by the formula

$$q_{ij} = \int_D \left[a_{ij}(x) - a_{il}(x) \frac{\partial}{\partial x_l} \chi^j(x) \right] dx, \quad D = [0, 1]^n$$

where the functions χ^j, $j = 1, \ldots, n$, called correctors, are the unique solutions of the problem

$$L(x_j - \chi^j) = 0, \quad \chi^j(x+z) = \chi^j(x), \qquad x \in \mathbb{R}^n, z \in \mathbb{Z}^n,$$
$$\int_D \chi^j(x)\, dx = 0, \quad D = [0, 1]^n.$$

The motivating example is the universal covering G of the group of Euclidean motions on the plane, which is a three dimensional solvable Lie group of polynomial growth. It turns out that every operator L, as in Theorem 1, in G, can be expressed as a second order differential operator in $\mathbb{R}^3$ with periodic coefficients.

More precisely, let $\mathfrak{g}$ denote the Lie algebra of G and identify its elements with the left invariant vector fields on G. Then, there is a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}$ such that

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = -X_2, \quad [X_2, X_3] = 0.$$

Identifying the simply connected analytic subgroups of G whose Lie algebras are generated by $\{X_2, X_3\}$ and $\{X_1\}$ with $\mathbb{R}^2$ and $\mathbb{R}$ respectively, we can see that G is isomorphic to the semidirect product $\mathbb{R}^2 \times_\tau \mathbb{R}$ where the action τ of $\mathbb{R}$ on $\mathbb{R}^2$ is given by $\tau\colon \mathbb{R} \to L(\mathbb{R}^2)\colon x \to \mathrm{rot}_x$, rot_x being the counterclockwise rotation by angle x and $L(\mathbb{R}^2)$ the space of linear transformations of $\mathbb{R}^2$.

Let us consider the exponential coordinates of the second kind (cf. **[12]**)

$$\varphi\colon \mathbb{R}^3 \to G, \quad \varphi\colon (x_3, x_2, x_1) \to \exp x_3 X_3 \exp x_2 X_2 \exp x_1 X_1.$$

If $x = (x_3, x_2, x_1)$, then we have (cf. §2)

$$d\varphi^{-1} X_1(x) = \frac{\partial}{\partial x_1}, \tag{0.5}$$
$$d\varphi^{-1} X_2(x) = \cos x_1 \frac{\partial}{\partial x_2} + \sin x_1 \frac{\partial}{\partial x_3},$$
$$d\varphi^{-1} X_3(x) = -\sin x_1 \frac{\partial}{\partial x_2} + \cos x_1 \frac{\partial}{\partial x_3}.$$

Let us now use φ to identify Q and $\mathbb{R}^3$ as differential manifolds.

Let

$$E_1 = X_1, \quad E_2 = X_1 + X_2, \quad E_3 = X_3 \quad \text{and} \quad L = -(E_1^2 + E_2^2 + E_3^2).$$

Then L becomes a uniformly elliptic differential operator on $\mathbb{R}^n$, which can be written in divergence form as

$$L = -\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j}$$

with

$$a_{11} = 2, \quad a_{22} = a_{33} = 1, \quad a_{12} = a_{21} = \cos x_1,$$
$$a_{13} = a_{31} = \sin x_1, \quad a_{23} = a_{32} = 0.$$

Moreover, the control distance $d_E(\cdot, \cdot)$ associated to the vector fields E_1, E_2, E_3 becomes equivalent to the Euclidean one; i.e., $\exists b \geq a > 0$ such that $a|x-y| \leq d_E(x, y) \leq b|x-y|$, $x, y \in \mathbb{R}^3$.

Let us now see why the inequalities (0.3) are not true for $l \geq 2$.

Let us put

$$L_\varepsilon = -\frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial}{\partial x_j}, \qquad 0 < \varepsilon \leq 1.$$

Then proving (0.3) for $l \geq 2$ and $i_1 = i_2 = 1$ is equivalent to proving that there is $c > 0$, independent of ε, such that

$$\left| \frac{\partial^2}{\partial x_1^2} u_\varepsilon(0) \right| \leq c u(0), \qquad 0 < \varepsilon \leq 1, \tag{0.6}$$

for all $u_\varepsilon \geq 0$ satisfying $L_\varepsilon u_\varepsilon = 0$ in $B(0, 1)$.

As we are going to see, (0.6) is not true.

In the example we consider, we have that

$$\chi^1(x) = 0, \quad \chi^2(x) = \frac{1}{2} \sin x_1, \quad \chi^3(x) = -\frac{1}{2} \cos x_1$$

and

$$L_0 = -\left(2\frac{\partial^2}{\partial x_1^2} + \frac{3}{4} \frac{\partial^2}{\partial x_2^2} + \frac{5}{4} \frac{\partial^2}{\partial x_3^2} \right).$$

Also L_ε can be written as

$$\begin{aligned} L_\varepsilon = & -2\frac{\partial^2}{\partial x_1^2} - 2\cos\frac{x_1}{\varepsilon} \frac{\partial^2}{\partial x_1 \partial x_2} - 2\sin\frac{x_1}{\varepsilon} \frac{\partial^2}{\partial x_1 \partial x_3} \\ & - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + \frac{1}{\varepsilon} \sin\frac{x_1}{\varepsilon} \frac{\partial}{\partial x_2} - \frac{1}{\varepsilon} \cos\frac{x_1}{\varepsilon} \frac{\partial}{\partial x_3}. \end{aligned} \tag{0.7}$$

Let us take $f = 0$ and $g = x_3 + 2$ in (0.4). Then $u_0 = x_3 + 2$. Hence $u_0 \geq 0$, $\partial u/\partial x_3 = 1$ and $\partial u_0/\partial x_1 = \partial u_0/\partial x_2 = 0$.

Since $L_\varepsilon \partial u_\varepsilon/\partial x_i = (\partial/\partial x_i) L_\varepsilon u_\varepsilon = 0$, $i = 2, 3$, it follows from Theorem 0.1 that

$$(0.8) \qquad u_\varepsilon \to u_0 \quad \text{and} \quad \frac{\partial}{\partial x_i} u_\varepsilon \to \frac{\partial}{\partial x_i} u_0, \qquad (\varepsilon \to 0), i = 2, 3,$$

uniformly on the compact subsets of $B(0, 1)$ and that there is $c > 0$ such that

$$(0.9) \qquad \left| \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} u_\varepsilon(x) \right| \leq c, \qquad x \in B(0, 1), i = 1, 2, 3, j = 2, 3.$$

Now, (0.7), (0.8) and (0.9) imply that

$$\frac{\partial^2}{\partial x_1^2} u_\varepsilon(0) \sim \frac{1}{\varepsilon}, \qquad (\varepsilon \to 0)$$

which disproves (0.6). So (0.3) is not true for $l \geq 0$.

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1. The structure of the Lie algebra. Let $\mathfrak{q}$ be a solvable Lie algebra and denote by $\mathfrak{n}$ its nil-radical. Then $\mathfrak{n}$ is a nilpotent ideal of $\mathfrak{q}$ and $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{n}$ (cf. **[12]**). We denote by π the natural map $\pi \colon \mathfrak{q} \to \mathfrak{q}/\mathfrak{n}$. We also put $k = \dim(\mathfrak{q}/\mathfrak{n})$.

Let $\operatorname{ad} X = S(X) + K(X)$ denote the Jordan decomposition of the derivation $\operatorname{ad} X(Y) = [X, Y]$, $X \in \mathfrak{q}$. $S(X)$ is the semisimple and $K(X)$ the nilpotent part. It is well known that

(i) $S(X)$ and $K(X)$ are derivations of $\mathfrak{q}$ (cf. **[12]**).

(ii) There are real polynomials $s(x)$ and $k(x)$ such that

(1.1) $\quad S(X) = s(\operatorname{ad} X)$ and $K(X) = k(\operatorname{ad} X)$ (cf. **[8]**).

(1.2) (iii) $[S(X), K(X)] = 0$.

Notice that the fact that $\operatorname{ad} X(X) = [X, X] = 0$, $X \in \mathfrak{q}$ implies that the constant coefficients of the polynomials $k(x)$, hence also of the polynomials $s(x)$, are zero.

LEMMA 1.1. *There are vectors* $Y_1, \dots, Y_k \in \mathfrak{q}$ *such that*
(a) $[S(Y_i), S(Y_j)] = 0$, $1 \leq i, j \leq k$,
(b) $\{\pi(Y_1), \dots, \pi(Y_k)\}$ *is a basis of* $\mathfrak{q}/\mathfrak{n}$.

Proof. Let $\{Z_1, \ldots, Z_k\}$ any choice of vectors of $\mathfrak{q}$ such that $\{\pi(Z_1), \ldots, \pi(Z_k)\}$ is a basis of $\mathfrak{q}/\mathfrak{n}$. To prove the lemma it is enough to prove that for every integer $1 \le m \le k$ we can choose vectors $Y_1, \ldots, Y_m \in \mathfrak{q}$ such that

$$(1.3) \qquad [S(Y_i), S(Y_j)] = 0, \qquad 1 \le i, j \le m,$$
$$\{\pi(Y_1), \ldots, \pi(Y_m), \pi(Z_{m+1}), \ldots, \pi(Z_k)\} \text{ basis of } \mathfrak{q}/\mathfrak{n}.$$

(1.3) will be proved by induction on m. For $m = 1$ it is enough to take $Y_1 = Z_1$. So assume that (1.3) is true for $m = j$, $1 \le j < k$. To prove that it is also true for $m = j + 1$ assume that the vectors $Y_1, \ldots, Y_j$ have been chosen and consider the linear space $\mathfrak{b}$ that is generated by $\mathfrak{n}$ and the vectors $Z_{j+1}, \ldots, Z_k$. It follows from the fact that $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{n}$ that $\mathfrak{b}$ is actually an ideal of $\mathfrak{q}$. By our induction hypothesis the derivations $S(Y_1), \ldots, S(Y_j)$ commute. They are also semisimple linear transformations and satisfy $S(Y_i)(\mathfrak{b}) \subseteq \mathfrak{n}$. This last assertion follows from the fact that the polynomials $k(x)$ and $s(x)$, in (1.1), have zero constant coefficients. Hence $\mathfrak{b}$ admits a subspace $\mathfrak{d}$ complementary to $\mathfrak{n}$, i.e., such that $\mathfrak{b} = \mathfrak{d} \oplus \mathfrak{n}$ and $S(Y_i)(\mathfrak{d}) = \{0\}$, $1 \le i \le j$. For Y_{j+1} we choose any non zero element of $\mathfrak{d}$ such that $\pi(Y_{j+1})$ is linearly independent of the vectors $\{\pi(Z_{j+2}), \ldots, \pi(Z_k)\}$. $S(Y_{j+1})$ will commute with the $S(Y_1), \ldots, S(Y_j)$ because of (1.1) and the fact that $S(Y_i)Y_{j+1} = 0$, $1 \le i \le j$. This proves (1.3) and the lemma follows.

PROPOSITION 1.2. *There are vectors $X_1, \ldots, X_k \in \mathfrak{q}$, such that*
(a) $S(X_i)X_j = 0$, $1 \le i, j \le k$.
(b) $\{\pi(X_1), \ldots, \pi(X_k)\}$ *is a basis of* $\mathfrak{q}/\mathfrak{n}$.

Proof. Let $\{Y_1, \ldots, Y_k\}$ be a set of elements of $\mathfrak{q}$ as in Lemma 1.1. Arguing in the same way as in the proof of that lemma we can see that $\mathfrak{q}$ has a subspace $\mathfrak{b}$ complementary to $\mathfrak{n}$, i.e. such that $\mathfrak{q} = \mathfrak{n} \oplus \mathfrak{b}$ and $S(Y_i)\mathfrak{b} = \{0\}$, $1 \le i \le k$.

Let $N_1, \ldots, N_k \in \mathfrak{n}$ such that $X_i = Y_i - N_i \in \mathfrak{b}$, $i = 1, \ldots, k$. The vectors $X_1, \ldots, X_k$ have all the properties required by the proposition: they satisfy (b) since they form a basis of $\mathfrak{b}$. To verify that they satisfy $S(X_i)X_j = 0$, $1 \le i, j \le k$ observe that if this weren't true then we would have $(\operatorname{ad} X_i)^n X_j \ne 0$, $n \in \mathbb{N}$. To see that this is not possible let us observe first that since $K(Y_i)$ is a derivation we have that $[K(Y_i), \operatorname{ad} N_i] = \operatorname{ad}(K(Y_i)N_i)$, which combined with the fact that $K(Y_i)N_i \in \mathfrak{n}$ implies that the linear transformation

$[K(Y_i), \operatorname{ad} N_i]$ is nilpotent. This in turn implies that although the $K(Y_i)$ and $\operatorname{ad} N_i$ do not commute, we can nevertheless find $m \in \mathbb{N}$ such that $(K(Y_i) + \operatorname{ad} N_i)^m = 0$, i.e. $K(Y_i) + \operatorname{ad} N_i$ is a nilpotent transformation.

Next we observe that

$$\operatorname{ad} X_i(X_j) = (\operatorname{ad} Y_i + \operatorname{ad} N_i)X_j = (K(Y_i) + \operatorname{ad} N_i)X_i$$

and that

$$\operatorname{ad} X_i(K(Y_i) + \operatorname{ad} N_i)^n = (K(Y_i) + \operatorname{ad} N_i)^{n+1} + S(Y_i)(K(Y_i) + \operatorname{ad} N_i)^n.$$

We also have that $[S(Y_i), \operatorname{ad} N_i] = 0$, since

$$0 = S(Y_i)X_i = S(Y_i)(Y_i - N_i) = -S(Y_i)N_i.$$

So using (1.2) we can conclude that

$$\begin{aligned} \operatorname{ad} X_i(K(Y_i) + \operatorname{ad} N_i)^n = (K(Y_i) + \operatorname{ad} N_i)^{n+1} \\ + (K(Y_i) + \operatorname{ad} N_i)^n S(Y_i), \qquad n \geq 0. \end{aligned}$$

From this observation we can easily see that it can be proved by induction that

$$(\operatorname{ad} X_i)^n X_j = (K(Y_i) + \operatorname{ad} N_i)^n X_j, \qquad n \in \mathbb{N}.$$

This contradicts the assumption that $(\operatorname{ad} X_i)^n X_j \neq 0$, $n \in \mathbb{N}$, because the transformation $K(Y_i) + \operatorname{ad} N_i$ as we have already seen is nilpotent.

In what follows we shall consider and fix, once and for all, vectors $X_1, \ldots, X_k \in \mathfrak{q}$ having the properties described in the above proposition.

The nil-shadow $\mathfrak{q}_N$ *of* $\mathfrak{q}$. We can easily see that the conditions

$$\begin{gathered} [X_i, X_j]_N = [X_i, X_j], [X_i, Y]_N = K(X_i)Y, \\ [Y, Z]_N = [Y, Z], \qquad 1 \leq i, j \leq k, Y, Z \in \mathfrak{n}, \end{gathered}$$

define a unique product $[\cdot, \cdot]_N$ on the linear space $\mathfrak{q}$. We can verify directly (writing the elements X of $\mathfrak{q}$ as a sum $X = X' + Y$ with X' a linear combination of the vectors $X_1, \ldots, X_k$ and $Y \in \mathfrak{n}$) that $[\cdot, \cdot]_N$ satisfies the Jacobi identity. So, $\mathfrak{q}_N = (\mathfrak{q}, [\cdot, \cdot]_N)$ is a Lie algebra, which is also nilpotent. $\mathfrak{q}_N$ is called the nil-shadow of $\mathfrak{q}$.

The filtration of $\mathfrak{q}$. We put $\mathfrak{r}_1 = \mathfrak{q}$ and $\mathfrak{r}_{i+1} = [\mathfrak{r}_1, \mathfrak{r}_i]_N$, $i \geq 1$. Then, since $\mathfrak{q}_N$ is nilpotent, we have the following filtration of $\mathfrak{q}$:

$$\mathfrak{q} = \mathfrak{r}_1 \supseteq \mathfrak{n} \supseteq \mathfrak{r}_2 \supseteq \cdots \supseteq \mathfrak{r}_m \supseteq \mathfrak{r}_{m+1} = \{0\}, \qquad \mathfrak{r}_m \neq \{0\}.$$

PROPOSITION 1.3. (1) $\mathfrak{r}_1 \supseteq \mathfrak{n} \supseteq \mathfrak{r}_2$.

(2) $\mathfrak{r}_i$ *is an ideal of* $\mathfrak{q}$, *i.e.* $[\mathfrak{q}, \mathfrak{r}_i] \subseteq \mathfrak{r}_i$, $i = 1, 2, \ldots$.

(3) *There are subspaces* $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ *of* $\mathfrak{q}$ *such that*

(a) $S(X_j)\mathfrak{a}_i \subseteq \mathfrak{a}_i$, $j = 1, \ldots, k$, $i = 1, \ldots, m$,

(b) $\mathfrak{r}_i = \mathfrak{a}_i \oplus \cdots \oplus \mathfrak{a}_m$ *and*

(c) $\mathfrak{a}_i = \mathfrak{a}_{0i} \oplus \mathfrak{a}_{1i}$, *where* $\mathfrak{a}_{0i} = \{Y \in \mathfrak{a}_i,\ S(X_j)Y = 0,\ 1 \le j \le k\}$, $S(X_j)\mathfrak{a}_{1i} \subseteq \mathfrak{a}_{1i}$, $1 \le j \le k$.

Proof. (1) follows from the fact that $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{n}$ and the way $[\cdot, \cdot]_N$ was defined. (2) can be proved by induction. It is trivially true for $i = 1$. So, assume that it is true for $i = n$. We are going to verify that it is also true for $i = n + 1$.

Let $X \in \mathfrak{q}$, $Y \in \mathfrak{r}_1$, $Z \in \mathfrak{r}_i$. If $X \in \mathfrak{n}$, then $\operatorname{ad} X([Y, Z]_N) = [X, [Y, Z]_N]_N \in \mathfrak{r}_{n+2} \subseteq \mathfrak{r}_{n+1}$. If $Z \in \mathfrak{n}$, $Y = X_j$ and $X = X_l$ for some $1 \le j,\ l \le k$, then $\operatorname{ad} X_l([X_j, Z]_N) = \operatorname{ad} X_l K(X_j)Z = K(X_l)K(X_j)Z + S(X_l)K(X_j)Z = K(X_l)K(X_j)Z + K(X_j)S(X_l)Z$, since $S(X_l)X_j = 0$, $S(X_l)$ is a derivation and $K(X_j)$ is a polynomial in $\operatorname{ad} X_j$. Hence $\operatorname{ad} X_l([X_j, Z]_N) = [X_l, [X_j, Z]_N]_N + [X_j, S(X_l)Z]_N \in \mathfrak{r}_{n+1}$. Finally, if $X = X_h$, $Y = X_l$ and $Z = X_j$ for some $1 \le h, l, j \le k$, then $\operatorname{ad} X_h([X_l, X_j]_N) = [X_h, [X_l, X_j]_N]_N \in \mathfrak{r}_{n+2} \subseteq \mathfrak{r}_{n+1}$. Since the general case is a linear combination of the cases examined above, we conclude that $\mathfrak{r}_{n+1}$ is also an ideal of $\mathfrak{q}$. This proves the inductive step and (2) follows.

(3a) and (3b) follow from the observation that, according to (2), the spaces $\mathfrak{r}_1, \ldots, \mathfrak{r}_m$ are invariant with respect to the transformations $S(X_i)$, $i = 1, \ldots, k$ (cf. **[8]**). Given (3a) and (3b), (3c) follows again from the observation that $\mathfrak{a}_{0i}$ is invariant with respect to the algebra of linear transformations of $\mathfrak{q}$ generated by the transformations $S(X_i)$, $i = 1, \ldots, k$.

We put $n = \dim \mathfrak{q}$, $n_0 = 0$ and $n_i = \dim(\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_i)$, $i = 1, \ldots, m$. Then

$$1 \le k \le n_1 < \cdots < n_m = n.$$

The choice of the basis of $\mathfrak{q}$. We assume now that $\mathfrak{q}$ is of type R, i.e. that all the eigenvalues of the derivations $\operatorname{ad} X$, $X \in \mathfrak{q}$ are purely imaginary (i.e. of the type ia, $a \in \mathbb{R}$).

PROPOSITION 1.4. *If* $\mathfrak{q}$ *is of type* R, *then there is a basis* $\{X_1, \ldots, X_n\}$ *of* $\mathfrak{q}$ *such that*

(1) $X_1, \ldots, X_k$ *are as in Proposition* 1.1 *and* $X_{k+1}, \ldots, X_n \in \mathfrak{n}$,

(2) $\{X_{n_{i-1}+1}, \ldots, X_{n_i}\}$ *is a basis of* $\mathfrak{a}_i$, $i = 1, \ldots, m$,

(3) $\{X_{n_{i-1}+1}, \dots, X_{n_{0i}}\}$ *and* $\{X_{n_{0i}+1}, \dots, X_{n_i}\}$ *are bases of* $\mathfrak{a}_{0i}$ *and* $\mathfrak{a}_{1i}$ *respectively,* $i = 1, \dots, m$ *and*

(4) *the number of the vectors* $\{X_{n_{0i}+1}, \dots, X_{n_i}\}$ *is even and they can be combined in pairs* $\{X_{n_{0i}+1}, X_{n_{0i}+2}\}, \dots, \{X_j, X_{j+1}\}, \dots, \{X_{n_i-1}, X_{n_i}\}$ *so that for every pair* $\{X_j, X_{j+1}\}$ *and every* $l = 1, \dots, k$ *there is* $a_l \in \mathbb{R}$ *such that*

$$\begin{aligned} e^{S(X_l)}X_j &= \cos a_l X_j + \sin a_l X_{j+1}, \\ e^{S(X_l)}X_{j+1} &= -\sin a_l X_j + \cos a_l X_{j+1}. \end{aligned} \tag{1.4}$$

Proof. For $\{X_{n_{i-1}+1}, \dots, X_{n_{0i}}\}$ we choose any basis of $\mathfrak{a}_{0i}$, so that (1) is satisfied. In order to choose $\{X_{n_{0i}+1}, \dots, X_{n_i}\}$ let us denote by $\mathfrak{a}_{1i,\mathbb{C}}$ the complexification of $\mathfrak{a}_{1i}$ and denote by $S(X_j)_\mathbb{C}$ the extension of $S(X_j)$ to $\mathfrak{a}_{1i,\mathbb{C}}$, $i = 1, \dots, k$. Since $S(X_j)_\mathbb{C}$ is also semisimple, we can decompose $\mathfrak{a}_{1i,\mathbb{C}}$ as $\mathfrak{a}_{1i,b_1} \oplus \cdots \oplus \mathfrak{a}_{1i,b_h}$ where $\mathfrak{a}_{1i,b_l} = \{Y \in \mathfrak{a}_{1i,\mathbb{C}}, S(X_j)_\mathbb{C}(Y) = ib_l Y\}$ and $ib_1, \dots, ib_h \in i\mathbb{R}$ are the different eigenvalues of $S(X_j)_\mathbb{C}$. Since $S(X_l)_\mathbb{C} S(X_j)_\mathbb{C} = S(X_j)_\mathbb{C} S(X_l)_\mathbb{C}$, $l = 1, \dots, k$, $S(X_l)_\mathbb{C}\mathfrak{a}_{1i,b_s} \subseteq \mathfrak{a}_{1i,b_s}$, $s = 1, \dots, h$. Applying the same procedure to $\mathfrak{a}_{1i,b_s}$ relative to any other $S(X_l)_\mathbb{C}$, we obtain a decomposition

$$\mathfrak{a}_{1i,\mathbb{C}} = \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_s \tag{1.5}$$

of $\mathfrak{a}_{1i,\mathbb{C}}$ into $\{S(X_j)_\mathbb{C}, j = 1, \dots, k\}$-invariant subspaces, such that the linear tranformations induced in the $\mathfrak{b}_l$ by every $S(X_j)_\mathbb{C}$ are scalar multiplications by some ia, $a \in \mathbb{R}$. Moreover the subspaces $\mathfrak{b}_l$ can be taken to be one-dimensional. Let us identify $\mathfrak{a}_{1i,\mathbb{C}}$ with $\{Z + iE, Z, E \in \mathfrak{a}_{1i}\}$ and put $\overline{Y} = Z - iE$, $\operatorname{Re} Y = Z$, $\operatorname{Im} Y = E$ for $Y = Z + iE \in \mathfrak{a}_{1i,\mathbb{C}}$, $Z, E \in \mathfrak{a}_{1i}$ and $\overline{A} = \{\overline{Y}, Y \in A\}$ for $A \subseteq \mathfrak{a}_{1i,\mathbb{C}}$. We observe that if ia, $a \in \mathbb{R}$, $a \neq 0$ is an eigenvalue of $S(X_j)_\mathbb{C}$ then $-ia$ is also an eigenvalue of the same multiplicity and that if Y is an eigenvector for ia, $Y \neq 0$ then $\operatorname{Re} Y \neq 0$, $\operatorname{Im} Y \neq 0$, $\operatorname{Re} Y \neq \operatorname{Im} Y$ and $\overline{Y}$ is an eigenvector for the eigenvalue $-ia$. Using this observation we can easily see that the subspaces $\mathfrak{b}_l$ can be chosen in such a way, that the decomposition (1.5) can be written as

$$\mathfrak{a}_{1i,\mathbb{C}} = \mathfrak{b}_{i_1} \oplus \overline{\mathfrak{b}}_{i_1} \oplus \cdots \oplus \mathfrak{b}_{i_r} \oplus \overline{\mathfrak{b}}_{i_r}$$

where $\mathfrak{b}_l = \{zY_l, z \in \mathbb{C}\}$ for some $Y_l \in \mathfrak{a}_{1i,\mathbb{C}}$, $Y_l = Z + iE$, $Z, E \in \mathfrak{a}_{1i}$, $Z \neq E$, $Z, E \neq 0$.

We take $X_{n_{0i}+1} = \operatorname{Re} Y_{j_1}$, $X_{n_{0i}+2} = \operatorname{Im} Y_{j_1}$, ..., $X_{n_i-1} = \operatorname{Re} Y_{j_r}$, $X_{n_i} = \operatorname{Im} Y_{j_r}$. We can easily see that the basis of $\mathfrak{q}$, constructed in this way, satisfies the requirements of the proposition.

2. The exponential coordinates of the second kind. Let Q a simply connected solvable Lie group of polynomial growth and denote by $\mathfrak{q}$ its Lie algebra. According to a well-known theorem of Y. Guivarc'h **[6]**, $\mathfrak{q}$ is of type R, i.e. all the eigenvalues of the derivations $\operatorname{ad} X(Y) = [X, Y]$, $X, Y \in \mathfrak{q}$ are of the type ia, $a \in \mathbb{R}$. We identify the elements of $\mathfrak{q}$ with the left invariant vector fields on Q.

The derivations $S(X)$, $K(X)$, $X \in \mathfrak{g}$ and the integers $n_1, \dots, n_m$ are as in §1. We put

$$\sigma(i) = j, \quad \text{if } n_{j-1} < i \leq n_j, \qquad i = 1, \dots, n.$$

We denote by N the nil-radical of Q i.e. the analytic subgroup of Q having the nil-shadow $\mathfrak{n}$ of $\mathfrak{q}$ as its Lie algebra. Note that N is nilpotent and that Q/N is abelian.

Using the basis $\{X_1, \dots, X_n\}$ of $\mathfrak{q}$ constructed in Proposition 1.4, we can consider the diffeomorphism

$$\varphi \colon \mathbb{R}^n \to Q, \quad \varphi \colon x = (x_n, \dots, x_1) \to \exp x_n X_n \cdots \exp x_1 X_1$$

which is called exponential coordinates of the second kind (cf. **[12]**).

We want to give an expression for $d\varphi^{-1}$. To this end, we shall need some notations.

We denote by $\overline{\operatorname{ad}} X_i$ and $\overline{K}(X_i)$ the linear transformations of $\mathfrak{q}$ defined by

$$\begin{aligned} &\overline{\operatorname{ad}}(X_i)X_j = 0, \quad \text{for } i \geq j \quad \text{and} \quad \overline{\operatorname{ad}}(X_i)X_j = \operatorname{ad}(X_i)X_j, \quad \text{for } i < j, \\ &\overline{K}(X_i)X_j = 0, \quad \text{for } i \geq j \quad \text{and} \quad \overline{K}(X_i)X_j = K(X_i)X_j, \quad \text{for } i < j. \end{aligned}$$

It follows from (1.1) and the fact that $S(X_i)X_j = 0$, $1 \leq i, j \leq k$, that

$$S(X_i)\overline{K}(X_j) = \overline{K}(X_j)S(X_i), \qquad 1 \leq i, j \leq k. \tag{2.1}$$

If $B(x) = b_n(x)\partial/\partial x_n + \cdots + b_1(x)\partial/\partial x_1$ is a vector field on $\mathbb{R}^n$, then we put $\operatorname{pr}_i B(x) = b_i(x)$. We also use the same notation for the left invariant vector fields on Q, i.e. if $E = c_n X_n + \cdots + c_1 X_1$, then we put $\operatorname{pr}_i E = c_i$.

Proposition 2.1. *With the above notations we have*

$$(2.2)\quad \begin{aligned} \mathrm{pr}_i\, d\varphi^{-1}E(x) &= \mathrm{pr}_i[e^{x_n \overline{\mathrm{ad}} X_n} \dots e^{x_1 \overline{\mathrm{ad}} X_1}](E) \\ &= \mathrm{pr}_i[e^{x_n \overline{K}(X_n)} \dots e^{x_1 \overline{K}(X_1)} e^{x_k S(X_k)} \dots e^{x_1 S(X_1)}](E) \\ &= \mathrm{pr}_i \left\{ \left[\sum_{\lambda_1\sigma(1)+\dots+\lambda_{i-1}\sigma(i-1)\le\sigma(i)-1} x_1^{\lambda_1}\cdots x_{i-1}^{\lambda_{i-1}} \right.\right. \\ &\qquad \left.\left. \cdot\, \overline{K}^{\lambda_{i-1}}(X_{i-1})\cdots\overline{K}^{\lambda_1}(X_1) \right] e^{x_k S(X_k)} \dots e^{x_1 S(X_1)} \right\} (E) \end{aligned}$$

Proof. Clearly, the third equality in (2.2) is a more explicit version of the second one and the second equality follows immediately from the first one using (2.1). So it is enough to prove the first equality in (2.2).

Let $g = \exp x_n X_n \cdots \exp x_1 X_1 \in Q$ and $\gamma(t) = g \exp tE$, $t > 0$ an integral curve of E. Then to prove the proposition it is enough to prove that

(2.3)
$$\begin{aligned} \gamma(t) = {} & \exp(x_n + t\,\mathrm{pr}_n\, e^{x_{n-1}\overline{\mathrm{ad}}X_{n-1}} \cdots e^{x_1 \overline{\mathrm{ad}} X_1} E + O(t^2))X_n \\ & \cdots \exp(x_2 + t\,\mathrm{pr}_2\, e^{x_1\overline{\mathrm{ad}}X_1} E + O(t^2))X_2 \exp(x_1 + t\,\mathrm{pr}_1\, E)X_1. \end{aligned}$$

(2.3) can be proved by induction on n: It is trivially true for $n = 1$. So assume that it is true for $n \le l$. To prove that it is also true for $n = l+1$, observe that it follows from the Campell-Hausdorff formula that

$$\begin{aligned} \exp tE &= \exp t(c_{l+1}X_{l+1} + \cdots + c_1 X_1) \\ &= \exp[(tc_{l+1} + O(t^2))X_{l+1} + \cdots + (tc_2 + O(t^2))X_2] \exp c_1 t X_1. \end{aligned}$$

Hence

$$(2.4)\quad \begin{aligned} \gamma(t) = {} & \exp x_{l+1}X_{l+1} \cdots \exp x_1 X_1 \\ & \cdot \exp[(tc_{l+1} + O(t^2))X_{l+1} + \cdots + (tc_2 + O(t^2))X_2] \\ & \cdot \exp -x_1 X_1 \exp x_1 X_1 \exp tc_1 X_1 \\ = {} & \exp x_{l+1} X_{l+1} \cdots \exp x_2 X_2 \\ & \cdot \exp e^{x_1 \overline{\mathrm{ad}} X_1}[(tc_{l+1} + O(t^2))X_{l+1} + \cdots + (tc_2 + O(t^2))X_2] \\ & \cdot \exp(x_1 + tc_1)X_1. \end{aligned}$$

Observing that the linear subspace of $\mathfrak{q}$ generated by the vectors $X_{l+1}, \dots, X_2$ is in fact an ideal of the Lie algebra $\mathfrak{q}$ we can see that it follows from (2.4) and the inductive hypothesis that (2.3) is also true for $n = l + 1$. This proves the inductive step and the proposition follows.

Let Q_N be a simply connected nilpotent Lie group that admits the nil-shadow $\mathfrak{q}_N$ of $\mathfrak{q}$ (cf. §1 for the definition) as its Lie algebra. Q_N is called the nil-shadow of Q.

We identify the elements of $\mathfrak{q}_N$ with the left invariant vector fields on Q_N and if $X \in \mathfrak{q}$ then we denote by ${}_NX$ the element of $\mathfrak{q}_N$ satisfying ${}_NX(e) = X(e)$. We extend the transformations $S(X)$, $X \in \mathfrak{q}$, to $\mathfrak{q}_N$ by putting $S(X)_NY = {}_N(S(X)Y)$.

Using again the exponential coordinates of the second kind

$$\varphi_N : \mathbb{R}^n \to Q_N, \quad \varphi : (x_n, \dots, x_1) \to \exp x_{nN}X_n \cdots \exp x_{1N}X_1$$

we can see that Q_N is diffeomorphic with $\mathbb{R}^n$.

From now on, using the exponential coordinates of the second kind φ and φ_N, we shall identify Q and Q_N as differential manifolds with $\mathbb{R}^n$.

It follows from (2.2) that if $x = (x_n, \dots, x_1) \in \mathbb{R}^n$ and $E \in \mathfrak{q}$ then

$$E(x) = (e^{x_k S(X_k)} \cdots e^{x_1 S(X_1)}{}_N E)(x). \tag{2.5}$$

3. The volume growth. Let Q be a simply connected solvable Lie group of polynomial growth and dg a left invariant Haar measure on Q.

We shall use the notations of §2. As it was explained in that section we identify Q and Q_N with $\mathbb{R}^n$.

Let $n_0, n_1, \dots, n_m$ as in §1 and $\sigma(1), \dots, \sigma(n)$ as in §2. We put

$$d = \sigma(1) + \cdots + \sigma(n).$$

Let $E_1, \dots, E_p$ as in Theorem 1, i.e. left invariant vector fields on Q that satisfy Hörmander's condition. The control distance $d_E(\cdot, \cdot)$ associated to these vector fields is defined as follows (cf. **[4]**, **[14]**):

We call an absolutely continuous path $\gamma : [0, 1] \to Q$ admissible if and only if $\dot\gamma(t) = a_1(t)E_1 + \cdots a_p(t)E_p$ for almost all $t \in [0, 1]$. It is a consequence of the Hörmander condition that all points $x, y \in Q$ can be joint with at least one admissible path. We put $|\dot\gamma(t)|^2 = a_1^2(t) + \cdots + a_p^2(t)$ and we define

$$d_E(x, y) = \inf\left\{ \int_0^1 |\dot\gamma(t)|\, dt, \gamma \text{ admissible path} \right.$$
$$\left. \text{such that } \gamma(0) = x, \gamma(1) = y \right\}.$$

We put $S_E(x, t) = \{y \in Q \colon d_E(x, y) < t\}$, $x \in Q$, $t > 0$.

We want to describe the shape of the balls $S_E(e, t)$, $t \geq 1$, and to estimate the dg-measure $(S_E(e, t))$. To this end we shall need some notations. If $x = (x_n, \ldots, x_1)$, then we put

$$x_t = (t^{\sigma(n)} x_n, \ldots, t^{\sigma(1)} x_1), \qquad t > 0.$$

$$D(x, t) = \{y = (y_n, \ldots, y_1) \in Q \colon \\ x_i - t^{\sigma(i)} < y_i < x_i + t^{\sigma(i)},\ 1 \leq i \leq n\}, \qquad t > 0.$$

We also put $D_t = D(e, t)$ and $D = D(e, 1)$.

Proposition 3.1. *Let $S_E(x, t)$ and D_t be as above. Then there is $c > 0$ such that*

$$S_E(e, c^{-1}t) \subseteq D_t \subseteq S_E(e, ct), \qquad t \geq 1,$$
$$c^{-1}t^d \leq dg\text{-}\mathrm{measure}(S_E(e, t)) \leq ct^d, \qquad t \geq 1.$$

Proof. As we see from (0.1), the balls $S_E(e, t)$, $t \geq 0$, behave, for large t, in the same way as the powers V^n, $n \in \mathbb{N}$, of a compact neighborhood V of e. Hence the vector fields $\{E_1, \ldots, E_p\}$ can be replaced with the basis $\{X_n, \ldots, X_1\}$ of $\mathfrak{q}$. Furthermore, it follows from (2.5), that $\{X_n, \ldots, X_1\}$ can be replaced by $\{{}_N X_n, \ldots, {}_N X_1\}$ and then the proposition becomes a well-known result (cf. **[5]**, **[6]**, **[15]**).

Arguing in the same way as in the above proposition, we can prove the following lemma which we shall need later on.

Lemma 3.2. *Let $S_E(x, t)$, $D(x, t)$ and D be as above. Then there is $A > 0$ and $\mu \in \mathbb{N}$ such that for all $x \in D$, $R \in (0, 1]$ and $t > t_0 = t_0(R)$, we have*

$$S_E(x_t, tR) \subseteq D(x_t, AtR^{1/\mu}), \quad D(x_t, tR) \subseteq S_E(x_t, AtR^{1/\mu}).$$

4. Generalization of some classical results of homogenization theory. Let Q be a simply connected solvable Lie group of polynomial growth and $E_1, \ldots, E_p$ and L as in Theorem 1, i.e. $E_1, \ldots, E_p$ are left invariant vector fields on Q that satisfy Hörmander's condition; let $L = -(E_1^2 + \cdots + E_p^2)$.

The purpose of this section is to generalize some classical results of the theory of homogenization (cf. **[2]**) in our context. In particular, we shall prove a homogenization formula for the operator L. The homogenized operator L_0 will be a left invariant sub-Laplacian defined on a limit group Q_H. Q_H is a homogeneous nilpotent Lie group and L_0 is invariant with respect to its dilation structure.

We fix a basis $\{X_n, \dots, X_1\}$ of $\mathfrak{q}$, as in Proposition 1.4. As it was explained in §2, we identify Q and Q_N with $\mathbb{R}^n$.

$n_0, n_1, \dots, n_m$ are as in §1, $\sigma(i)$, $i = 1, \dots, n$, as in §2 and $D(x, t)$, D_t, D as in §3.

To simplify the notations, we shall use the summation convention for repeated indices.

The dilation. We denote by τ_e, $\varepsilon > 0$, the dilation of $\mathbb{R}^n$, hence of Q and Q_N, defined by

$$\tau_\varepsilon\colon \mathbb{R}^n \to \mathbb{R}^n, \quad \tau_\varepsilon\colon (x_n, \dots, x_1) \to (\varepsilon^{\sigma(n)}x_n, \dots, \varepsilon^{\sigma(1)}x_1).$$

We put

$$\begin{aligned} E_{\varepsilon, i} &= \frac{1}{\varepsilon} d\tau_\varepsilon(E_i), \qquad i = 1, \dots, p \quad \text{and} \\ L_\varepsilon &= -(E_{\varepsilon,1}^2 + \cdots + E_{\varepsilon,p}^2), \qquad 0 < \varepsilon \le 1. \end{aligned}$$

The compactness. We recall the following Moser type Harnack inequality due to N. Th. Varopoulos [**13**]:

THEOREM 4.1 (*cf. N. Th. Varopoulos* [**13**]). *For all $a \in (0, 1)$ there is a constant $c > 0$ such that for all $t > 0$ and $u \ge 0$ such that $Lu = 0$ in $S_E(x, t)$ we have*

$$\sup_{y \in S_E(x, at)} u(y) \le c \inf_{y \in S_E(x, at)} u(y).$$

The above theorem provides a compactness on families of functions u_ε, satisfying

$$\|u_\varepsilon\|_\infty \le 1, \quad L_\varepsilon u_\varepsilon = 0 \text{ in } D, \qquad 0 < \varepsilon \le 1. \tag{4.1}$$

More precisely we have the following

PROPOSITION 4.2. *Let u_ε, $0 < \varepsilon \le 1$, be a family of functions satisfying* (4.1). *Then there is a subsequence, also denoted by u_ε, such that*

$$u_\varepsilon \to u_0 \qquad (\varepsilon \to 0)$$

uniformly on the compact subsets of D.

Proof. The first thing to observe is that if $L_\varepsilon u_\varepsilon = 0$ in D then the function $u(x) = u(\tau_\varepsilon(x))$ satisfies $Lu = 0$ in D_t, for $t = \varepsilon^{-1}$. Using this observation and Lemma 3.2, we can easily see that it follows from Theorem 4.1 that for every compact $U \subseteq D$ there are sequences

$$r_1 > r_2 > \cdots, r_i \to 0 \ (i \to \infty)$$
$$1 > \varepsilon_1 > \varepsilon_2 > \cdots, \varepsilon_i \to 0 \ (i \to \infty)$$

and a constant $c > 0$ such that

$$\sup_{y \in D(x, r_{i+1})} v_\varepsilon(y) \le c \inf_{y \in D(x, r_{i+1})} v_\varepsilon(y), \qquad x \in U \tag{4.2}$$

for all $v_\varepsilon \ge 0$ satisfying

$$L_\varepsilon v_\varepsilon = 0 \quad \text{in } D(x, r_i), \qquad \varepsilon \le \varepsilon_i.$$

Now, let r_i such that $D(x, r_i) \subseteq D$, $x \in U$, $\varepsilon < \varepsilon_i$, u_ε satisfying (4.1) and put

$$v_\varepsilon = 1 + u_\varepsilon,$$

$$M = \sup_{y \in D(x, r_{i+1})} v_\varepsilon(y), \quad M' = \sup_{y \in D(x, r_i)} v_\varepsilon(y),$$
$$m = \inf_{y \in D(x, r_{i+1})} v_\varepsilon(y), \quad m' = \inf_{y \in D(x, r_i)} v_\varepsilon(y).$$

Then it follows from (4.2) that

$$\begin{aligned} M' - m &= \sup_{y \in D(x, r_{i+1})} (M' - v_\varepsilon(y)) \\ &\le c \inf_{y \in D(x, r_{i+1}} (M' - v_\varepsilon(y)) = c(M' - M), \\ M - m' &= \sup_{y \in D(x, r_{i+1})} (v_\varepsilon(y) - m') \\ &\le c \inf_{y \in D(x, r_{i+1}} (v_\varepsilon(y) - m') = c(m - m') \end{aligned}$$

and from this that

$$M - m \le \frac{c-1}{c+1}(M' - m').$$

It follows from the above argument that for every compact $U \subseteq D$ and $\delta > 0$ there is $r = r(U, \delta) > 0$ and $\varepsilon_0 \in (0, 1)$ such that

$$|u_\varepsilon(y) - u_\varepsilon(z)| \le \delta, \qquad y, z \in D(x, r), x \in U,$$

for all u_ε satisfying (4.1), with $\varepsilon \le \varepsilon_0$ and the proposition follows by standard arguments.

The limit group Q_H. Let $[\cdot, \cdot]_N$ as in §1 and $\mathfrak{r}_1, \ldots, \mathfrak{r}_m$ and $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ as in Proposition 1.2. Making use of the direct sum decomposition

$$\mathfrak{q} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_m$$

we consider the projection $\mathrm{pr}_{\mathfrak{a}_i}$ of $\mathfrak{q}$ on $\mathfrak{a}_i$.

We denote by $[\cdot, \cdot]_H$ the unique product on the linear space $\mathfrak{q}$ satisfying for $X \in \mathfrak{a}_i$ and $Y \in \mathfrak{a}_j$

$$\begin{aligned} &[X, Y]_H = \mathrm{pr}_{\mathfrak{a}_{i+j}}[X, Y]_N, \quad \text{if } i+j \leq m \quad \text{and} \\ &[X, Y]_H = 0, \quad \text{if } i+j > m. \end{aligned}$$

It is easy to see that $[\cdot, \cdot]_H$ satisfies the Jacobi identity (observe that if $Z \in \mathfrak{a}_h$ and X, Y are as above then it follows from the way the spaces $\mathfrak{r}_i$, $\mathfrak{a}_i$, $i = 1, \ldots, m$, were defined that $[X, [Y, Z]_H]_H = \mathrm{pr}_{\mathfrak{a}_{i+j+h}}[X, [Y, Z]_N]_N)$. So, $\mathfrak{q}_H = (\mathfrak{q}, [\cdot, \cdot]_H)$ is a nilpotent Lie algebra which is also stratified.

The limit group Q_H is defined to be a simply connected Lie group that admits $\mathfrak{q}_H$ as its Lie algebra.

If $X \in \mathfrak{q}_H$ then we denote by ${}_HX(e)$ the left invariant vector field on Q_H satisfying ${}_HX(e) = X(e)$ (e is the identity element of Q_H). Using the exponential coordinates of the second kind

$$\varphi_H \colon \mathbb{R}^n \to Q_H, \quad \varphi \colon (x_n, \ldots, x_1) \to \exp x_{nH}X_n \cdots \exp x_{1H}X_1$$

we identify Q_H with $\mathbb{R}^n$.

Having done this identification, we should notice that the family of dilations τ_ε, $\varepsilon > 0$, introduced in the beginning of this section, is exactly the natural family of dilations which is compatible with the Lie group structure of Q_H (cf. **[5]**).

The coefficients of the operator L. Let us fix a vector field E_h, $1 \leq h \leq p$. Then from (2.2) and with the same notations we have that

$$E_h = (a_n^h + b_n^h)\frac{\partial}{\partial x_n} + \cdots + (a_1^h + b_1^h)\frac{\partial}{\partial x_1}$$

where

$$a_i^h(x) = \alpha_i^h(x, x), \quad b_i^h(x) = \beta_i^h(x, x),$$

$$(4.3)\quad \alpha_i^h(x, y) = \mathrm{pr}_i \left\{ \left[\sum_{\lambda_1\sigma(1)+\cdots+\lambda_{i-1}\sigma(i-1)=\sigma(i)-1} x_1^{\lambda_1}\cdots x_{i-1}^{\lambda_{i-1}} \cdot \overline{K}^{\lambda_{i-1}}(X_{i-1})\cdots\overline{K}^{\lambda_1}(X_1) \right] \cdot e^{y_k S(X_k)}\cdots e^{y_1 S(X_1)} \right\}(E_h)$$

and

(4.4)

$$\beta_i^h(x, y) = \mathrm{pr}_i \left\{ \left[\sum_{\lambda_1\sigma(1)+\cdots+\lambda_{i-1}\sigma(i-1)<\sigma(i)-1} x_1^{\lambda_1}\cdots x_{i-1}^{\lambda_{i-1}} \cdot \overline{K}^{\lambda_{i-1}}(X_{i-1})\cdots\overline{K}^{\lambda_1}(X_1) \right] \cdot e^{y_k S(X_k)}\cdots e^{y_1 S(X_1)} \right\}(E_h),$$

$$x = (x_n, \ldots, x_1), \; y = (y_n, \ldots, y_1), \qquad x, y \in \mathbb{R}^n, \; 1 \le i \le n.$$

We have the following proposition which is a direct consequence of the above definitions and the way the vectors $X_1, \ldots, X_n$ were chosen (cf. Propositions 1.3).

PROPOSITION 4.3. *The coefficients $\alpha_i^h(x, y)$ and $\beta_i^h(x, y)$ have the following properties*:

(1) *$\alpha_i^h(x, y) = \text{constant}$, for $1 \le i \le k$,*

(2) *if $k < i \le n_1$, then $\alpha_i^h(x, y) = \alpha_i^h(y)$ and it is periodic with respect to y,*

(3) *if $n_1 < i \le n$, then $\alpha_i^h(x, y)$ and $\beta_i^h(x, y)$ can be written as finite sums of terms of the form $p(x)\varphi(y)$, where $p(x) = cx_{i_1}\cdots x_{i_l}$, $c \in \mathbb{R}$, $1 \le i_j < i$, $1 \le j \le l$ and $\varphi(y) = \cos ay_j$ or $\sin ay_j$ for some $1 \le j \le k$, hence a periodic function and*

(4) *$\beta_i^h(x, y) = 0$, $1 \le i \le n_1$.*

Let $\overline{K}_H(X_i)$, $1 \le i \le n$, be the linear transformations of $\mathfrak{q}$ defined by

$$\overline{K}_H(X_i)X_j = 0, \quad j \le i, \quad \text{and} \quad \overline{K}_H(X_i)X_j = [X_i, X_j]_H, \quad i \le j.$$

Then (4.3) becomes

(4.5) $\alpha_i^h(x, y) = \mathrm{pr}_i[e^{x_{i-1}\overline{K}_H(X_{i-1})} \dots e^{x_1\overline{K}_H(X_1)} e^{y_k S(X_k)} \dots e^{y_1 S(X_1)}](E_h)$

and from this we have

$$(4.6) \qquad \alpha_i^h(x, y) = \sum_{1\le j\le n_1} \alpha_j^h(y)\,\mathrm{pr}_i[e^{x_{i-1}\overline{K}_H(X_{i-1})} \dots e^{x_1\overline{K}_H(X_1)}](X_j).$$

Let us put, for $1 \le i, j \le n$

$$\alpha_{ij}(x, y) = \sum_{1\le h\le p} \alpha_i^h(x, y)\alpha_j^h(x, y),$$

$$\beta_{ij}(x, y) = \sum_{1\le h\le p} [\alpha_i^h(x, y)\beta_j^h(x, y) + \beta_i^h(x, y)\beta_j^h(x, y) + \beta_i^h(x, y)\alpha_j^h(x, y)]$$

$$a_{ij}(x) = \alpha_{ij}(x, x), \quad b_{ij}(x) = \beta_{ij}(x, x).$$

Then we have (we use the summation convention for repeated indices)

$$L = A + B, \quad \text{where } A = -\frac{\partial}{\partial x_i} a_{ij}(x)\frac{\partial}{\partial x_j} \text{ and } B = -\frac{\partial}{\partial x_i} b_{ij}(x)\frac{\partial}{\partial x_j}.$$

In the following proposition we have gathered some properties of the coefficients $\alpha_{ij}(x, y)$ and $\beta_{ij}(x, y)$ which are immediate consequences of the definitions.

PROPOSITION 4.4. (1) *The coefficients* $\alpha_{ij}(x, y)$ *and* $\beta_{ij}(x, y)$ *are finite sums of terms of the form* $p(x)\varphi(y)$, *where* $p(x) = cx_{i_1}\cdots x_{i_l}$, $c \in \mathbb{R}$, $1 \le i_h < \max(i, j)$, $1 \le h \le l$, *and* $\varphi(y) = \cos ay_j$ *or* $\sin ay_j$ *for some* $1 \le j \le k$, *hence a periodic function.*

(2) $\alpha_{ij}(x, y) = \alpha_{ij}(y)$, $1 \le i, j \le n_1$.

(3) $\alpha_{ij}(x, y) = $ *constant*, $1 \le i, j \le k$.

(4) $\beta_{ij}(x, y) = 0$, $-1 \le i, j \le n_1$.

The correctors. We put

$$A(x) = -\frac{\partial}{\partial y_i}\alpha_{ij}(x, y)\frac{\partial}{\partial y_j}.$$

If $f(x, y)$ is a finite sum of functions periodic with respect to the variable y then we denote by $\mathfrak{M}(f)(x)$ the mean of f, defined by

$$\mathfrak{M}(f)(x) = \lim_{t\to\infty} \frac{1}{|D_t|}\int_{D_t} f(x, y)\,dy$$

where $|D_t|$ denotes the volume of D_t.

The correctors $\chi^j(x, y)$, $1 \leq j \leq n$, are defined to be C^∞ functions satisfying

$$(4.7) \qquad A(x)\chi^j(x, y) = -\frac{\partial}{\partial y_i}\alpha_{ij}(x, y), \quad \mathfrak{M}(\chi^j) = 0.$$

They are defined as follows:
For $1 \leq j \leq n_1$ they are defined to be the unique solutions of the problem

$$A(x)\chi^j(x, y) = -\frac{\partial}{\partial y_i}\alpha_{ij}(x, y), \quad \mathfrak{M}(\chi^j) = 0.$$

Notice that, in view of Proposition 4.4,

$$\sum_{1\leq i\leq n} \frac{\partial}{\partial y_i}\alpha_{ij}(x, y) = \sum_{1\leq i\leq k} \frac{\partial}{\partial y_i}\alpha_{ij}(y_k, \dots, y_1), \qquad 1 \leq j \leq n_1,$$

which is a periodic function with mean zero and therefore the correctors χ^j, $1 \leq j \leq n_1$, are well defined.

For $n_1 < j \leq n$ the correctors χ^j are defined by

$$\chi^j(x, y) = \sum_{1\leq l\leq n_1} \chi^l(y)\, \mathrm{pr}_j[e^{x_{j-1}\overline{K}_H(X_{j-1})} \dots e^{x_1\overline{K}_H(X_1)}](X_l).$$

An immediate consequence of the definition is the following

PROPOSITION 4.5. (1) $A(x)(\chi^j(x, y) - y_j) = 0$, $1 \leq j \leq n$.
(2) $\chi^j(x, y) = \chi^j(x, (y_k, \dots, y_1))$, $1 \leq j \leq n$.
(3) $\chi^j = 0$, $1 \leq j \leq k$.
(4) *If* $k < j \leq n_1$, *then* $\chi^j(x, y) = \chi^j(y)$ *and is periodic with respect to* y.

The homogenized operator L_0. We put

$$q_{ij}(x) = \mathfrak{M}\left\{\alpha_{ij}(x, y) - \alpha_{il}(x, y)\frac{\partial}{\partial y_l}\chi^j(x, y)\right\}.$$

The homogenized operator L_0 is defined by

$$L_0 = -\frac{\partial}{\partial x_i}q_{ij}(x)\frac{\partial}{\partial x_j}.$$

PROPOSITION 4.6. (1) $q_{ij}(x) = q_{ji}(x)$, $1 \leq i, j \leq n$.
(2) $q_{ij}(x) = constant$, $1 \leq i, j \leq n_1$.

(3)

$$q_{ij}(x) = \sum_{1\le l,\mu\le n_1} \{\mathrm{pr}_i[e^{x_{i-1}\overline{K}_H(X_{i-1})}\cdots e^{x_1\overline{K}_H(X_1)}](X_l)\}q_{l\mu} \cdot \{\mathrm{pr}_j[e^{x_{j-1}\overline{K}_H(X_{j-1})}\cdots e^{x_1\overline{K}_H(X_1)}](X_\mu)\}, \quad 1\le i, j\le n.$$

Proof. (2) and (3) follow from the definitions and Propositions 4.4 and 4.5. To prove (1) let us observe that

$$q_{ij}(x) = \mathfrak{M}\left\{\left(\frac{\partial}{\partial y_h}y_i\right)\alpha_{hl}(x, y)\frac{\partial}{\partial y_l}[y_j - \chi^j(x, y)]\right\}$$

and that from the definition of the correctors χ^j, $1\le j\le n$, we have that

$$\mathfrak{M}\left\{\left[\frac{\partial}{\partial y_h}\chi^i(x, y)\right]\alpha_{hl}(x, y)\frac{\partial}{\partial y_l}[y_j - \chi^j(x, y)]\right\} = 0.$$

Hence

$$q_{ij}(x) = \mathfrak{M}\left\{\frac{\partial}{\partial y_h}[y_i - \chi^i(x, y)]\alpha_{hl}(x, y)\frac{\partial}{\partial y_l}[y_j - \chi^j(x, y)]\right\} \tag{4.8}$$

and the proposition follows.

LEMMA 4.7. *The operator*

$$L_0' = -\sum_{1\le i, j\le n_1}\frac{\partial}{\partial x_i}q_{ij}(x)\frac{\partial}{\partial x_j}$$

is an elliptic operator with constant coefficients in $\mathbb{R}^{n_1}$.

Proof. Let $\xi = (\xi_1, \dots, \xi_{n_1}) \in \mathbb{R}^{n_1}$, $\xi \ne 0$, and (cf. Proposition 4.5)

$$f(y) = \xi_1[y_1 - \chi^1(y)] + \cdots + \xi_{n_1}[y_{n_1} - \chi^{n_1}(y)].$$

Then, from (4.7) we have that

$$\sum_{1\le i, j\le n_1} q_{ij}\xi_i\xi_j = \mathfrak{M}\left\{\left[\frac{\partial}{\partial y_h}f(y)\right]\alpha_{hl}(y)\frac{\partial}{\partial y_l}f(y)\right\}$$

and from Proposition 4.4 that

$$\mathfrak{M}\left\{\left[\frac{\partial}{\partial y_l}f(y)\right]\alpha_{l\mu}(y)\frac{\partial}{\partial y_\mu}f(y)\right\} = \mathfrak{M}\{(E_1f)^2 + \cdots + (E_pf)^2\}.$$

So to prove the lemma it is enough to prove that

$$\mathfrak{M}\{(E_1f)^2 + \cdots + (E_pf)^2\} \ne 0.$$

To do this, since the function $(E_1 f)^2 + \cdots + (E_p f)^2$ is a finite sum of C^∞ periodic functions, it is enough to prove that there is an open set $U \subseteq \mathbb{R}^n$ and $1 \leq i \leq p$ such that $E_i f(y) \neq 0$, $y \in U$. This follows from the observation that if $E_i f(y) = 0$, $\forall y \in \mathbb{R}^n$ then, since the vector fields $E_1, \dots, E_p$ satisfy Hörmander's condition, we would have that $f(y) = c$, $\forall y \in \mathbb{R}^n$ and hence that

$$\xi_1 y_1 + \cdots + \xi_{n_1} y_{n_1} = \xi_1 \chi^1(y) + \cdots + \xi_{n_1} \chi^{n_1}(y) + c$$

which is absurd since the second member of the above equality is a sum of periodic functions.

It follows from the above proposition that there are linearly independent vector fields $Y_1, \dots, Y_{n_1}$ in $\mathbb{R}^{n_1}$, with constant coefficients, such that $L_0' = -(Y_1^2 + \cdots + Y_{n_1}^2)$. Let us denote by $W_1, \dots, W_{n_1}$, respectively the images of $Y_1, \dots, Y_{n_1}$ under the linear isomorphism of $\mathbb{R}^{n_1}$ with $\mathfrak{a}_1$ that maps $\partial/\partial x_i \to X_i$, $1 \leq i \leq n_1$, and denote by ${}_H W_1, \dots, {}_H W_{n_1}$ the left invariant vector fields on the limit group Q_H satisfying ${}_H W_i(e) = W_i$, $i = 1, \dots, n_1$. Since Q_H (as well as Q) has been identified as differential manifold with $\mathbb{R}^n$, ${}_H W_1, \dots, {}_H W_{n-1}$ can also be viewed as vector fields on $\mathbb{R}^n$ (as well as on Q). Then it follows from Proposition 4.6(3) that the limit operator L_0 satisfies

$$L_0 = -\frac{\partial}{\partial x_i} q_{ij}(x) \frac{\partial}{\partial x_j} = -(W_1^2 + \cdots + W_{n_1}^2),$$

i.e. L_0 is a left invariant sub-Laplacian on Q_H, which is also invariant with respect to the natural dilation structure of Q_H (cf. [**5**]).

The homogenization formula. Now we can state the following

Proposition 4.8. *Let u_0 be as in Proposition 4.2 and L_0 as above. Then*

$$L_0 u_0 = 0 \quad \text{in } D.$$

The proof of the above proposition is exactly the same with the proof of the homogenization formula in the classical case of uniformly elliptic second order differential operators with periodic coefficients (cf. [**2**]).

The only modification is that, since in our case we deal with hypoelliptic and not uniformly elliptic operators we have to replace D with a neighborhood U of 0 which is very regular, in the sense of Bony [**4**], i.e. it is such that

(i) $U = B_1 \cap B_2$, where B_1 and B_2 are two Euclidean balls of $\mathbb{R}^n$ and

(ii) if $x \in \partial U$, hence $x \in B_i$ for some $i \in \{1, 2\}$, $v = (v_n, \ldots, v_1)$ is the vertical unit vector to the ball B_i at the point x and the operators L_ε, $0 < \varepsilon \leq 1$, are written in divergence form as $L_\varepsilon = -(\partial/\partial x_i)a^\varepsilon_{ij}(\partial/\partial x_j)$ then

$$\sum_{1 \leq i, j \leq n} a^\varepsilon_{ij}(x)v_i v_j > 0.$$

Observe that since D can be scaled down to a subset of U, we can indeed replace it by U.

To see that not only 0 but every $y = (y_n, \ldots, y_1) \in \mathbb{O}$ has such a very regular neighborhood U let us observe that $a^\varepsilon_{iij} = \text{const.}$, $1 \leq i$, $j \leq k$. Hence, if $\xi \neq 0$, $\xi = (\xi_n, \ldots, \xi_1)$, $\xi_{k+1} = \cdots = \xi_n = 0$, then

$$\sum_{1 \leq i, j \leq n} a^\varepsilon_{ij}\xi_i\xi_j > 0, \qquad 0 < \varepsilon \leq 1.$$

So the intersection $U = B_1 \cap B_2$ of the balls B_1 and B_2 of radius $M + \delta$, centered at the points $y + M\xi$ and $y - M\xi$ respectively, for M large and δ small enough is a very regular neighborhood of y.

Apart from this modification the energy proof of the homogenization formula (cf. [2]) can be carried through without any change at all.

5. The proof of Theorem 1. The proof of Theorem 1 will be based on a rescaling argument of M. Avellaneda and F. H. Lin [1] that we shall adapt in our context.

We shall use the notations of §4.

LEMMA 5.1. *For all $\mu \in (0, 1)$ there are $\theta \in (0, 1)$, $\varepsilon_0 \in (0, 1)$ and $c > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ and all functions u_ε satisfying*

$$L_\varepsilon u_\varepsilon = 0 \quad in\ D, \qquad \|u_\varepsilon\|_\infty \leq 1$$

we have that

$$\sup_{x \in D_\theta} |u_\varepsilon(x) - A^\varepsilon_0 - \sum_{1 \leq j \leq n_1} A^\varepsilon_j(x_j - \varepsilon\chi^j(\tau_{\varepsilon^{-1}}x))| < \theta^{1+\mu} \tag{5.1}$$

where, A^ε_j, $0 \leq j \leq n_1$, are constants satisfying $|A^\varepsilon_j| < c$, $0 \leq j \leq n_1$.

Proof. First we observe that there is $\mu' > \mu$, $\theta \in (0, 1)$ and $c > 0$ such that for all u satisfying

$$L_0 u = 0 \quad \text{in } D, \qquad \|u\|_\infty \leq 1$$

we have that

$$\sup_{x\in D_\theta}\left|(u(x)-A_0^0-\sum_{1\le j\le n_1} x_j\right|<\theta^{1+\mu'} \tag{5.2}$$

where A_j^0, $0\le j\le n_1$, are constants satisfying $|A_j^0|<c$, $0\le j\le n_1$. This follows from the fact that the homogenized operator L_0 is hypoelliptic (cf. [**4**]).

Let us fix these values of θ and c. If (5.1) weren't true then there would be a sequence of functions u_{ε_m}, $\varepsilon_m\to 0$ $(m\to\infty)$ not satisfying (5.1). We can assume, by extracting a subsequence if necessary, that $u_{\varepsilon_m}\to u_0$ $(m\to\infty)$ uniformly on the compact subsets of D, and then u would satisfy (5.2).

Let us take $A_j^{\varepsilon_m}=A_j^0$, $0\le j\le n_1$. Then using the assumption that the functions u_{ε_m} do not satisfy (5.1) and passing to the limit we have that

$$\theta^{1+\mu}<\sup_{x\in D_\theta}\left|u(x)-A_0^0-\sum_{1\le j\le n_1}A_j^0x_j\right|<\theta^{1+\mu'}$$

which is absurd. Hence the lemma.

Lemma 5.2. *Let θ, μ and ε_0 be as in Lemma 5.1. Then there is a constant $c>0$ such that for all $m\in\mathbb{N}$ and $\varepsilon\in(-1,1)$ such that $\varepsilon\le\theta^{m-1}\varepsilon_0$ and all u_ε satisfying*

$$L_\varepsilon u_\varepsilon=0\quad in\ D,\qquad \|u_\varepsilon\|_\infty\le 1$$

we have that

$$\sup_{x\in D_{\theta^m}}\left|u_\varepsilon x-A_0^{\varepsilon,m}-\sum_{1\le j\le n_1}A_j^{\varepsilon,m}(x_j-\varepsilon\chi^j(\tau_{\varepsilon^{-1}}x))\right|<\theta^{m(1+\mu)} \tag{5.3}$$

where $A_j^{\varepsilon,m}$, $0\le j\le n_1$, are constants satisfying $|A_j^{\varepsilon,m}|<c$, $0\le j\le n_1$.

Proof. The lemma will be proved by induction. For $m=1$ we are in the case of Lemma 5.1. So assume that (5.3) is true for some $m\in\mathbb{N}$. We put

$$w_\varepsilon(x)=\theta^{m(1+\mu)}\left[u_\varepsilon(\tau_{\theta^m}x)-A_0^{\varepsilon,m}\right.$$
$$\left.-\sum_{1\le j\le n_1}A_j^{\varepsilon,m}(\theta^m x_j-\varepsilon\chi^j(\tau_{\varepsilon^{-1}\theta^m}x))\right]. \tag{5.4}$$

Then we have that

$$L_{\varepsilon^{\theta-m}} w_\varepsilon = 0 \quad \text{in } D, \qquad \|w_\varepsilon\|_\infty \le 1$$

Therefore it follows from Lemma 5.1 that, for $\varepsilon\theta^{-m} \le \varepsilon_0$ we have that

$$\text{(5.5)} \quad \sup_{x\in D_\theta} \left| w_\varepsilon(x) - B_0^\varepsilon - \sum_{1\le j\le n_1} B_j^\varepsilon(\theta^m x_j - \varepsilon\theta^{-m}\chi^j(\tau_{\varepsilon^{-1}\theta^m} x)) \right| < \theta^{1+\mu}$$

with $|B_j^\varepsilon| < c$, $0 \le j \le n_1$ (the constant c being as in Lemma 5.1).

Let us put

$$\begin{aligned} A_0^{\varepsilon, m+1} &= A_0^{\varepsilon, m} + \theta^{m(1+\mu)} B_0^\varepsilon, \\ A_j^{\varepsilon, m+1} &= A_j^{\varepsilon, m} + \theta^{m\mu} B_j^\varepsilon, \qquad 1 \le j \le n_1. \end{aligned}$$

Then putting (5.4) and (5.5) together we have that

$$\sup_{x\in D_\theta} \theta^{-m(1+\mu)} \left| u_\varepsilon(\tau_{\theta^m} x) - A_0^{\varepsilon, m+1} \right.$$
$$\left. - \sum_{1\le j\le n_1} A_j^{\varepsilon, m+1}(\theta^m x_j - \varepsilon\chi^j(\tau_{\varepsilon^{-1}\theta^m} x)) \right| < \theta^{1+\mu}$$

and from this

$$\sup_{x\in D_{\theta^{m+1}}} \left| u_\varepsilon(x) - A_0^{\varepsilon, m+1} \right.$$
$$\left. - \sum_{1\le j\le n_1} A_j^{\varepsilon, m+1}(x_j - \varepsilon\chi^j(\tau_{\varepsilon^{-1}} x)) \right| < \theta^{(m+1)(1+\mu)}$$

which proves the inductive step and the lemma follows.

COROLLARY 5.3. *Let ε_0 be as in Lemma 5.2. Then there is $c > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ and all u_ε satisfying*

$$L_\varepsilon u_\varepsilon = 0 \quad \text{in } D, \qquad \|u_\varepsilon\|_\infty \le 1$$

we have that

$$\text{(5.6)} \qquad \sup_{x\in D_{\varepsilon/\varepsilon_0}} |u_\varepsilon(x) - A_0^\varepsilon| < c\frac{\varepsilon}{\varepsilon_0}$$

where A_0^ε is a constant such that $A_0^\varepsilon < c$.

COROLLARY 5.4. *There is a constant $c > 0$ such that for all u satisfying*

$$Lu = 0 \quad in\ D_t, \qquad t \geq 1$$

we have that

$$\sup_{x\in D} |u(x) - A_0| < \frac{c}{t}\|u\|_\infty \tag{5.7}$$

where A_0 is a constant such that $|A_0| < c\|u\|_\infty$.

Proof. The corollary follows from Corollary 5.3 and the observation that if u satisfies

$$Lu = 0 \quad \text{in } D_t, \qquad t \geq 1$$

then the function u_ε defined by $u_\varepsilon(x) = u(\tau_t x)$, $\varepsilon = 1/t$ satisfies

$$L_\varepsilon u_\varepsilon = 0 \quad \text{in } D.$$

Proof of Theorem 1. It is enough to prove Theorem 1 when Q is simply connected. In that case it is an immediate consequence of Corollary 5.4 and Theorem 4.1.

REFERENCES

[1] M. Avellaneda and F. H. Lin, *Compactness methods in the theory of homogenization*, Comm. Pure Appl. Math., **40** (1987), 803–847.

[2] A. Bensoussan, J. L. Lions and G. Papanicolaou, *Asymptotic Analysis of Periodic Structures*, North-Holland, Amsterdam, 1978.

[3] A. S. Besicovitch, *Almost Periodic Functions*, Dover Publications, 1954.

[4] J. M. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs élliptiques dégénérés*, Ann. Inst. Fourier, **19** (1), (1969), 277–304.

[5] G. B. Folland and E. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton University Press, 1982.

[6] Y. Guivarc'h, *Croissance polynômiale et périodes de fonctions harmoniques*, Bull. Sci. Math. France, **101** (1973), 149–152.

[7] L. Hörmander, *Hypoelliptic second order operators*, Acta Math., **119** (1967), 147–171.

[8] N. Jacobson, *Lie Algebras*, Wiley, 1962.

[9] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math., **17** (1964), 577–591.

[10] E. Stein, *Singular integrals and differentiability properties of functions*, Ann. of Math. Studies, 1970.

[11] ——, *Topics in Harmonic Analysis*, Princeton University Press, 1970.

[12] V. S. Varadarajan, *Lie Groups, Lie Algebras and Their Representations*, Springer-Verlag, 1984.

[13] N. Th. Varopoulos, *Fonctions harmoniques positives sur les groupes de Lie*, C.R. Acad. Sci. Paris, **309**, serie I, (1987), 519–521.
[14] ——, *Analysis on Lie groups*, J. Funct. Anal., **76** no. 2, (1988), 346–410.
[15] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon, *Harmonic Analysis on Lie Groups*, Cambridge University Press, to appear.

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