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MÖBIUS-INVARIANT HILBERT SPACES IN POLYDISCS

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We define the Dirichlet space $\mathscr{D}$ on the unit polydisc $\mathbb{U}^n$ of $\mathbb{C}^n$. $\mathscr{D}$ is a semi-Hilbert space of of holomorphic functions, contains the holomorphic polynomials densely, is invariant under compositions with the biholomorphic automorphisms of $\mathbb{U}^n$, and its semi-norm is preserved under such compositions. We show that $\mathscr{D}$ is unique with these properties. We also prove $\mathscr{D}$ is unique if we assume that the semi-norm of a function in $\mathscr{D}$ composed with an automorphism is only equivalent in the metric sense to the semi-norm of the original function. Members of a subclass of $\mathscr{D}$ given by a norm can be written as potentials of $\mathscr{L}^2$-functions on the n-torus $\mathbb{T}^n$. We prove that the functions in this subclass satisfy strong-type inequalities and have tangential limits almost everywhere on $\partial\mathbb{U}^n$. We also make capacitory estimates on the size of the exceptional sets on $\partial\mathbb{U}^n$.

1. Introduction. *Möbius-invariant spaces.* Let $\mathbb{U}$ be the open *unit disc* in $\mathbb{C}$ and $\mathbb{T}$ be the *unit circle* bounding it. The open *unit polydisc* $\mathbb{U}^n$ and the *torus* $\mathbb{T}^n$ in $\mathbb{C}^n$ are the cartesian products of n unit discs and n unit circles, respectively. $\mathbb{T}^n$ is the distinguished boundary of $\mathbb{U}^n$ and forms only a small part of the topological boundary $\partial\mathbb{U}^n$ of $\mathbb{U}^n$. We denote by $\mathscr{M}$ the group of all *biholomorphic automorphisms* of $\mathbb{U}^n$ (the *Möbius group*). The subgroup of *linear* automorphisms in $\mathscr{M}$ is denoted by $\mathscr{U}$. The space of holomorphic functions with domain $\mathbb{U}^n$ will be called $\mathscr{H}(\mathbb{U}^n)$ and will carry the topology of *uniform convergence on compact subsets* of $\mathbb{U}^n$.

A *semi-inner product* on a complex vector space $\mathscr{H}$ is a sesquilinear functional on $\mathscr{H}\times\mathscr{H}$ with all the properties of an inner product except that it is possible to have $\langle\langle a, a\rangle\rangle = 0$ when $a \neq 0$. $\|a\| = \sqrt{\langle\langle a, a\rangle\rangle}$ is the associated *semi-norm*. We assume $\langle\langle \cdot\,, \cdot\rangle\rangle$ is not identically zero.

DEFINITION 1.1. $\mathscr{H}$ is called a *Hilbert space* of holomorphic functions on $\mathbb{U}^n$ if

(i) $\mathscr{H}$ is a linear subspace of $\mathscr{H}(\mathbb{U}^n)$,
(ii) the semi-inner product $\langle\langle \cdot\,, \cdot\rangle\rangle$ of $\mathscr{H}$ is complete,
(iii) $\mathscr{H}$ contains all (holomorphic) polynomials,
(iv) polynomials are dense in $\mathscr{H}$ in the topology of the semi-norm $\|\cdot\|$ of $\mathscr{H}$.

A space $\mathscr{H}$ of functions on $\mathbb{U}^n$ is *$\mathscr{M}$-invariant* if $f \circ \Psi \in \mathscr{H}$ whenever $f \in \mathscr{H}$ and $\Psi \in \mathscr{M}$. An $\mathscr{M}$-invariant Hilbert space $\mathscr{H}$ of holomorphic functions on $\mathbb{U}^n$ will be called an *$\mathscr{M}$-space* for brevity. *$\mathscr{U}$-invariance* and *$\mathscr{U}$-space* have similar definitions.

$\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of nonnegative integers, positive integers, integers, and real numbers, respectively. A *multi-index* $\alpha = (\alpha_1, \dots, \alpha_n)$ is a point in $\mathbb{N}^n$. $\sum_\alpha$ indicates a summation with α running over all the points in $\mathbb{N}^n$, and $\sum'_{\alpha\in\mathbb{I}}$ is a summation where we consider only those α in the index set $\mathbb{I}$ with all positive components. Let also $D_j = \partial/\partial z_j$ and $\bar{D}_j = \partial/\partial \bar{z}_j$. The following abbreviated notations will be used:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$
$$\alpha! = \alpha_1! \cdots \alpha_n!, \qquad D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}.$$

The *Dirichlet space* $\mathscr{D}(\mathbb{U}^n)$ is the class of $f(z) = \sum_\alpha f_\alpha z^\alpha \in \mathscr{H}(\mathbb{U}^n)$ with

$$\|f\|_{\mathscr{D}}^2 = \sum_\alpha \alpha_1 \cdots \alpha_n \, |f_\alpha|^2 = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \alpha_1 \cdots \alpha_n \, |f_\alpha|^2 < \infty. \tag{1.1}$$

Equivalently, $\mathscr{D}(\mathbb{U}^n)$ is the class of those $f \in \mathscr{H}(\mathbb{U}^n)$ with

$$\|f\|_{\mathscr{D}}^2 = \int_{\mathbb{U}^n} |D_1 \cdots D_n f|^2 \, d\mu_n < \infty,$$

where μ_n is the Lebesgue maesure on $\mathbb{U}^n$ normalized so that $\mu_n(\mathbb{U}^n) = 1$. The semi-norm $\|\cdot\|_{\mathscr{D}}$ is obtained from the semi-inner product

$$\langle\langle f, g \rangle\rangle_{\mathscr{D}} = \sum_\alpha \alpha_1 \cdots \alpha_n \, f_\alpha \bar{g}_\alpha = \int_{\mathbb{U}^n} (D_1 \cdots D_n f) \, \overline{(D_1 \cdots D_n g)} \, d\mu_n.$$

Main results. In this work, we first prove *two* theorems which show that the Dirichlet space is unique among $\mathscr{M}$-spaces that have certain properties.

THEOREM A. *Let $\mathscr{H}$ be an $\mathscr{M}$-space and suppose that*

$$\|f\| = \|f \circ \Psi\| \qquad (f \in \mathscr{H}, \ \Psi \in \mathscr{M}). \tag{1.2}$$

Then

$$\|f\| = C \, \|f\|_{\mathscr{D}} \qquad (f \in \mathscr{H}),$$

where $C = \|z_1 \cdots z_n\|^2$. Thus $\mathscr{H}$ is $\mathscr{D}(\mathbb{U}^n)$.

Note that the assumption on the semi-norm is equivalent to $\langle\langle f \circ \Psi, g \circ \Psi \rangle\rangle = \langle\langle f, g \rangle\rangle$, and the conclusion implies that $\langle\langle f, g \rangle\rangle = C\langle\langle f, g \rangle\rangle_{\mathscr{D}}$, for all $f, g \in \mathscr{H}$ and $\Psi \in \mathscr{M}$.

For the second theorem, we need a strengthening of condition (ii) of Definition 1.1. To derive it, we write the Taylor series expansion at 0 of an $f \in \mathscr{H}$ by seperating the higher and lower dimensional terms. For example, when $n = 2$, using (z, w) for (z_1, z_2) and (k, l) for (α_1, α_2), we write

$$f(z, w) = f_{00} + \sum_{k=1}^{\infty} f_{k0} z^k + \sum_{l=1}^{\infty} f_{0l} w^l + \sum_{k, l=1}^{\infty} f_{kl} z^k w^l.$$

Since we assume conditions (iii) and (iv) of Definition 1.1, we can define a *norm* on $\mathscr{H}$ by

$$|||f|||^2 = |f_{00}|^2 + \sum_{k=1}^{\infty} |f_{k0}|^2 \|z^k\|_1^2 + \sum_{l=1}^{\infty} |f_{0l}|^2 \|w^l\|_1^2 + \sum_{k, l=1}^{\infty} |f_{kl}|^2 \|z^k w^l\|^2,$$

where $\|\cdot\|$ is still the semi-norm of $\mathscr{H} \subset \mathscr{H}(\mathbb{U}^n)$ and $\|\cdot\|_1$ denotes the semi-norm of a similar $\mathscr{H}' \subset \mathscr{H}(\mathbb{U})$. We already know (see **[1]**) that $\|\cdot\|_1$ is equivalent to $\|\cdot\|_{\mathscr{D}}$ in $\mathbb{U}$. Since our proof of Theorem B is by induction on the dimension of the polydisc, the above definition of $|||\cdot|||$ makes sense. Now we make the alternate assumption

(ii)′ $\mathscr{H}$ is complete in the norm $|||\cdot|||$.

A similar condition was assumed in **[1]**, whereas **[4]** assumes (ii).

THEOREM B. *Let $\mathscr{H}$ be an $\mathscr{M}$-space in the sense modified by* (ii)′ *and assume that there is a positive constant $\delta < 1$ such that*

$$\delta\|f\| \le \|f \circ \Psi\| \le \frac{1}{\delta}\|f\| \qquad (f \in \mathscr{H}, \ \Psi \in \mathscr{M}). \tag{1.3}$$

Then there exists positive constants K_1 and K_2 such that

$$K_1\|f\|_{\mathscr{D}} \le \|f\| \le K_2\|f\|_{\mathscr{D}} \qquad (f \in \mathscr{H}).$$

Thus $\mathscr{D}(\mathbb{U}^n)$ is unique again.

The proofs of these theorems will be presented in §2.

Next, we consider a subspace of the Dirichlet space, one that is defined by a genuine norm similar to $|||\cdot|||$. This space is not $\mathscr{M}$-invariant any more, but the stronger conditions on it allow us to prove that it has *tangential* limits as we approach $\partial\mathbb{U}^n$. In fact, tangential limits exist for a wider class of functions which are potentials of certain functions in $\mathscr{L}^2(\mathbb{T}^n)$. The precise definitions and theorems are stated in §3. Theorems C and D at the end of that section are the major results in this direction.

In earlier work, Arazy and Fisher **[1]** proved, under slightly different hypotheses, the analogs of Theorem A and Theorem B in $\mathbb{U}$. Zhu **[5]** found the equivalent of Theorem A for the unit ball in $\mathbb{C}^n$ when $n \geq 2$. Nagel, Rudin and Shapiro [3] obtained the unit-disc versions of Theorems C and D.

After the submission of the manuscript, we were informed by a referee that in the preprint *Invariant Hilbert Spaces of Analytic Functions on Bounded Symmetric Domains* by J. Arazy and S. D. Fisher, results analogous to Theorems A and B were established for all irreducible bounded symmetric domains.

NOTATION. λ_n is the Lebesgue measure on $\mathbb{T}^n$ both normalized to have mass 1; i.e., it is the Haar measure on the compact abelian group $\mathbb{T}^n$. If $p \in [1, \infty)$, its *conjugate* is $q = p/(p-1)$. The $\mathscr{L}^p$- and ℓ^p-spaces will have their usual meaning. z_j will usually be an element of $\mathbb{U}$ and ζ_j of $\mathbb{T}$. Apart from the usual big $\mathscr{O}$ notation, we will use $u \sim v$ to mean both $u = \mathscr{O}(v)$ and $v = \mathscr{O}(u)$, and $u \approx v$ to mean u/v has a finite positive limit.

The *Poisson integral* of an $f \in \mathscr{L}^1(\mathbb{T}^n)$ is

$$P[f](z) = \int_{\mathbb{T}^n} f(\zeta) \prod_{j=1}^{n} \frac{1-|z_j|^2}{|1-z_j\bar{\zeta}_j|^2}\, d\lambda_n(\zeta) \qquad (z \in \mathbb{U}^n),$$

and its *Cauchy integral* is

$$C[f](z) = \int_{\mathbb{T}^n} f(\zeta) \prod_{j=1}^{n} \frac{1}{1-z_j\bar{\zeta}_j}\, d\lambda_n(\zeta) \qquad (z \in \mathbb{U}^n),$$

where the products are called the *Poisson kernel* $P(z, \zeta)$ and the *Cauchy kernel* $C(z, \zeta)$ for $\mathbb{U}^n$, respectively. These transforms have the following invariance properties: If $f \in \mathscr{L}^1(\lambda_n)$, $\Psi \in \mathscr{M}$, and $U \in \mathscr{U}$, then

$$P[f \circ \Psi] = P[f] \circ \Psi \qquad \text{and} \qquad C[f \circ U] = C[f] \circ U.$$

The automorphisms of $\mathbb{U}^n$ for $n \geq 2$ are generated by the following three subgroups: *rotations* in each variable separately

$$R_\theta(z) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n),$$

Möbius transformations in each variable separately

$$\Phi_w(z) = (\phi_{w_1}(z_1), \dots, \phi_{w_n}(z_n)),$$

and the *coordinate permutations*. Here $\theta \in [-\pi, \pi]^n$ and $w \in \mathbb{U}^n$ are fixed, Möbius transformations are in the form

$$\phi_w(z) = \frac{w - z}{1 - \bar{w} z} \qquad (w \in \mathbb{U},\ z \in \overline{\mathbb{U}}), \tag{1.4}$$

and the coordinate permutations are nothing but the $n!$ members of the symmetric group $\mathscr{S}_n$ on n objects. Thus an arbitrary $\Psi \in \mathscr{M}$ can be written in the form

$$\Psi(z) = \left(e^{i\theta_1}\phi_{w_1}(z_{\sigma(1)}), \dots, e^{i\theta_n}\phi_{w_n}(z_{\sigma(n)})\right),$$

for some $w \in \mathbb{U}^n$ and $\theta \in [-\pi, \pi]^n$, and $\sigma \in \mathscr{S}_n$. $\mathscr{U}$ is generated by $\sigma \in \mathscr{S}_n$ and the rotations R_θ. Each Möbius transformation Φ_w is an *involution* (its inverse is itself) exchanging 0 and w. $\mathscr{M}$ acts *transitively* on $\mathbb{U}^n$: if $a, b \in \mathbb{U}^n$, then $\Phi_b \circ \Phi_a \in \mathscr{M}$ moves a to b (and b to a). Finally, $\mathscr{M}^*$ denotes the component of the identity in $\mathscr{M}$; i.e., $\mathscr{M}^*$ is $\mathscr{M}$ without the action of $\mathscr{S}_n$.

2. Uniqueness of the Dirichlet space. We start by showing that $\mathscr{D}(\mathbb{U}^n)$ has all the properties of a Hilbert space in the sense of Definition 1.1. Clearly the polynomials are in $\mathscr{D}(\mathbb{U}^n)$ and $\|z^\alpha\|^2 = \alpha_1 \cdots \alpha_n$ for all $\alpha \in \mathbb{N}^n$. A quick look at (1.1) shows that the polynomials are dense in $\mathscr{D}(\mathbb{U}^n)$ with respect to $\|\cdot\|_{\mathscr{D}}$. Again from (1.1), identifying g by $\{g_\alpha\}$, we see that $\mathscr{D}(\mathbb{U}^n)$ is a weighted ℓ^2-space, hence every Cauchy sequence $\{f_j\}$ in $\mathscr{D}(\mathbb{U}^n)$ converges in $\|\cdot\|_{\mathscr{D}}$ to some $f \in \mathscr{D}(\mathbb{U}^n)$ represented by $\{f_\alpha\}$ for $\alpha \in \mathbb{Z}_+^n$. To show that f is holomorphic, let $f_m(z) = \sum_{k=1}^m \sum'_{|\alpha|=k} f_\alpha z^\alpha$ and pick $\varepsilon > 0$. For any $0 < r < 1$ and positive integers $m > l > n$,

$$\begin{aligned}
\sup_{z \in r\overline{\mathbb{U}}^n} |(f_m - f_l)(z)| &= \sup_{z \in r\overline{\mathbb{U}}^n} \left| \sum_{k=l+1}^m {\sum_{|\alpha|=k}}' f_\alpha z^\alpha \right| \le \sum_{k=l+1}^m {\sum_{|\alpha|=k}}' |f_\alpha| r^{|\alpha|} \\
&\le \left(\sum_{k=l+1}^m {\sum_{|\alpha|=k}}' |f_\alpha|^2 \right)^{1/2} \left(\sum_{k=l+1}^m {\sum_{|\alpha|=k}}' r^{2|\alpha|} \right)^{1/2} \\
&< \left(\sum_{k=l+1}^m \sum_{|\alpha|=k} \alpha_1 \cdots \alpha_n |f_\alpha|^2 \right)^{1/2} \left(\sum_{k=l+1}^m k^n r^{2k} \right)^{1/2}.
\end{aligned}$$

The first factor is less than ε when l and m are large enough because $f \in \mathscr{D}(\mathbb{U}^n)$, and the second factor is bounded as $l, m \to \infty$. Hence

$f(z) = \sum'_\alpha f_\alpha z^\alpha$ is uniformly convergent on compact subsets of $\mathbb{U}^n$, and this proves $f \in \mathscr{H}(\mathbb{U}^n)$. Note that we need not know f_α if $\alpha \in \mathbb{N}^n \backslash \mathbb{Z}_+^n$. These coefficients of f can be taken arbitrarily as long as f remains holomorphic.

LEMMA 2.1. *$\mathscr{M}$ is generated by $\mathscr{S}_n$, rotations $R_\omega(z) = (e^{i\omega} z_1, z_2, \ldots, z_n)$ with $\omega \in [-\pi, \pi]$, and Möbius transformations of the form $\Phi_t(z) = (\phi_t(z_1), z_2, \ldots, z_n)$ with $0 \le t < 1$.*

PROPOSITION 2.2. *$\mathscr{D}(\mathbb{U}^n)$ is $\mathscr{M}$-invariant.*

Proof. The integral form of the Dirichlet semi-norm uses the measure μ_n which is invariant under rotations and permutations. Thus $\mathscr{D}(\mathbb{U}^n)$ is $\mathscr{U}$-invariant. To prove invariance under Möbius transformations, in view of Lemma 2.1, it suffices to consider

$$w = \Phi_r(z) = \left(\frac{r - z_1}{1 - r z_1}, z_2, \ldots, z_n \right).$$

Then $D_1^z(f \circ \Phi_r) = (D_1^w f) dw_1/dz_1$ and

$$\begin{aligned} |D_1^z D_2 \cdots D_n(f \circ \Phi_r)|^2 &= |D_1^w D_2 \cdots D_n f|^2 \frac{(r^2-1)^2}{|1 - r z_1|^4} \\ &= |D_1^w D_2 \cdots D_n f|^2 J_{\Re}\Phi_r(z), \end{aligned}$$

since $dw_1/dz_1 = (r^2 - 1)/(1 - r z_1)^2$, where $J_{\Re}\Phi_r$ is the real Jacobian of Φ_r. Therefore

$$\begin{aligned} \|f \circ \Phi_r\|_{\mathscr{D}}^2 &= \int_{\mathbb{U}^n} \left| D_1^z D_2 \cdots D_n(f \circ \Phi_r)(z) \right|^2 d\mu_n(z) \\ &= \int_{\mathbb{U}^n} \left| D_1^w D_2 \cdots D_n f(w) \right|^2 \frac{(r^2-1)^2}{|1 - r z_1|^4} \frac{1}{J_{\Re}\Phi_r(z)} d\mu_n(w) \\ &= \int_{\mathbb{U}^n} \left| D_1^w D_2 \cdots D_n f(w) \right| d\mu_n(w) = \|f\|_{\mathscr{D}}^2. \end{aligned}$$ □

Note that when $n \ge 2$, $\mathscr{D}(\mathbb{U}^n)$ does not put any conditions on the infinitely many power series coefficients of f, those with at least one $\alpha_j = 0$, i.e., those in $\mathbb{N}^n \backslash \mathbb{Z}_+^n$. Thus if each term in the Taylor expansion of some $f \in \mathscr{H}(\mathbb{U}^n)$ depends on fewer than n variables, then $\|f\|_{\mathscr{D}} = 0$ and $f \in \mathscr{D}$. The Dirichlet space can also be thought of as a quotient space of holomorphic functions satisfying (1.1) where the functions whose Taylor series differ by terms depending on at most $n-1$ variables are identified. Trivially any holomorphic function f of fewer than n variables or any constant f has $\|f\|_{\mathscr{D}} = 0$ and is in $\mathscr{D}(\mathbb{U}^n)$. For comparison, when $n = 1$, only constants (a one-dimensional subspace) have zero Dirichlet semi-norm.

We can define a *modified* Dirichlet space $\tilde{\mathscr{D}}(\mathbb{U}^n)$ similar to $\mathscr{D}(\mathbb{U}^n)$ by considering a norm instead of a semi-norm. This requires some control on *all* the power series coefficients of $f \in \mathscr{H}(\mathbb{U}^n)$. For simplicity let's look at the case $n = 2$. With notation as before, let

$$\begin{aligned}\|f\|_{\tilde{\mathscr{D}}}^2 &= |f_{00}|^2 + \sum_{k=1}^{\infty} k|f_{k0}|^2 + \sum_{l=1}^{\infty} l|f_{0l}|^2 + \sum_{k,l=1}^{\infty} kl|f_{kl}|^2 \\ &= \left|f(0,0)\right|^2 + \int_{\mathbb{U}} \left|D_1 f(z,0)\right|^2 d\mu_1(z) \\ &\quad + \int_{\mathbb{U}} \left|D_2 f(0,w)\right|^2 d\mu_1(w) \\ &\quad + \int_{\mathbb{U}^2} \left|D_1 D_2 f(z,w)\right|^2 d\mu_2(z,w).\end{aligned}$$

This norm is $\mathscr{U}$-invariant, but not $\mathscr{M}$-invariant; in fact, none of its first three terms is preserved under compositions with Φ_r.

Proof of Theorem A. First, $\langle\langle z^\alpha, z^\beta \rangle\rangle = 0$ if $\alpha \neq \beta$. To see this, assume, without loss of generality, $\alpha_1 \neq \beta_1$. Let ω be an irrational multiple of π and consider the rotation $R_\omega(z) = (e^{i\omega} z_1, z_2, \dots, z_n)$. By the $\mathscr{M}$-invariance of $\langle\langle \cdot, \cdot \rangle\rangle$,

$$\begin{aligned}\langle\langle z^\alpha, z^\beta \rangle\rangle &= \langle\langle z^\alpha \circ R_\omega, z^\beta \circ R_\omega \rangle\rangle \\ &= \langle\langle e^{i\alpha_1\omega} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}, e^{i\beta_1\omega} z_1^{\beta_1} z_2^{\beta_2} \cdots z_n^{\beta_n} \rangle\rangle \\ &= e^{i(\alpha_1 - \beta_1)\omega} \langle\langle z^\alpha, z^\beta \rangle\rangle,\end{aligned}$$

and the desired orthogonality result follows.

Put $C_\alpha = \langle\langle z^\alpha, z^\alpha \rangle\rangle$. Note that C_α is defined only for $\alpha \in \mathbb{N}^n$. If β is another multi-index and $\beta = \sigma(\alpha)$ for some $\sigma \in \mathscr{S}_n$, then by the $\mathscr{M}$-invariance of $\langle\langle \cdot, \cdot \rangle\rangle$ again, $C_\alpha = C_\beta$.

Now let $0 < r_j < 1$, $\Psi = (\phi_{r_1}, \dots, \phi_{r_n})$, and consider $f(z) = \prod_{j=1}^n (1 - r_j z_j) \in \mathscr{H}$. Then since z^α is orthogonal to z^β for $\alpha \neq \beta$, we get

$$\langle\!\langle f, f\rangle\!\rangle = \left\|\prod_{j=1}^{n}(1-r_j z_j)\right\|^2 = \left\|1-\sum_{j=1}^{n} r_j z_j + \cdots + (-1)^m \prod_{j=1}^{n} r_j z_j\right\|^2$$

$$= C_{(0,\dots,0)} + \left(\sum_{j=1}^{n} r_j^2\right) C_{(1,0,\dots,0)}$$

$$+ \left(\sum_{l>j=1}^{n} r_j^2 r_l^2\right) C_{(1,1,0,\dots,0)} + \cdots + \left(\prod_{j=1}^{n} r_j^2\right) C_{(1,\dots,1)}.$$

On the other hand,

$$(f\circ\Psi)(z) = \prod_{j=1}^{n}\left(1 - r_j \frac{r_j - z_j}{1-r_j z_j}\right) = \prod_{j=1}^{n} \frac{1-r_j^2}{1-r_j z_j}$$

$$= \prod_{j=1}^{n}\left[(1-r_j^2)\sum_{\alpha_j=0}^{\infty} r_j^{\alpha_j} z_j^{\alpha_j}\right] = \left(\prod_{j=1}^{n}(1-r_j^2)\right)\left(\sum_{\alpha} r^{\alpha} z^{\alpha}\right);$$

now the density of the polynomials and the axioms of a Hilbert space imply

$$\langle\!\langle f\circ\Psi, f\circ\Psi\rangle\!\rangle = \left(\prod_{j=1}^{n}(1-r_j^2)^2\right)\sum_{\alpha} r^{2\alpha} C_{\alpha}.$$

Putting $x_j = r_j^2$ and using the $\mathcal{M}$-invariance of the semi-norm gives

$$\text{(2.1)}\quad C_{(0,\dots,0)} + \left(\sum_{j=1}^{n} x_j\right) C_{(1,0,\dots,0)} + \left(\sum_{l>j=1}^{n} x_j x_l\right) C_{(1,1,0,\dots,0)}$$

$$+ \cdots + \left(\prod_{j=1}^{n} x_j\right) C_{(1,\dots,1)} = \left(\prod_{j=1}^{n}(1-x_j)^2\right)\sum_{\alpha} x^{\alpha} C_{\alpha}.$$

The constant terms $(C_{(0,\dots,0)})$ cancel, and if we set the coefficients of x_1, x_1x_2, $\dots$, and $x_1x_2\cdots x_n$ on either side equal to each other, we obtain, respectively,

$$C_{(1,0,\dots,0)} = C_{(1,0,\dots,0)} - 2C_{(0,\dots,0)},$$

$$C_{(1,1,0,\dots,0)} = C_{(1,1,0,\dots,0)} - 2^2 C_{(1,0,\dots,0)},$$

$$\vdots$$

$$C_{(1,\dots,1)} = C_{(1,\dots,1)} - 2^n C_{(1,\dots,1,0)}.$$

These imply $C_{(0,\dots,0)} = 0$, $C_{(1,0,\dots,0)} = 0$, $\dots$, $C_{(1,\dots,1,0)} = 0$. After the elimination of the terms that are zero, (2.1) simplifies to

$$\begin{aligned}\sum_{\alpha}{}' x^\alpha C_\alpha &= \frac{x_1 \cdots x_n C_{(1,\dots,1)}}{(1-x_1)^2 \cdots (1-x_n)^2} \\ &= C_{(1,\dots,1)} \left(\sum_{\alpha_1=1}^{\infty} \alpha_1 x_1^{\alpha_1} \right) \cdots \left(\sum_{\alpha_n=1}^{\infty} \alpha_n x_n^{\alpha_n} \right);\end{aligned}$$

and this implies

$$C_\alpha = \alpha_1 \cdots \alpha_n C_{(1,\dots,1)} \qquad (\alpha \in \mathbb{Z}_+^n).$$

Thus the norm of a monomial of fewer than n variables is zero. Since the polynomials are dense in $\mathscr{H}$, the same result is true for any $f \in \mathscr{H}$ whose Taylor expansion consists of monomials depending on fewer than n variables. But $C_{(1,\dots,1)} \neq 0$, because otherwise, since the polynomials are dense in $\mathscr{H}$, $\langle\langle \cdot , \cdot \rangle\rangle$ would be identically zero contrary to hypothesis. Then renaming $C_{(1,\dots,1)} = C$ completes the proof. □

Proof of Theorem B. We will only show how the two-variable case is obtained from the one-variable case. This then can be adapted to prove by induction the case for arbitrary $\mathbb{U}^n$. Unless explicitly stated, subscripted C's will denote positive constants that are independent of any parameters.

Step 1. We begin by introducing two other semi-inner products on $\mathscr{H}$. For $f, g \in \mathscr{H}$, let

$$[f, g] = \int_{\mathbb{T}^2} \langle\langle f \circ R_\theta , g \circ R_\theta \rangle\rangle \, d\lambda_2(\theta)$$

and

$$\langle f, g \rangle = m \langle\langle f \circ \Phi, g \circ \Phi \rangle\rangle,$$

where m is an *invariant mean* (see [2]) on the abelian subgroup

$$\mathscr{N} = \{ \Phi = (\phi_s, \phi_t) : 0 \le s, t < 1 \}$$

of $\mathscr{M}$. To actually make $\mathscr{N}$ abelian, in this proof we change our definition of a Möbius transformation so that $\phi_w(z)$ is the negative of what is given in (1.4). The earlier definition was adopted to make ϕ_w an involution, since it simplified calculations involving ϕ_w^{-1}. The required boundedness condition for the existence of this nonunique mean is furnished by (1.3). Rotations and $\mathscr{N}$, along with $\mathscr{S}_2$, suffice to generate $\mathscr{M}$, by Lemma 2.1.

Now $[\cdot\,,\,\cdot]$ is rotation-invariant

$$[f\,,\,g] = [f \circ R_\theta\,,\, g \circ R_\theta] \qquad (\theta \in [-\pi\,,\,\pi]^n)\,, \tag{2.2}$$

and $\langle\cdot\,,\,\cdot\rangle$ is $\mathscr{N}$-invariant

$$\langle f\,,\,g\rangle = \langle f \circ \Phi\,,\, g \circ \Phi\rangle \qquad (\Phi \in \mathscr{N}). \tag{2.3}$$

Moreover, (1.3) implies

$$\delta^2\|f\|^2 \le [f\,,\,f] \le \frac{1}{\delta^2}\|f\|^2 \qquad (f \in \mathscr{H})\,, \tag{2.4}$$

and

$$\delta^2\|f\|^2 \le \langle f\,,\,f\rangle \le \frac{1}{\delta^2}\|f\|^2 \qquad (f \in \mathscr{H})\,; \tag{2.5}$$

and combining these two, we further obtain

$$\delta^4[f\,,\,f] \le \langle f\,,\,f\rangle \le \frac{1}{\delta^4}[f\,,\,f] \qquad (f \in \mathscr{H}). \tag{2.6}$$

(2.4) and (2.5) show that the semi-norms associated to $[\cdot\,,\,\cdot]$ and $\langle\cdot\,,\,\cdot\rangle$ are both equivalent to $\|\cdot\|$.

As in the proof of Theorem A, the rotation-invariance of $[\cdot\,,\,\cdot]$ gives the orthogonality condition

$$[z^{k_1}w^{l_1}\,,\, z^{k_2}w^{l_2}] = 0 \qquad ((k_1\,,\,l_1) \ne (k_2\,,\,l_2))\,, \tag{2.7}$$

which leads to

$$[f\,,\,f] = \sum_{k\,,\,l=0}^{\infty} |f_{kl}|^2\,[z^k w^l\,,\, z^k w^l] \qquad (f \in \mathscr{H}).$$

Therefore, to prove the theorem, it suffices to show that

$$K_1\,kl \le [z^k w^l\,,\, z^k w^l] \le K_2\,kl \qquad ((k\,,\,l) \in \mathbb{N}^2)$$

or, equivalently,

$$K_3\,kl \le \langle z^k w^l\,,\, z^k w^l\rangle \le K_4\,kl \qquad ((k\,,\,l) \in \mathbb{N}^2)$$

for some positive constants K_1, K_2, K_3, and K_4. Clearly $K_2 \ge K_1$ and $K_4 \ge K_3$.

Step 2. Claim:

$$\langle z^k w^l\,,\, z^k w^l\rangle \ne 0 \qquad \text{if } k \ge 1 \text{ and } l \ge 1. \tag{2.8}$$

Suppose it is zero for some (N, M); then $[z^N w^M, z^N w^M] = 0$ also. Then for $0 \leq s, t < 1$, if we use (2.3), (2.6) and power series expansion

$$\begin{aligned} 0 &= \langle z^N w^M, z^N w^M \rangle \\ &= \left\langle \left(\frac{z-s}{1-sz}\right)^N \left(\frac{w-t}{1-wt}\right)^M, \left(\frac{z-s}{1-sz}\right)^N \left(\frac{w-t}{1-wt}\right)^M \right\rangle \\ &\geq \delta^8 \left[\left(\frac{z-s}{1-sz}\right)^N \left(\frac{w-t}{1-wt}\right)^M, \left(\frac{z-s}{1-sz}\right)^N \left(\frac{w-t}{1-wt}\right)^M \right] \\ &= \left[\sum_{k=0}^{\infty} c_{kN}(s) z^k \sum_{l=0}^{\infty} c'_{lM}(t) w^l, \sum_{k=0}^{\infty} c_{kN}(s) z^k \sum_{l=0}^{\infty} c'_{lM}(t) w^l \right] \\ &= \sum_{k,l=0}^{\infty} |c_{kN}(s)|^2 |c'_{lM}(t)|^2 [z^k w^l, z^k w^l]. \end{aligned}$$

A tedious computation shows that the coefficients $c_{kN}(s) \neq 0$ for any k, N, and s as given above; the same is obviously true for $c'_{lM}(t)$. Thus $[z^k w^l, z^k w^l] = 0$. This means that every element in $\mathscr{H}$ has zero norm and contradicts our basic assumption that $\langle\!\langle \cdot , \cdot \rangle\!\rangle$ is not identically zero. Hence the claim is proved.

The one-variable result can be stated as

$$C_1 k \leq \langle z^k, z^k \rangle_1 \leq C_2 k \qquad (k \in \mathbb{N}). \tag{2.9}$$

It is a consequence of condition (ii)′ of Definition 1.1 and of (2.8) that the subspace of $\mathscr{H}$ consisting of functions whose Taylor series expansion at 0 depend only on z is closed. Then (2.9) implies that, for *fixed* $M \in \mathbb{N}$,

$$C_3 k \leq \langle z^k w^M, z^k w^M \rangle \leq C_4 k \qquad (k \in \mathbb{N}), \tag{2.10}$$

and we have a similar equation when the power of z is held constant. Of course, the constants C_3 and C_4 are different for different M. It is our aim to find their explicit dependence on M. If we had only finitely many M, we could pick C_3 and C_4 independently of M and the proof would be over. In the sequel, whenever we have only finitely many n or M, we will use this fact without further reference.

Step 3 (*upper bound*). Let $M \in \mathbb{N}$ be *fixed* and $k, j \in \mathbb{N}$. Put

$$\alpha_{kM}^{jM} = \langle z^k w^M, z^j w^M \rangle, \qquad \beta_{kM} = \langle z^k w^M, z^k w^M \rangle,$$
$$b_{kM} = [z^k w^M, z^k w^M].$$

By (2.3), $\alpha_{kM}^{jM} = \langle \phi_s^k w^M, \phi_s^j w^M \rangle$ for any $s \in [0, 1)$. Differentiate both sides of this equality with respect to s and set $s = 0$. Then take $k = j+1$ and add the resulting expressions from $j = 0$ to $j = N \geq 1$; and finally divide both sides by $N+1$. The result is

$$\frac{\beta_{N+1,M}}{N+1} = \frac{2S_{NM}}{N(N+1)} - \frac{\alpha_{N+2,M}^{NM}}{N}, \tag{2.11}$$

where $S_{NM} = \frac{1}{2}\beta_{0M} + \sum_{k=1}^{N} \beta_{kM}$. In particular, $\alpha_{N+2,M}^{NM}$ is *real*. Now using (2.3) and (2.5), and letting $s^2 = \frac{N}{N+1}$, we obtain

$$\begin{aligned}\frac{1}{\delta^4}\beta_{1M} &= \frac{1}{\delta^4}\langle \phi_s w^M, \phi_s w^M \rangle \geq [\phi_s w^M, \phi_s w^M] \\ &= s^2 b_{0M} + (1-s^2)^2 \sum_{k=0}^{\infty} s^{2k} b_{k+1,M} \\ &\geq \frac{\delta^4}{(N+1)^2} \sum_{k=0}^{N} \left(\frac{N}{N+1}\right)^k \beta_{k+1,M} \geq \frac{\delta^4}{(N+1)^2 e} \sum_{k=0}^{N} \beta_{k+1,M},\end{aligned}$$

which implies

$$S_{NM} \leq \left(\frac{e\beta_{1M}}{\delta^8} + \frac{\beta_{0M}}{2}\right) N^2.$$

Using (2.10) twice on the right side gives

$$S_{NM} \leq C_5 M N^2. \tag{2.12}$$

It is this inequality and its pair (2.15) below that allow us to pass from one variable to several variables.

As a special case, when $M = 0$, we get $S_{N0} = 0$ for all $N \geq 1$. Symmetric nature of the calculation shows also $S_{0N} = 0$ for all $N \geq 1$. It follows that

$$\beta_{N0} = \langle z^N, z^N \rangle = 0 \quad \text{and} \quad \beta_{0N} = \langle w^N, w^N \rangle = 0 \qquad (N \in \mathbb{Z}_+). \tag{2.13}$$

The inequality

$$\delta^8(\beta_{NM} + \beta_{N+2,M}) \leq \langle (z^N + z^{N+2}) w^M, (z^N + z^{N+2}) w^M \rangle$$

is a direct consequence of (2.6) and (2.7). Using this, after some routine calculation, (2.11) can be written as

$$\begin{aligned}2\frac{\beta_{N+1,M}}{N+1} &\leq \frac{4S_{NM}}{N(N+1)} + \frac{2\beta_{N+2,M}}{N(N+2)} + (1-\delta^8)\left(\frac{\beta_{NM}}{N} + \frac{\beta_{N+2,M}}{N+2}\right) \\ &\leq 6C_5 M + (1-\delta^8)\left(\frac{\beta_{NM}}{N} + \frac{\beta_{N+2,M}}{N+2}\right),\end{aligned}$$

which is equivalent to $2\gamma_{N+1,M} \le (1-\delta^8)(\gamma_{NM} + \gamma_{N+2,M})$ if we let

$$\gamma_{NM} = \frac{\beta_{NM}}{N} - \frac{3C_5 M}{\delta^8}.$$

A result in [1] shows $\gamma_{N+1,M} \le |\gamma_{NM}|$ for positive N. Then using (2.8), we get

$$\frac{\beta_{NM}}{N} - \frac{3C_5 M}{\delta^8} \le \left|\frac{\beta_{1M}}{1} - \frac{3C_5 M}{\delta^8}\right| \le C_6 M.$$

Multiplying both sides by M, we conclude

$$\beta_{NM} \le \left(C_6 + \frac{3C_5}{\delta^8}\right) NM = K_4 NM,$$

which holds for $N \ge 1$ and $M \ge 0$, and for $M \ge 1$ and $N \ge 0$, by the symmetry of the computation.

Step 4 (*lower bound*). If we combine the result of Step 1 with (2.6), we also get

$$b_{kM} \le \frac{K_4}{\delta^4} kM. \tag{2.14}$$

We use (2.6), (2.7), (2.3), (2.14), and take $s^2 = \frac{N}{N+1}$ to calculate

$$\begin{aligned}\delta^8 s^2 b_{1M} &\le \delta^8 [(1+sz)w^M, (1+sz)w^M] \\ &= \delta^4 \langle (1+s\phi_s)w^M, (1+s\phi_s)w^M \rangle \\ &\le (1-s^2)^2 \left[\frac{w^M}{1-sz}, \frac{w^M}{1-sz}\right] = \left(\frac{1}{N+1}\right)^2 \sum_{k=0}^{\infty} s^{2k} b_{kM} \\ &\le \frac{1}{(N+1)^2}\left(\sum_{k=0}^{mN} b_{kM} + \frac{K_2}{\delta^4} M \sum_{k=mN+1}^{\infty} k\left(\frac{N}{N+1}\right)^k\right),\end{aligned}$$

where m will be determined shortly. After approximating the second sum by an integral, we have

$$\delta^8 \left(\frac{N}{N+1}\right) b_{1M} \le \frac{1}{(N+1)^2} \sum_{k=0}^{mN} b_{kM} + \frac{K_2}{\delta^4} M (m+1) e^{-m}.$$

Because of (2.10), M in the last term can be replaced with $C_8 b_{1M}$. Now choose m so large that $K_2 \delta^{-4}(m+1)e^{-m} \le \delta^8/3$. Using (2.10) once again and some simplification yields

$$\sum_{k=0}^{N} b_{kM} \ge C_9 M N^2, \tag{2.15}$$

which is the reverse inequality for (2.12). Combining (2.15) with (2.11), we obtain

$$\delta^4 C_9 M \le \frac{\beta_{N+1,M}}{N+1} + \left(\frac{\beta_{NM}}{N}\right)^{1/2}\left(\frac{\beta_{N+2,M}}{N+2}\right)^{1/2}. \tag{2.16}$$

Now let

$$\phi_s^N(z) = \left(\frac{z-s}{1-sz}\right)^N = \sum_{k=0}^{\infty} c_{kN}(s) z^k$$

and consider

$$b_{NM} \ge \delta^4 \beta_{NM} = \delta^4 \langle \phi_s^N w^M, \phi_s^N w^M \rangle \ge \delta^8 [\phi_s^N w^M, \phi_s^N w^M]$$
$$= \delta^8 \sum_{k=1}^{\infty} |c_{kN}(s)|^2 b_{kM} \ge \delta^8 |c_{N+1,N}(s)|^2 b_{N+1,M}.$$

A calculation in [1] shows that $|c_{N+1,N}(s)|^2 \ge 1/2$ for all $s \in [0, 1)$ and $N \ge 1$. Thus there is a constant C_{10} such that

$$\beta_{N+1,M} \le C_{10} \beta_{NM}. \tag{2.17}$$

Now (2.16) and (2.17) together imply

$$\beta_{NM} \ge \left(\frac{\delta^4 C_9}{4C_{10}}\right) NM = K_3 NM,$$

for $N \ge 1$ and $M \ge 0$, and for $M \ge 1$ and $N \ge 0$.

Step 5. The only term we have not yet accounted for is $\beta_{00} = \langle 1, 1 \rangle$. Since it represents a one-dimensional subspace of $\mathscr{H}$, we now know $\mathscr{H} = \mathscr{D}(\mathbb{U}^2)$. To complete the proof, we will also show $\beta_{00} = b_{00} = 0$. To obtain a contradiction, suppose $b_{00} = [1, 1] \ne 0$. Let $f \in \mathscr{H}$, Φ be a Möbius transformation, and denote the power series coefficients of f and $f \circ \Phi$ by f_{kl} and f'_{kl}, respectively. Because of (2.13)

$$\|f\|^2 \ge \delta^2 [f, f] = \delta^2 \left(b_{00} |f(0, 0)|^2 + \sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl} \right)$$

and

$$[f \circ \Phi, f \circ \Phi] = b_{00} |f(\Phi(0, 0))|^2 + \sum_{k,l=1}^{\infty} |f'_{kl}|^2 b_{kl}.$$

We have

$$\sum_{k,l=1}^{\infty} |f'_{kl}|^2 b_{kl} \ge K_1 \sum_{k,l=1}^{\infty} |f_{kl}|^2 kl = K_1 \|f \circ \Phi\|^2_{\mathscr{D}} = K_1 \|f\|^2_{\mathscr{D}}$$
$$= K_1 \sum_{k,l=1}^{\infty} |f_{kl}|^2 kl \ge \frac{K_1}{K_2} \sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl}.$$

Hence

$$b_{00}|f(0,0)|^2 + \sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl} = [f, f] \geq \delta^8[f\circ\Phi, f\circ\Phi]$$
$$= \delta^8\left(b_{00}\,|f(\Phi(0,0))|^2 + \sum_{k,l=1}^{\infty} |f'_{kl}|^2 b_{kl}\right)$$
$$= \delta^8\left(b_{00}\,|f(\Phi(0,0))|^2 + \frac{K_1}{K_2}\sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl}\right),$$

from which we obtain

$$b_{00}|f(0,0)|^2 + \left(1-\delta^8\frac{K_1}{K_2}\right)\sum_{k,l=1}^{\infty} |f_{kl}|^2 b_{kl} \geq \delta^8 b_{00}\,|f(\Phi(0,0))|^2.$$

The left hand side of this equation is finite since it is equivalent to $\|f\|^2$. Since $\Phi(0,0)$ can be any point in $\mathbb{U}^2$, it follows that every element of $\mathscr{H}$, *i.e.*, of $\mathscr{D}(\mathbb{U}^2)$, is bounded. But a Dirichlet space contains unbounded elements. In $\mathbb{U}$, this is seen most easily by the Area Theorem; in $\mathbb{U}^2$, we take an unbounded function depending only on one variable. Therefore $b_{00} = 0$ and we are done. □

COROLLARY 2.3. *Theorem* B *is true even if* (1.3) *holds only for* $\Psi \in \mathscr{M}^*$. *Theorem* A *is true even if* (1.2) *holds only for* $\Psi \in \mathscr{M}^*$.

Proof. The proof of Theorem B uses coordinate permutations nowhere. Theorem A is a consequence of Theorem B. □

3. Boundary behavior. *Dirichlet-type spaces.* This section requires some new notions that were studied in $\mathbb{U}$ in [3] and [4]. For each $\delta = (\delta_1, \ldots, \delta_n)$ with each $0 \leq \delta_j \leq 1$, we define the *Dirichlet-type spaces* $\mathscr{D}_\delta(\mathbb{U}^n)$ to consist of those $f(z) = \sum_\alpha f_\alpha z^\alpha \in \mathscr{H}(\mathbb{U}^n)$ that satisfy

$$\|f\|_{\mathscr{D}_\delta} = \sum_\alpha \alpha_1^{2\delta_1}\cdots\alpha_n^{2\delta_n}|f_\alpha|^2 < \infty.$$

This definition makes sense even if some $\delta_j = 0$ if we interpret $0^0 = 1$. In fact, if all the $\delta_j = 0$, then $\mathscr{D}_\delta(\mathbb{U}^n) = \mathscr{H}^2(\mathbb{U}^n)$. The space corresponding to $\delta_1 = \cdots = \delta_n = 1$ consists of functions f with $D_1\cdots D_n f \in \mathscr{H}^2(\mathbb{U}^n)$. When $\delta_1 = \cdots = \delta_n = 1/2$, we have the Dirichlet space. For $n = 1$, all Dirichlet-type spaces are contained in $\mathscr{H}^2(\mathbb{U})$, but this is not true if $n > 1$.

Some subclasses of $\mathcal{D}_\delta(\mathbb{U}^n)$ have certain integral representations: If $F \in \mathcal{L}^2(\mathbb{T}^n)$, for $0 \le \delta_j < 1$ and $z \in \mathbb{U}^n$, set

$$f(z) = \int_{\mathbb{T}^n} F(\zeta) \prod_{j=1}^{n} \frac{1}{(1 - z_j\bar{\zeta}_j)^{1-\delta_j}}\, d\lambda_n(\zeta). \tag{3.1}$$

The product is the Cauchy kernel each of whose factors is raised to a fractional power. Omitting j, each factor can be expanded as

$$(1 - z\bar{\zeta})^{\delta-1} = \sum_{\alpha=0}^{\infty} b_\alpha\, z^\alpha \bar{\zeta}^\alpha$$

where

$$b_\alpha = \frac{\Gamma(1-\delta+\alpha)}{\Gamma(1-\delta)\,\Gamma(1+\alpha)} \sim \frac{1}{\alpha^\delta}.$$

In particular $b_0 = 1$. Let c_α be the $(\alpha_1, \dots, \alpha_n)$*th Fourier coefficient* of F; i.e.,

$$c_\alpha = \int_{\mathbb{T}^n} \bar{\zeta}^\alpha\, F(\zeta)\, d\lambda_n(\zeta).$$

Setting $f_\alpha = b_{\alpha_1} \cdots b_{\alpha_n} c_\alpha$, we get

$$\begin{aligned} f(z) &= \int_{\mathbb{T}^n} \prod_{j=1}^{n} \sum_{\alpha_j=0}^{\infty} b_{\alpha_j} z_j^{\alpha_j} \bar{\zeta}_j^{\alpha_j} F(\zeta)\, d\lambda_n(\zeta) \\ &= \sum_\alpha b_{\alpha_1} \cdots b_{\alpha_n} z^\alpha \int_{\mathbb{T}^n} \bar{\zeta}^\alpha\, F(\zeta)\, d\lambda_n(\zeta) \\ &= \sum_\alpha b_{\alpha_1} \cdots b_{\alpha_n} z^\alpha\, c_\alpha = \sum_\alpha f_\alpha z^\alpha. \end{aligned}$$

Now

$$\|f\|_{\mathcal{D}_\delta} = \sum_\alpha \alpha_1^{2\delta_1} \cdots \alpha_n^{2\delta_n} |b_{\alpha_1}|^2 \cdots |b_{\alpha_n}|^2 |c_\alpha|^2 \sim \sum_\alpha |c_\alpha|^2 < \infty.$$

Hence $f \in \mathcal{D}_\delta(\mathbb{U}^n)$; i.e., any f given by (3.1) is in a Dirichlet-type space.

But not all $f \in \mathcal{D}_\delta(\mathbb{U}^n)$ have integral representations as in (3.1), because a Dirichlet-type space does not control all the power series coefficients of its members. However, we can define a space $\tilde{\mathcal{D}}_\delta(\mathbb{U}^n)$ similar to $\tilde{\mathcal{D}}(\mathbb{U}^n)$ in which an integral representation is possible. Let's concentrate on the case $n = 2$ again for simplicity. With obvious notation, $f \in \tilde{\mathscr{D}}_\delta(\mathbb{U}^2)$ if and only if

$$\|f\|^2_{\tilde{\mathscr{D}}_\delta} = |f_{00}|^2 + \sum_{k=1}^{\infty} k^{2\delta_1}|f_{k0}|^2 + \sum_{l=1}^{\infty} l^{2\delta_2}|f_{0l}|^2 + \sum_{k,l=1}^{\infty} k^{2\delta_1} l^{2\delta_2}|f_{kl}|^2 < \infty. \tag{3.2}$$

Given $f(z) = \sum_{kl} f_{kl} z^k w^l \in \tilde{\mathscr{D}}_\delta(\mathbb{U}^2)$, let $c_{kl} = f_{kl}/b_k b_l$ if $(k, l) \in \mathbb{N}^2$ (recall that $b_0 = 1$), and let $c_{kl} = 0$ otherwise. Then, using (3.2),

$$\sum_{(k,l)\in\mathbb{Z}^2} |c_{kl}|^2 = |c_{00}|^2 + \sum_{k=1}^{\infty} |c_{k0}|^2 + \sum_{l=1}^{\infty} |c_{0l}|^2 + \sum_{k,l=1}^{\infty} |c_{kl}|^2 \sim \|f\|^2_{\tilde{\mathscr{D}}_\delta} < \infty.$$

Thus there is an $F \in \mathscr{L}^2(\mathbb{T}^2)$ such that $\hat{F}(k, l) = c_{kl}$. Therefore

$$\begin{aligned} f(z) &= \sum_{k,l=0}^{\infty} f_{kl} z^k w^l = \sum_{k,l=0}^{\infty} c_{kl}\, b_k b_l\, z^k w^l \\ &= \sum_{k,l=0}^{\infty} b_k b_l\, z^k w^l \int_{\mathbb{T}^2} \bar{\zeta}_1^k \bar{\zeta}_2^l F(\zeta)\, d\lambda_2(\zeta) \\ &= \int_{\mathbb{T}^2} \frac{F(\zeta)\, d\lambda_2(\zeta)}{(1 - \bar{\zeta}_1 z)^{1-\delta_1}(1 - \bar{\zeta}_2 w)^{1-\delta_2}}. \end{aligned}$$

Clearly F is not unique. In fact, c_{kl} can be defined arbitrarily for $(k, l) \notin \mathbb{N}^2$ as long as we retain $\sum_{(k,l)\in\mathbb{Z}^2} |c_{kl}|^2 < \infty$.

Kernels and potentials. From now on, we will also use $e^{i\theta_j}$ for $\zeta_j \in \mathbb{T}$, $e^{i\varphi_j}$ for $\eta_j \in \mathbb{T}$, and $r_j e^{i\theta_j} = r_j \zeta_j$ for $z_j \in \mathbb{U}$. The point $(1, \ldots, 1) \in \mathbb{T}^n$ corresponding to $\theta_1 = \cdots = \theta_n = 0$ will act like the origin in $\mathbb{R}^n$. Now the Poisson kernel takes the more familiar form

$$P_r(\theta) = \prod_{j=1}^{n} \frac{1 - r_j^2}{1 - 2r_j \cos\theta_j + r_j^2},$$

and it is considered as a function of θ indexed by r. So the $\mathscr{L}^p$-norm of a Poisson integral will be obtained by an integration on the θ-variable and will still depend on r.

A *kernel* K is a nonnegative $\mathscr{L}^1$-function on $\mathbb{T}^n$ which is even and decreasing in each $|\theta_j|$ when the other variables are kept fixed. We will also have $K(1, \ldots, 1) = \infty$ and normalize as $\|K\|_1 = 1$. A *potential* is the convolution of an $\mathscr{L}^p$-function F on $\mathbb{T}^n$ with a kernel. Thus (3.1) defines $f(z) \in \tilde{\mathscr{D}}_\delta(\mathbb{U}^n)$ as a potential. The Poisson integral is simply the convolution with the Poisson kernel.

Let's define the *Bessel kernels* on the torus. For $0 < \delta_j \le 1$, let

$$G_\delta(\zeta) = \prod_{j=1}^{n} g_{\delta_j}(\theta_j) = \prod_{j=1}^{n}\Big(1 + \frac{1}{2}\sum_{\alpha_j \ne 0} |\alpha_j|^{-\delta_j}\zeta_j^{\alpha_j}\Big)$$
$$= \prod_{j=1}^{n}\Big(1 + \sum_{\alpha_j=1}^{\infty} \alpha_j^{-\delta_j}\cos(\alpha_j\theta_j)\Big),$$

where each g_{δ_j} is a Bessel kernel on the unit circle. $g_\delta(0) = \infty$, g_δ is a decreasing function of $|\omega|$ for $\omega \ne 0$, $g_\delta(\omega) > 0$, and $g_\delta(-\omega) = g_\delta(\omega)$. Each $g_{\delta_j} \in \mathscr{L}^1(\mathbb{T})$, so $G_\delta \in \mathscr{L}^1(\mathbb{T}^n)$, and $\|G_\delta\|_1 = \prod_{j=1}^n \|g_{\delta_j}\|_1 = 1$. When $0 < \delta < 1$,

$$g_\delta(\omega) \approx \left|\sin\frac{\omega}{2}\right|^{\delta-1} \quad \text{as } \omega \to 0.$$

Also $g_1(\omega) = 1 - \log|2\sin(\omega/2)|$. $P_r[g_\delta] = P_r * g_\delta$ is the harmonic extension of g_δ to $\mathbb{U}$. As $r \to 1$, it satisfies the following:

$$\text{(3.3)} \qquad \|P_r * g_\delta\|_q \sim (1-r)^{\delta - 1/p} \qquad (\delta p < 1),$$

$$\text{(3.4)} \qquad \|P_r * g_\delta\|_q \sim \left(\log\frac{1}{1-r}\right)^{1/q} \qquad (\delta p = 1,\ p > 1),$$

$$\|P_r * g_1\|_\infty \sim \log\frac{1}{1-r} \qquad (\delta = 1,\ p = 1).$$

$P_r[G_\delta]$ possesses these properties in each variable seperately.

On the unit circle, for $0 < \delta < 1$, the *modified* Bessel kernels are

$$\tilde g_\delta(\omega) = (1 - e^{i\omega})^{\delta-1} \quad \text{and} \quad \tilde g_1(\omega) = \log\frac{1}{1-e^{i\omega}}.$$

On the torus, let $\tilde G_\delta(\zeta) = \prod_{j=1}^n \tilde g_{\delta_j}(\theta_j)$. These functions are not positive, so they are not properly kernels, but they are dominated by the Bessel kernels: There are constants $C_\delta > 0$ such that $|\tilde G_\delta| \le C_\delta G_\delta$. If each δ_j is less than 1,

$$P_r[\tilde G_\delta](\theta) = \prod_{j=1}^{n} P_{r_j}[\tilde g_\delta](\theta_j) = \prod_{j=1}^{n}(1 - z_j)^{\delta_j - 1},$$

with a logarithmic term if some $\delta_k = 1$. For $F \in \mathscr{L}^p(\mathbb{T}^n)$, the map that takes F to $G_\delta * F$ is one-to-one, and the Cauchy integral of $\tilde G_\delta * F$ is the same as its Poisson integral:

$$P_r[\tilde G_\delta * F](\theta) = (P_r * \tilde G_\delta * F)(\theta)$$
$$= \int_{\mathbb{T}^n} F(\zeta)\prod_{j=1}^{n}\frac{1}{(1 - z_j\bar\zeta_j)^{1-\delta_j}}\, d\lambda_n(\zeta) = f(z).$$

Thus we have obtained the integral in (3.1). From now on, F and f will always be related as in this equation. $\delta_j = 1$ does not give rise to a Cauchy-type integral; so we will not pay any attention to this case any more.

Tangential limits. Define the *tangential approach regions* to the unit circle:

$$(3.5)\qquad A_{\gamma,c}(\varphi) = \left\{ re^{i\theta} : 1 - r > c\left|\sin\frac{\theta - \varphi}{2}\right|^{\gamma} \right\},$$

$$E_{\gamma,c}(\varphi) = \left\{ re^{i\theta} : 1 - r > \exp\left(-c\left|\sin\frac{\theta - \varphi}{2}\right|^{-\gamma}\right) \right\}.$$

$A_{\gamma,c}(\varphi)$ has (*polynomial*) *order of contact* γ, and $E_{\gamma,c}(\varphi)$ makes *exponential contact*, with $\mathbb{T}$. A function f defined in $\mathbb{U}$ has $A_\gamma(E_\gamma)$-*limit* L at $e^{i\varphi}$ if $f(z) \to L$ as $z \to e^{i\varphi}$ within $A_{\gamma,c}(E_{\gamma,c})$ for every $c > 0$. In [3], it was shown that Poisson integrals of the modified Bessel potentials have A_γ-limits a.e. on $\mathbb{T}$ if $\delta p < 1$ for $\gamma = \frac{1}{1-\delta p}$, and E_γ-limits a.e. on $\mathbb{T}$ if $\delta p = 1$ for $\gamma = q - 1$.

Let $Q = Q(\eta, s)$ be the cube centered at $\eta \in \mathbb{T}^n$ with sides $s = (s_1, \dots, s_n)$, where each s_j has the same order as $\max\{s_j : 1 \le j \le n\} \to 0$. Its volume is $\lambda_n(Q) = s_1 \cdots s_n$. If $F \in \mathscr{L}^p(\mathbb{T}^n)$, its *Hardy-Littlewood maximal function* is

$$(M_pF)(\eta) = \sup_{0<s_1,\dots,s_n\le 1} \left(\frac{1}{\lambda_n(Q)} \int_Q |f|^p \, d\lambda_n \right)^{1/p}.$$

M_1 is of weak type $(1, 1)$; and since $M_pF = \left(M_1|F|^p\right)^{1/p}$, M_p is of weak type (p, p). Thus there are C_p such that

$$\lambda_n(\{M_pF > t\}) \le \frac{C_p}{t^p}\|F\|_p^p \qquad (F \in \mathscr{L}^p(\mathbb{T}^n),\ t \in (0, \infty)).$$

The proofs of the following assertions are similar to the proofs given in [3] for $n = 1$ and will be omitted. Some of them are valid in more general situations. The first result is obtained using the straightforward inequality

$$\int_{\mathbb{T}^n} |F|\, G_\delta \, d\lambda_n \le (M_1F)(1, \dots, 1) \int_{\mathbb{T}^n} G_\delta \, d\lambda_n = (M_1F)(1, \dots, 1),$$

which holds for any $F \in \mathscr{L}^1(\mathbb{T}^n)$, and whose proof is also in [3].

THEOREM 3.1. *There is a $C_p < \infty$ such that for $F \in \mathscr{L}^p(\mathbb{T}^n)$ and $\zeta, \eta \in \mathbb{T}^n$,*

$$|(G_\delta * F)(\zeta)| \le C_p\left[(M_pF)(\eta)\left(\prod_{j=1}^n |\zeta_j - \eta_j|^{1/p}\right)\|G_\delta\|_q + (M_1F)(\eta)\right].$$

Convolution of two kernels is a kernel; so Theorem 3.1 holds with $P_r * G_\delta = P_r[G_\delta]$ in place of G_δ, which has the desired properties $\|P_r * G_\delta\|_1 = \|P_r\|_1 \, \|G_\delta\|_1 = 1$ and $\|P_r * G_\delta\|_q = \prod_{j=1}^n \|P_{r_j} * g_{\delta_j}\|_q$. The Hölder inequality gives $M_1 F \le M_p F$. In addition, $|\zeta_j - \eta_j|$ can be replaced by $|\theta_j - \varphi_j|$, or even by $|\sin((\theta_j - \varphi_j)/2)|$, since they are all of the same order as $\zeta \to \eta$. Lastly, we can put $\tilde{G}_\delta$ in place of G_δ on the left side of the inequality since the latter dominates the former. Hence Theorem 3.1 yields

THEOREM 3.2. *If $F \in \mathscr{L}^p(\mathbb{T}^n)$, then, using $z = re^{i\theta}$, for all $e^{i\theta}$, $e^{i\varphi} \in \mathbb{T}^n$,*

$$(3.6) \qquad |f(z)| \le C_p\,(M_p F)(\varphi)\left[1 + \prod_{j=1}^n \left|\sin\frac{\theta_j - \varphi_j}{2}\right|^{1/p} \|P_{r_j} * g_{\delta_j}\|_q\right].$$

For given φ, any bound on the product on the right side gives a bound on $f(z)$. This leads us to the tangential approach regions to $\mathbb{T}^n$. So fix an $\eta \in \mathbb{T}^n$. As $z \to \zeta$, all $r_j \to 1$; and because of (3.3) and (3.4), $\|P_r * G_\delta\|_q \sim \prod_{j=1}^n b_j$, where $b_j = (1 - r_j)^{\delta_j - 1/p}$ or $b_j = (\log(1/(1 - r_j)))^{1/q}$ depending on whether $\delta_j p < 1$ or $\delta_j p = 1$, respectively. In other words, an approach region should be determined by

$$\prod_{j=1}^n b_j \left|\sin\frac{\theta_j - \varphi_j}{2}\right|^{1/p} < c.$$

So define $B_{\gamma,c}(\eta)$ by

$$(3.7) \qquad B_{\gamma,c}(\eta) = B_{\gamma,c}(\varphi) = \left\{ z \in \mathbb{U}^n : \prod_{j=1}^n b_j^{-1/\gamma_j} \left|\sin\frac{\theta_j - \varphi_j}{2}\right| < c \right\}.$$

Each of the factors in the above product is related to one of the regions in (3.5). In particular, points in a cartesian product of one-dimensional approach regions such as $B_{\gamma_1,c_1}(\varphi_1) \times \cdots \times B_{\gamma_n,c_n}(\varphi_n)$, where each $B_{\gamma_j,c_j}(\varphi_j)$ is either $A_{\gamma_j,c_j}(\varphi_j)$ or $E_{\gamma_j,c_j}(\varphi_j)$, satisfy the criterion for being in $B_{\gamma,c}(\eta)$. Hence an approach region can make exponential contact with $\mathbb{T}^n$ in one (complex) direction and polynomial contact in another. For $\eta \in \mathbb{T}^n$, the *maximal functions* associated to these approach regions are defined as

$$(M_{G_\delta,\gamma,c} f)(\eta) = \sup\{\, |f(z)| : z \in B_{\gamma,c}(\eta)\}.$$

A function f defined in $\mathbb{U}^n$ is said to have *B_γ-limit* L at $\eta \in \mathbb{T}^n$ if $f(z) \to L$ as $z \to \eta$ within $B_{\gamma,c}(\eta)$ for every $c > 0$.

THEOREM 3.3. *If* $F \in \mathscr{L}^p(\mathbb{T}^n)$ *and* $t \in (0, \infty)$, *there is a* $C = C(p, c)$ *such that*

$$\lambda_n(\{M_{G_\delta, \gamma, c} f > t\}) < \frac{C}{t^p} \|F\|_p^p,$$

where

$$\gamma_j = \frac{1}{1 - \delta_j p} \quad \text{if } \delta_j p < 1 \quad \text{and} \quad \gamma_j = q - 1 \quad \text{if } \delta_j p = 1. \tag{3.8}$$

This follows from the weak-type-(p, p) estimate for M_p. The weak-type estimate gives rise to a convergence theorem via classical arguments. This is the content of part (i) of Theorem C. The case $p = 1$ of part (ii) of that theorem also follows from Theorem 3.3. Henceforth, p, δ, and γ will always be related by (3.8).

Capacities. For $E \subset \mathbb{T}^n$, let $T(G_\delta, p, E)$ be the set of all nonnegative $F \in \mathscr{L}^p(\mathbb{T}^n)$ such that $(G_\delta * F)(\zeta) \geq 1$ for all $\zeta \in E$. The *p-capacity* of E is

$$\Sigma_{G_\delta}(E) = \inf\{ \|F\|_p^p : F \in T(G_\delta, p, E) \}.$$

$\Sigma_{G_\delta}(E) = 0$ implies $\lambda_n(E) = 0$. The functions $G_\delta * F$ are defined Σ_{G_δ}-almost everywhere. If $F \in T(P_r * G_\delta, p, E)$, then $\Sigma_{P_r}(E) \leq \Sigma_{P_r * G_\delta}(E)$. For $\eta \in \mathbb{T}^n$ and *fixed* $\rho > 1$, let

$$\Gamma(\eta) = \{ z \in \mathbb{U}^n : |z_j - \eta_j| < \rho(1 - |z_j|), \ 1 \leq j \leq n \},$$

and set $S(E) = \mathbb{U}^n \setminus \bigcup_{\eta \notin E} \Gamma(\eta)$. $\Gamma(\eta)$ is the cartesian product of n sets each of which is asymptotic, as $z_j \to \eta_j$, to an angle-shaped approach region in $\mathbb{U}$ with vertex at η_j. For $\eta \in \mathbb{T}^n$, the *nontangential maximal function* is

$$(Nf)(\eta) = \sup\{ |f(z)| : z \in \Gamma(\eta)\}.$$

For $W \subset \mathbb{U}^n$, $J_c^\gamma(W)$ is the set of $\eta \in \mathbb{T}^n$ for which W intersects $B_{\gamma, c}(\eta)$.

LEMMA 3.4. *There exists a constant* $b = b(n) > 0$ *such that if* $F \geq 0$ *on* $\mathbb{T}^n$, $F \geq 1$ *on* $E \in \mathbb{T}^n$, *and* $z \in S(E)$, *then* $P_r[F](z) > b$.

We will use this lemma with $G_\delta * F$ in place of F. It leads to the following lower estimate for capacities.

PROPOSITION 3.5. *If* $F \in \mathscr{L}^1(\mathbb{T}^n)$ *and* $0 < t < \infty$, *then* $\{M_{G_\delta, \gamma, c} f > t\}$ *is contained in* $J_c^\gamma(S(\{Nf > t\}))$. *Thus, there is a* $C = C(p, c)$ *such that if* $t \in (0, \infty)$ *and* $F \in \mathscr{L}^1(\mathbb{T}^n)$, *we have*

$$\lambda_n(\{M_{G_\delta, \gamma, c} f > t\}) \le C \Sigma_{G_\delta}(\{Nf > t\}).$$

THEOREM 3.6. *For* $1 < p < \infty$, *there is a constant* $C_p < \infty$ *such that if* $F \in \mathscr{L}^p(\mathbb{T}^n)$ *and* $F \ge 0$, *then*

$$\int_0^\infty \Sigma_{G_\delta}(\{G_\delta * F > t\})\, d(t^n) \le C_p \|F\|_p^p.$$

Combining Lemma 3.4, Proposition 3.5, and Theorem 3.6 with the fact that G_δ dominates $\tilde{G}_\delta$ and that $N(G_\delta * F) = G_\delta * NF$, we obtain the strong-type estimates in part (ii) of Theorem C.

THEOREM C. *Let* $1 \le p < \infty$, $F \in \mathscr{L}^p(\mathbb{T}^n)$, $0 < \delta_j < 1$, *define* f *as in* (3.1), *pick* γ_j *as in* (3.8), *and for* $\zeta \in \mathbb{T}^n$, *construct* $B_{\gamma, c}(\zeta)$ *as in* (3.7).

(i) *The* B_γ*-limit of* f *exists a.e.* $[\lambda_n]$ *on* $\mathbb{T}^n$
(ii) *There are positive constants* C_p *such that*

$$\|M_{G_\delta, \gamma, c} f\|_p \le C_p \|F\|_p \qquad (1 < p < \infty),$$
$$\lambda_n(\{ M_{g_\delta, \gamma, c} f > t \}) \le \frac{C_1}{t} \|F\|_1 \qquad (p = 1,\ 0 < t < \infty).$$

If $\zeta \in \partial\mathbb{U}^n \backslash \mathbb{T}^n$, then only one component of ζ, say the nth, has $|\zeta_n| = 1$. Then the first $n-1$ factors in the product in (3.6) are bounded as $z \to \zeta$. So in this case, it suffices to apply the one-variable result in the nth variable. The approach regions are restricted only in the nth component as in (3.5), and $(z_1, \dots, z_{n-1})$ can approach $(\zeta_1, \dots, \zeta_{n-1}) \in \partial\mathbb{U}^n$ in any manner whatsoever. Theorem C remains valid except that in part (ii), we would use one-dimensional norm and Lebesgue measure.

When $p = 2$ and all the $\delta_j = 1/2$, this theorem takes care of $\tilde{\mathscr{D}}(\mathbb{U}^n)$, but cannot deal with $\mathscr{D}(\mathbb{U}^n)$. Thus the functions in the modified Dirichlet space have tangential ($B_{(1, \dots, 1)}$-) limits at almost every boundary point of the unit polydisc. When $n = 1$, since $\mathscr{H}^2(\mathbb{U})$ includes all Dirichlet-type spaces, elements of $\mathscr{D}(\mathbb{U})$ have nontangential limits a.e. on $\mathbb{T}$.

Now we will look at the size of the exceptional sets. From Lemma 3.4, part (ii) of Theorem C, and the first part of Proposition 3.5, we obtain

LEMMA 3.7. *If* $1 < p < \infty$, *then for some* $C = C(p, c)$

$$\Sigma_{P_r}(J_c^\gamma(S(E))) \leq C\,\Sigma_{P_r * G_\delta}(E).$$

Hence for $F \in \mathscr{L}^p(\mathbb{T}^n)$,

$$\Sigma_{P_r}(\{M_{G_\delta, \gamma, c} f > t\}) \leq C\,\Sigma_{P_r * G_\delta}(\{Nf > t\}).$$

THEOREM 3.8. *If* $1 < p < \infty$ *and* $F \in \mathscr{L}^p(\mathbb{T}^n)$, *then*

$$\int_0^\infty \Sigma_{P_r}(\{M_{G_\delta, \gamma, c} f > t\})\, d(t^n) \leq C\,\|F\|_p^p,$$

and thus

$$\Sigma_{P_r}(\{M_{G_\delta, \gamma, c} f > t\}) \leq C \frac{\|F\|_p^p}{t^p}.$$

This theorem is an analog of Theorem C in the language of capacities and proved similarly.

THEOREM D. *Let* $1 < p < \infty$, $F \in \mathscr{L}^p(\mathbb{T}^n)$, *and* f *be as in* (3.1).

(i) *There is a set* $E_1 \subset \mathbb{T}^n$ *with* $\Sigma_{P_r * G_\delta}(E_1) = 0$ *such that the nontangential limit of* f *exists at every point of* $\mathbb{T}^n \backslash E_1$.

(ii) *There is a set* $E_2 \subset \mathbb{T}^n$ *with* $\Sigma_{P_r}(E_2) = 0$ *such that the* B_γ*-limit of* f *exists at every point of* $\mathbb{T}^n \backslash E_2$.

This result is a consequence of the basic properties of capacities and Theorem 3.10. For points on $\partial \mathbb{U}^n \backslash \mathbb{T}^n$, the one-variable result can again be used to reach a similar conclusion. Hence if $p = 2$ and all the $\delta_j = 1/2$, the points on $\partial \mathbb{U}^n$ where the modified Dirichlet space does not have nontangential limits have zero capacity in some sense.

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