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CORRECTION TO "TRACE RINGS FOR VERBALLY PRIME ALGEBRAS"

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In **[1]** and **[2]** we incorrectly state a theorem of Razmyslov from **[3]**. We quoted Razmyslov as saying:

For all k and l, $M_{k,l}$ satisfies a trace identity of the form

$$(*)\qquad p(x_1, \dots, x_n, a) = c(x_1, \dots, x_n)\mathrm{tr}(a)$$

where $p(x_1, \dots, x_n, a)$ and $c(x_1, \dots, x_n)$ are central polynomials.

This statement is true if $k \neq l$ and false if $k = l$. We will indicate why this is true and what effect it has on the results of **[1]** and **[2]**. It turns out that **[1]** needs only a very minor comment, but that **[2]** requires a modification to the main theorem and a longer proof in the case of $k = l$.

First, here is a correct version of Razmyslov's theorem:

For all k and l, $M_{k,l}$ satisfies a trace identity of the form

$$(**)\qquad p(x_1, \dots, x_n, a) = \mathrm{tr}(c'(x_1, \dots, x_n))\mathrm{tr}(a)$$

where $p(x_1, \dots, x_n, a)$ is a central polynomial and $c'(x_1, \dots, x_n)$ does not involve any traces.

If $k \neq l$, then the trace of the identity matrix equals $k - l$ which is not zero. So, if we set $a = I$ in $(**)$ we get

$$\mathrm{tr}(c'(x_1, \dots, x_n)) = (k-l)^{-1}p(x_1, \dots, x_n, I).$$

Hence, in this case $\mathrm{tr}(c'(x_1, \dots, x_n))$ equals a central polynomial modulo the identities for $M_{k,l}$, and so $(*)$ is true in this case. To see that $(*)$ is false if $k \neq l$ it is useful to have the following lemma.

LEMMA 1. *Let $f(x_1, \dots, x_n)$ be a pure trace identity for $M_{k,k}$ and write $f(x_1, \dots, x_n) = f_0(x_1, \dots, x_n) + f_1(x_1, \dots, x_n)$, where each monomial in f_0 involves an even number of traces and each monomial in f_1 involves an odd number of traces. Then $f_0(x_1, \dots, x_n)$ and $f_1(x_1, \dots, x_n)$ are each trace identities for $M_{k,k}$.*

Proof. We define an automorphism on $M_{k,k}$. Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an element of $M_{k,k}$, where A, B, C and D are $k \times k$ blocks, and define $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^*$ to be the matrix $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$. Then $-^*$ is an automorphism and $\mathrm{tr}(x^*) = -\mathrm{tr}(x)$ for any matrix x. Hence $M_{k,k}$ satisfies the trace identity $f(x_1^*, \ldots, x_n^*) = f_0(x_1^*, \ldots, x_n^*) + f_1(x_1^*, \ldots, x_n^*) = f_0(x_1, \ldots, x_n) - f_1(x_1, \ldots, x_n)$. The lemma follows.

COROLLARY. *$M_{k,k}$ does not satisfy* $(*)$.

Proof. Multiply $(*)$ by a new variable x_{n+1} and take trace. The left-hand side becomes a product of two traces which is not an identity, and the right-hand side becomes a product of three traces, contradicting Lemma 1.

To fix up the proof in [1] in the case $k = l$ all that is required is this simple remark: Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be $2n$ variables. Then $M_{k,k}$ satisfies the identity

$$(1) \qquad \mathrm{tr}(c'(x_1, \ldots, x_n))\mathrm{tr}(c'(y_1, \ldots, y_n)) = p(x_1, \ldots, x_n, c'(y_1, \ldots, y_n)).$$

Hence,

$$c(x_1, \ldots, x_n)c(y_1, \ldots, y_n) = \mathrm{tr}(c'(x_1, \ldots, x_n))\mathrm{tr}(c'(y_1, \ldots, y_n))$$

is a central polynomial for $M_{k,l}$ even if $k = l$. This is all that **[1]** requires. (We will now resume using the shorthand notation from **[2]** and we will write $p(x, a)$, $c(x)$, $c'(x)$, $p(y, a)$, etc.)

DEFINITION. Let R be any ring and let J^2 be the ideal of R generated by all evaluations of $p(x_1, \ldots, x_n, c'(y_1, \ldots, y_n))$ on R.

We remark for future reference this easy consequence of (1): $M_{k,k}$ satisfies the identity

$$(2) \qquad p(x, c'(y)) = p(y, c'(x)).$$

Hence we may denote it as $c(x)c(y)$ to emphasize its symmetric nature.

Here is the main result:

THEOREM 3. *Assume that R is p.i. equivalent to some $M_{k,k}$ and that the annihilator of J^2 is* (0). *Then there is an embedding of R into a $\mathbf{Z}/2\mathbf{Z}$-graded ring with trace $\overline{R} = R_0 + R_1$, such that $R \subset R_0$,* $\mathrm{tr}(R_0) \subset R_1$ *and* $\mathrm{tr}(R_1) = (0)$; *such that $\overline{R}$ is generated by R and* $\mathrm{tr}(R)$; *and such that*

(a) *the trace on $\overline{R}$ is a non-degenerate,*

(b) *there is a faithful R-submodule of* R_1, J *such that for all homogeneous* r *in* $\overline{R}$ *there exists an integer* n *such that* $J^n r \subset R$, *and*
(c) $\overline{R}$ *satisfies the same trace identities as* $M_{k,k}$.

Proof. The construction of $\overline{R}$ will be in two parts, first R_0 and then R_1. Much of the construction will be very similar to [2] and so we will omit a number of details.

For any $a, b \in R$ we construct an R-map $t(a, b)\colon J^2 \to R$ via $t(a, b)(c(x)c(y)r) = p(x, a)p(y, b)r$. The reader should think of $t(a, b)$ as $\operatorname{tr}(a)\operatorname{tr}(b)$. The proof that $t(a, b)$ is well-defined is similar to the corresponding proof in [2]. We note that $t(a, b)$ is symmetric, bilinear and vanishes if either argument is a commutator. Here are a few of its other properties:

$$\text{(3)}\quad \text{if } \sum_i r_i t(a_i, b_i) = 0, \quad \text{then for all } s, \quad \sum_i r_i s t(a_i, b_i) = 0,$$

$$\text{(4)}\qquad t(a, b)t(c, d) = t(c, b)t(a, d),$$

$$\text{(5)}\qquad t(t(a, b), c) = 0.$$

Finally, as in [2], R_0 can be constructed as the subring of $\varprojlim \hom_R((J^2)^n, R)$ generated by R and all $t(a, b)$. Note that t extends to a map from $R_0 \times R_0$ to its center.

To define R_1 we start with the free R_0-module on the symbols $\operatorname{tr}(a)$, $a \in R$ and then mod out by the relation (&)

$$\text{if } \sum_i \alpha_i t(a_i, b) = 0 \quad \text{for all } b \in R \quad \text{then } \sum_i \alpha_i \operatorname{tr}(a_i) = 0,$$

where the α_i are in R_0 and the a_i are in R.

This relation has a number of implications for tr. Regarded as a map from R_0 to R_1 it is linear over the center of R_0 and it vanishes on commutators. Equations (3)–(5) all have counterparts for tr:

$$\text{(3}'\text{)}\quad \text{if } \sum_i \alpha_i \operatorname{tr}(a_i) = 0 \quad \text{then for all } s, \quad \sum_i \alpha_i s \operatorname{tr}(a_i) = 0,$$

$$\text{(4}'\text{)}\qquad t(a, b)\operatorname{tr}(c) = t(c, b)\operatorname{tr}(a),$$

$$\text{(5}'\text{)}\qquad \operatorname{tr}(t(a, b)) = 0.$$

It follows from (3′) that we may define a bimodule structure on R_1 via $(\sum_i \alpha_i \operatorname{tr}(a_i))s = \sum_i \alpha_i s \operatorname{tr}(a_i)$. Then we define a bilinear pairing $R_1 \times R_1 \to R_0$ via $(\alpha \operatorname{tr}(a))(\beta \operatorname{tr}(b)) = \alpha\beta t(a, b)$. Using (&) it is straightforward to show that this pairing is well-defined. Finally, we construct a multiplicative structure on $\overline{R} = R_0 + R_1$ via

$$\left(a + \sum_i b_i \operatorname{tr}(c_i)\right)\left(d + \sum_j e_j \operatorname{tr}(f_j)\right)$$
$$= \left(ad + \sum_{i,j} b_i e_j t(c_i, f_j)\right) + \left(\sum_j a e_j \operatorname{tr}(f_j) + \sum_i b_i d \operatorname{tr}(c_i)\right).$$

That it is associative follows from (4′). We now prove that $\overline{R}$ has the properties (a), (b) and (c) that we claimed in the statement of the theorem.

It is useful at this point to prove that $\overline{R}$ satisfies the identity $(**)$, namely

$$(**) \qquad p(x, a) = \operatorname{tr}(c'(x))\operatorname{tr}(a).$$

In order to prove this it suffices to take x and a in R. Consider $\operatorname{tr}(c'(x))\operatorname{tr}(a) = t(c'(x), a)$ as a map from J^2 to R. This map takes $c(y)c(z)$ to

$$\begin{aligned} p(y, c'(x))p(z, a) &= \qquad \text{(by (2))} \\ p(x, c'(y))p(z, a) &= \qquad \text{(by (2) of [2])} \\ p(x, a)p(z, c'(y)) &= \end{aligned}$$

$p(x, a)$ times $c(y)c(z)$. This proves $(**)$.

Let $J = R\operatorname{tr}(c'(R^n)) \subset R_1$. Note that the square of J equals the ideal of R we denoted J^2 by $(**)$, and so $\operatorname{ann}(J) = (0)$. Continuing the proof of (b), let $r \in R_0$. It follows from the construction of R_0 that $(J^2)^n r = J^{2n} r$ is contained in R, for some n. And, if $r \in R_1$ then we may assume without loss of generality that $r = \alpha \operatorname{tr}(a)$ for some $\alpha \in R_0$, $a \in R$. But then, $J^{2n}\alpha \subset R$ for some n as above, and $J\operatorname{tr}(\alpha) \subset R$ by $(**)$. Hence $J^{2n+1} r \subset R$.

The proof of (c) follows from (b) as in [2]. Let $f(x) = f(x_1, \dots, x_m)$ be a trace polynomial in which either term has an even number of traces or each term has an odd number of traces. Then it follows from (b) that $M_{k,k}$ and R satisfy an identity of the form $j(y)f(x) = g(x, y)$, where x and y are disjoint sets of variables and $g(x, y)$ doesn't involve any traces. Since $M_{k,k}$ is verbally prime, $f(x)$ is a trace identity for $M_{k,k}$ if and only if $g(x, y)$ is a p.i. for $M_{k,k}$. Moreover, since $\overline{R}$ is a central extension of R, they satisfy the same p.i.'s. Hence, if $f(x)$ is a trace identity for $\overline{R}$ then $p(x, y)$ will be an identity for R and so for $M_{k,k}$, and so $f(x)$ will also be an identity for $M_{k,k}$. Conversely, if $f(x)$ is a trace identity for $M_{k,k}$, then it follows that $j(y)f(x)$ is a trace identity for $\overline{R}$. But this implies that the evaluations of $f(x)$ would annihilate some power of J and so $f(x)$ is forced to be an identity.

The proof of (a) is also similar to the corresponding proof in **[2]** and we omit it.

References

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