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CONDITIONAL WIENER INTEGRALS II

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In this paper we establish various results involving conditional Wiener integrals, $E(F|X)$, for very general conditioning functions X. Most related results in the literature, including the case when the conditioning function X is vector-valued, then follow as corollaries of this more general theory. A simple formula is given for converting these generalized conditional Wiener integrals into ordinary Wiener integrals and then this formula is used to evaluate $E(F|X)$ for various classes of functionals F. Finally these results are used to obtain a generalized conditional form of the Cameron-Martin translation theorem.

1. Introduction. Let $(C[0,T],\mathcal{F}^*,m_w)$ denote Wiener space, where $C[0,T]$ is the space of all continuous functions x on $[0,T]$ vanishing at the origin. Let $F(x)$ be a Wiener integrable function on $C[0,T]$ (i.e., $E[|F(x)|]<\infty$) and let $X(x)$ be a Wiener measurable function on $C[0,T]$. In [**13**], Yeh introduced the concept of conditional Wiener integrals. He defined the conditional Wiener integral of F given X as a function on the value space of X and derived a Fourier transform inversion formula for computing conditional Wiener integrals. Using this formula for the case $X(x)=x(T)$, Yeh [**13, 14**] obtained some very useful results including a Kac-Feynman integral equation and a conditional Cameron-Martin translation theorem.

In [**4**], for certain functions F, Chang and Chang, using Yeh's inversion formula, evaluated the conditional Wiener integral of F given $X(x)=(x(t_1),\dots,x(t_n))$ where $0<t_1<t_2<\dots<t_n=T$. In [**8**], the current authors obtained a very simple formula for the conditional Wiener integral of F given $X(x)=(x(t_1),\dots,x(t_n))$. In particular we expressed the conditional Wiener integral directly in terms of an ordinary (i.e., nonconditional) Wiener integral. Using this formula it was relatively simple to generalize the Kac-Feynman formula and to obtain a conditional Cameron-Martin translation theorem involving vector-valued conditioning functions.

In this paper we consider much more general conditioning functions. In particular they need not depend upon the values of x at only finitely many points in $(0, T]$. A major thrust of this paper is to develop a useful formula to convert these generalized conditional Wiener integrals into ordinary (i.e., nonconditional) Wiener integrals and then to obtain the corresponding Cameron-Martin translation theorem for these generalized conditional Wiener integrals. We also use this simple formula to compute the generalized conditional Wiener integral for various functions $F(x)$ on $C[0, T]$. Most of the results in [**4**, **8**, **13**, and **14**] then follow as special cases of the results obtained in this paper.

2. Preliminaries and definitions. Let $\mathcal{H}$ be an infinite dimensional subspace of $L_2[0, T]$ with a complete orthonormal basis $\{\alpha_j\}$. Then the corresponding stochastic integrals

$$\gamma_j(x) = \int_0^T \alpha_j(t)dx(t), \; j = 1, 2, \dots \tag{2.1}$$

form a set of independent standart Gaussian variables on $C[0, T]$ with

$$E[x(t)\gamma_j(x)] = \int_0^t \alpha_j(s)ds \equiv \beta_j(t). \tag{2.2}$$

For each $n \in \mathbb{N}$ let $\mathcal{H}_n$ be the subspace of $\mathcal{H}$ spanned by $\{\alpha_1, \dots, \alpha_n\}$, and let $X_n : C[0, T] \to \mathbb{R}^n$ and $X_\infty : C[0, T] \to \mathbb{R}^n$ be defined by

$$X_n(x) = (\gamma_1(x), \dots, \gamma_n(x)), \; X_\infty(x) = (\gamma_1(x), \gamma_2(x), \dots). \tag{2.3}$$

If $\mathcal{B}^n$ denotes the σ-algebra of Borel sets in $\mathbb{R}^n$, then a set of the type

$$I = \{x \in C[0, T] : X_n(x) \in B\} \equiv X_n^{-1}(B), \; B \in \mathcal{B}^n$$

is called a quasi-Wiener interval (or a Borel cylinder). It is well known that

$$m_w(I) = \int_B K_n(\vec{\xi})d\vec{\xi}, \tag{2.4}$$

where

$$K_n(\vec{\xi}) = (2\pi)^{-\frac{n}{2}} \exp\left\{ -\frac{1}{2}\sum_{j=1}^n \xi_j^2 \right\}. \tag{2.5}$$

Let $\mathcal{F}_n$ be the σ-algebra formed by the sets $\{X_n^{-1}(B) : B \in \mathcal{B}^n\}$, and let $\mathcal{F}$ be the σ-algebra generated by $\cup_{n=1}^{\infty}\mathcal{F}_n$. Then, by the definition of conditional expectations (see Doob [**5**], Tucker [**10**] and Yeh [**12**]) for each $F \in L_1(C[0,T], m_w)$,

(2.6)
$$\begin{aligned}\mu(B) &\equiv \int_{X_n^{-1}(B)} F(x)m_w(dx) = \int_{X_n^{-1}(B)} E(F|\mathcal{F}_n)m_w(dx)\\ &= \int_B E(F(x)|X_n(x) = \vec{\xi})P_{X_n}(d\vec{\xi})\\ &= \int_B E(F(x)|\gamma_j(x) = \xi_j,\ j = 1, \dots, n)P_{X_n}(d\vec{\xi}),\ B \in \mathcal{B}^n,\end{aligned}$$

where $P_{X_n}(B) = m_w(X_n^{-1}(B))$, and $E(F(x)|X_n(x) = \vec{\xi})$ is a Lebesgue measurable function for $\vec{\xi}$ which is unique up to null sets in $\mathbb{R}^n$.

Since $\{\mathcal{F}_n\}$ is an increasing sequence of σ-algebras of Weiner measurable sets, for $F \in L_1(C[0,T], m_w)$, $\{E(F|\mathcal{F}_n)\}$ is a martingale sequence. Thus, $E\,|E(F|\mathcal{F}_n)| \le E|F|$ for every n, and so by the martingale convergence theorem, $\lim E(F|\mathcal{F}_n) = E(F|\mathcal{F})$ almost surely and for each $A \in \cup_{n=1}^{\infty}\mathcal{F}_n$,

$$\int_A E(F(x)|\mathcal{F})m_w(dx) = \lim \int_A E(F(x)|\mathcal{F}_n)m_w(dx). \tag{2.7}$$

From this and (2.6), it follows that for every $B \in \cup_{n=1}^{\infty}\mathcal{B}^n$,

$$\begin{aligned}&\int_B E(F(x)|\gamma_j(x) = \xi_j,\ j = 1, 2, \dots)P_{X_\infty}(d\vec{\xi})\\ &= \lim \int_B E(F(x)|\gamma_j(x) = \xi_j,\ j = 1, \dots, n)P_{X_n}(d\vec{\xi}),\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}P_{X_n}(d\vec{\xi}) &= \prod_{j=1}^{n}\left\{(2\pi)^{-\frac{1}{2}}\exp(-\xi_j^2/2)d\xi_j\right\},\\ P_{X_\infty}(d\vec{\xi}) &= \prod_{j=1}^{\infty}\left\{(2\pi)^{-\frac{1}{2}}\exp(-\xi_j^2/2)d\xi_j\right\}.\end{aligned} \tag{2.9}$$

In (2.8) we used the convention that if $B \in \mathcal{B}^n$, then $B \in \mathcal{B}^{n+k}$ by identifying B and $B \times \mathbb{R}^k$ in $\mathcal{B}^{n+k}$ for $k = 1, 2, \dots$. Thus if

$B \in \cup_{n=1}^{\infty}\mathcal{B}^n$, then there exists $N \in \mathbb{N}$ such that $B \in \mathcal{B}^n$ for all $n \geq N$, and hence by the martingale property

$$\begin{aligned}(2.10) \quad &\int_B E(F(x)|\gamma_j(x) = \xi_j,\ j = 1, 2, \dots)P_{X_\infty}(d\vec{\xi}) \\ &= \int_B E(F(x)|\gamma_j(x) = \xi_j,\ j = 1, \dots, n)P_{X_n}(d\vec{\xi}), \text{ for all } n \geq N,\end{aligned}$$

from which (2.8) follows.

In the next section we develop quite simple formulas for converting the generalized conditional Wiener integrals of the types $E(F(x)|X_n(x) = \vec{\xi}) = E(F(x)|\gamma_j(x) = \xi_j,\ j = 1, \dots, n)$ and $E(F(x)|\gamma_j(x) = \xi_j, j = 1, 2, \dots)$ into ordinary Weiner integrals which can often be computed explicitly. It then turns out that all the conditional Weiner integrals that occur in [**4**, **8**, **13**, and **14**] are special cases of conditional expectations given in this paper.

3. Useful formulas for conditional Wiener integrals. Let $\mathcal{H}$, $\{\alpha_j\}$, $\mathcal{H}_n$ and $\{\gamma_j(x)\}$ be as in Section 2. Define projection maps $\mathcal{P}$ and $\mathcal{P}_n$ from $L_2[0, T]$ into $\mathcal{H}$ and $\mathcal{H}_n$, respectively, by

$$(3.1) \qquad \mathcal{P}h(t) = \sum_{j=1}^{\infty}(h, \alpha_j)\alpha_j(t),$$

$$\mathcal{P}_n h(t) = \sum_{j=1}^{n}(h, \alpha_j)\alpha_j(t).$$

For $x \in C[0, T]$ and $\vec{\xi} = (\xi_1, \xi_2, \dots)$, let

$$(3.2) \qquad x_n(t) = \int_0^T \mathcal{P}_n I_{[0,t]}(s)dx(s) = \sum_{j=1}^{n} \gamma_j(x) \int_0^t \alpha_j(s)ds,$$

$$\vec{\xi}_n(t) = \sum_{j=1}^{n} \xi_j(\alpha_j, I_{[0,t]}),$$

where $I_{[0,t]}$ is the indicator function of the interval $[0, t]$. Similarly, define

$$(3.3) \qquad x_\infty(t) = \int_0^T \mathcal{P}I_{[0,t]}(s)dx(s) = \sum_{j=1}^{\infty} \gamma_j(x) \int_0^t \alpha_j(s)ds,$$

$$\vec{\xi}_\infty(t) = \sum_{j=1}^{\infty} \xi_j(\alpha_j, I_{[0,t]}).$$

We note here that since $\{\gamma_j(x)\}$ is a sequence of i.i.d. standard Gaussian random variables, the series $x_\infty(t)$ converges m_w-a.e. x(see Shepp [**9**, p.324]). Since $\vec{\xi}_\infty(t)$ is the evaluation of the random variable $x_\infty(t)$ for $\gamma_j(x) = \xi_j$, $j = 1, 2, \ldots$, $\vec{\xi}_\infty(t)$ converges P_{x_∞} - a.e. $\vec{\xi}$.

Our first theorem plays a key role throughout this paper.

THEOREM 1. *If $\{x(t),\ 0 \le t \le T\}$ is the standart Wiener process, then the processes $\{x(t) - x_\infty(t),\ 0 \le t \le T\}$ and $\gamma_j(x)$ are (stochastically) independent for $j = 1, 2, \ldots$. Also, $\{x(t) - x_n(t),\ 0 \le t \le T\}$ and $\gamma_j(x)$ are independent for $j = 1, \ldots, n$.*

Proof. For each j, using (2.2), (3.1) and (3.2)

$$E[\gamma_j(x)\{x(t) - x_\infty(t)\}] = \int_0^t \alpha_j(s)ds - \sum_{i=1}^{\infty} \delta_{ij} \int_0^t \alpha_j(s)ds = 0.$$

Since both $\gamma_j(x)$ and $x(t) - x_\infty(t)$ are Gaussian and uncorrelated, it follows that they are independent. The second claim follows in similar manner. □

COROLLARY 1. *The processes $\{x(t) - x_\infty(t),\ 0 \le t \le T\}$ and $\{x_\infty(t),\ 0 \le t \le T\}$ are independent, and so are $\{x(t) - x_n(t),\ 0 \le t \le T\}$ and $\{x_n(t),\ 0 \le t \le T\}$.*

The following theorem is one of our main results.

THEOREM 2. *Let $F \in L_1(C[0,T],\ m_w)$. Then*

(3.4)
$$E[F(x)|\gamma_j(x) = \xi_j,\ j = 1, 2, \ldots] = E[F(x - x_\infty + \vec{\xi}_\infty)], \text{ and}$$
$$E[F(x)|\gamma_j(x) = \xi_j,\ j = 1, \ldots, n] = E[F(x - x_n + \vec{\xi}_n)].$$

Proof. Since $x - x_\infty$ and x_∞ are independent processes, and $\gamma_j(x)$ and $x - x_\infty$ are independent by Theorem 1, we may write

$$\begin{aligned} &E[F(x)|\gamma_j(x) = \xi_j,\ j = 1, 2, \ldots] \\ &= E[F((x - x_\infty) + x_\infty)|\gamma_j(x) = \xi_j,\ j = 1, 2, \ldots] \\ &= E_y\{E_x[F((y - y_\infty) + x_\infty)|\gamma_j(x) = \xi_j,\ j = 1, 2, \ldots]\}, \end{aligned}$$

where y is a standart Wiener process independent of x. Thus, we have

$$\begin{aligned}&E[F(x)|\gamma_j(x)=\xi_j,\ j=1,2,\dots]\\&=E_y\{F((y-y_\infty)+\vec{\xi}_\infty)\}=E[F(x-x_\infty+\vec{\xi}_\infty)],\end{aligned}$$

as $x_\infty=\vec{\xi}_\infty$ under the condition $\gamma_j=\xi_j,\ j=1,2,\dots$. The second formula of (3.4) follows by the same reasoning. □

COROLLARY 2. *Let $F\in L_1(C[0,T],\ m_w)$. If $\mathcal{H}=L_2[0,T]$, then $E[F(x)|\gamma_j(x)=\xi_j,\ j=1,2,\dots]=F(\vec{\xi}_\infty)$.*

Proof. This follows from (3.4) by the fact that if $\mathcal{H}=L_2[0,T]$, then $x(t)=\int_0^T I_{[0,t]}(s)ds(s)=\sum_{j=1}^\infty(\alpha_j,I_{[0,t]})\gamma_j(x)=x_\infty(t)$ for m_w - a.e. x. □

COROLLARY 3. *Let $F\in L_1(C[0,T],m_w)$. Then for every $B\in\mathcal{B}^n$,*

$$\int_{X_n^{-1}(B)}F(x)m_w(dx)=\int_B E[F(x-x_n+\vec{\xi}_n)P_{X_n}(d\vec{\xi}).$$

The above corollary is a simple consequence of the second formula in (3.4). In addition Theorem 4 on page 114 of [**2**] is a special case of Corollary 3 above with $B=\mathbb{R}^n$.

REMARKS.

(i) For each partition $\tau\equiv\tau_n=\{t_1,\dots,t_n\}$ of $[0,T]$ with $0=t_0<t_1<\dots<t_n=T$, let $X_\tau:C[0,T]\to\mathbb{R}^n$ be defined by $X_\tau(x)=(x(t_1),\dots,x(t_n))$. In [**8**], the current authors considered vector-valued conditional Wiener integrals of the type $E(F(x)|X_\tau(x)=\vec{\xi})$ for $F\in L_1(C[0,T],m_w)$. We note that these can be rewritten in the form

$$\begin{aligned}(3.5)\quad &E(F(x)|X_\tau(x)=\vec{\xi})=E(F(x)|x(t_j)=\xi_j,\ j=1,\dots,n)\\&=E(F(x)|x(t_j)-x(t_{j-1})=\xi_j-\xi_{j-1},\ j=1,\dots,n)\\&=E\left(F(x)|\int_0^T\alpha_j(t)dx(t)=\frac{\xi_j-\xi_{j-1}}{\sqrt{t_j-t_{j-1}}},\ j=1,\dots,n\right)\end{aligned}$$

where $\xi_0=t_0=0$ and

$$(3.6)\qquad \alpha_j(t)=I_{[t_{j-1},t_j]}(t)/\sqrt{t_j-t_{j-1}},\ j=1,\dots,n.$$

Since $\{\alpha_1(t), \dots, \alpha_n(t)\}$ is obviously an orthonormal set of functions in $L_2[0,T]$, the vector-valued conditional Wiener integral $E(F(x)|X_\tau(x) = \vec{\xi})$ is a special case of the general conditional Wiener integrals of the type $E(F(x)|X_n(x) = \vec{\xi})$ considered in this paper. Thus the conditional Wiener integrals that occur in [**4**], [**8**], [**13**] and [**14**] are all special cases of those of the type $E(F(x)|X_n(x) = \vec{\xi})$ for appropriate n and $\alpha_1, \dots, \alpha_n$.

It is also interesting to note that for each $x \in C[0,T]$ the polygonal function $[x]$ defined by

$$[x](t) = x(t_{j-1}) + \frac{t - t_{j-1}}{t_j - t_{j-1}}(x(t_j) - x(t_{j-1})),$$

$$t_{j-1} \le t \le t_j,\ j = 1, \dots, n$$

has another representation, namely

$$[x](t) = x_n(t),\ 0 \le t \le T$$

where the α_j's are given by (3.6) and $x_n(t)$ is given by (3.2). The formula in [**8**], p.385, corresponding to (3.4) above is

$$E(F(x)|X_\tau(x) = \vec{\xi}) = E[F(x - [x] + [\vec{\xi}])]$$

where for $\vec{\xi} \in \mathbb{R}^n$, $[\vec{\xi}](t)$ is the polygonal function

$$\begin{aligned}[\vec{\xi}](t) &= \xi_{j-1} + \frac{t - t_{j-1}}{t_j - t_{j-1}}(\xi_j - \xi_{j-1}),\ t_{j-1} \le t \le t_j,\ j = 1, \dots, n \\ &= \vec{\xi}_n(t)\end{aligned}$$

where the α_j's are given by (3.6) and $\vec{\xi}_n(t)$ is given by (3.2).

(ii) Thanks to the referee's suggestions, this paper has gone through a number of improvements. The expressions given by (3.2) and (3.3) were suggested by the referee. This in turn, strengthened Theorems 1 and 2. Another suggestion made by the referee was the possibility of generalizing Theorem 2 to other Gaussian processes. This question is perhaps best handled by using the representation of a Gaussian process using Wiener processes; see [**7**] and example 3 below.

We close this section with some examples which illustrate that formulas (3.4) are indeed very useful and easy to apply. In particular, the third example deals with the Ornstein-Uhlenbeck process to show that our formulas can be applied to other useful Gaussian processes.

EXAMPLE 1. For $x \in C[0,T]$ let $F(x) = \int_0^T x^2(t)dt$. Then using (3.4) we obtain

$$\begin{aligned} E&\left[\int_0^T x^2(t)dt | X_\alpha(x) = \vec{\xi}\right] \\ &= E\left[\int_0^T (x(t) - x_n(t) + \vec{\xi}_n(t))^2 dt\right] \\ &= \int_0^T E\left[(x(t) - x_n(t))^2 + (\vec{\xi}_n(t))^2 + 2\vec{\xi}_n(t)(x(t) - x_n(t))\right] dt. \end{aligned}$$

Since $x - x_n$ and x_n are independent by Corollary 1, $E[x_n(t)(x(t) - x_n(t))] = 0$, and using (2.2) and the fact that $E[x(s)x(t)] = \min\{s,t\}$, we obtain

$$E\left[\int_0^T x^2(t)dt | X_n(x) = \vec{\xi}\right] = \int_0^T \left\{t + (\vec{\xi}_n(t))^2 - \sum_{j=1}^n \beta_j^2(t)\right\} dt.$$

In particular, if $n = 1$ and $\alpha(s) = 1/\sqrt{T}$, we see that

$$\begin{aligned} E\left[\int_0^T x^2(t)dt | X_1(x) = \xi\right] &= E\left[\int_0^T x^2(t)dt | x(T) = \xi\right] \\ &= \int_0^T \left\{t + \frac{\xi^2 t^2}{T^2} - \frac{t^2}{T}\right\} dt = \frac{T^2}{6} + \frac{\xi^2 T}{3} \end{aligned}$$

which agrees with the results in [**4**], [**8**] and [**13**].

EXAMPLE 2. For $x \in C[0,T]$ let $F(x) = \exp\left\{\int_0^T x(t)dt\right\}$. Then

$$\begin{aligned} E&\left[\exp\left\{\int_0^T x(t)dt\right\} | X_n(x) = \vec{\xi}\right] \\ &= E\left[\exp\left\{\int_0^T (x(t) - x_n(t) + \vec{\xi}_n(t))dt\right\}\right] \\ &= \exp\left\{\int_0^T \vec{\xi}_n(t)dt\right\} E\left[\exp\left\{\int_0^T (x(t) - x_n(t))dt\right\}\right]. \end{aligned}$$

In particular, if we choose the complete orthonormal cosine sequence $\alpha_j(t) = \sqrt{2/T}\cos[(j-1/2)\pi t/T]$, $j = 1,2,\ldots$, on $[0,T]$, then it is well known (see Shepp [**9**], p.325) that the corresponding $x_n(t)$ converges to $x(t)$ uniformly in t with probability one, and for each $u \in C[0,T]$,

$$\sum_{j=1}^{\infty}\int_0^T\left\{\int_0^t \alpha_j(s)ds\int_0^T \alpha_j(s)du(s)\right\}dt = \int_0^T u(t)dt.$$

Thus

$$\lim_{n\to\infty} E\left[\exp\left\{\int_0^T x(t)dt\right\}|X_n(x) = X_n(u)\right] = \exp\left\{\int_0^T u(t)dt\right\}$$

as expected. Since the orthonormal cosine sequence given above is complete on $[0,T]$, Corollary 2 can be applied to get

$$E\left[\exp\left\{\int_0^T x(t)dt\right\}|\gamma_j(x) = \gamma_j(u),\ j = 1,2,\ldots\right] = \exp\left\{\int_0^T u(t)dt\right\}$$

for a.e. $u \in C[0,T]$.

EXAMPLE 3. Consider the Ornstein-Uhlenbeck process $y(t)$ with mean zero and covariance function $R(s,t) = \sigma^2\exp\{-\beta|t-s|\}$ where $\beta > 0$. If we take $\sigma = \beta = 1$ for convenience, then $y(t)$ can be expressed in terms of the standart Wiener process $x(t)$ (see p.414 of [**7**]),

$$y(t) = e^{-t}x(e^{2t}),\ 0 \le t \le T. \tag{3.7}$$

Suppose $F(y)$ is an integrable function of y. Let $\tau = \{0 = t_0, t_1, \ldots, t_n = T\}$ be a partition of $[0,T]$. Then, the conditional expectation

$$E[F(y)|y(t_j) = \xi_j,\ j = 0,1,\ldots,n]$$

can be expressed as a non-conditional expectation by utilizing (3.7). Since $e^t y(t) = x(e^{2t})$ and $x(\cdot)$ has independent increments, we write

$$\begin{aligned}&E[F(y)|y(t_j) = \xi_j,\ j = 0,1,\ldots,n]\\ &= E[F(y)|e^{t_j}y(t_j) - e^{t_{j-1}}y(t_{j-1}) = e^{t_j}\xi_j - e^{t_{j-1}}\xi_{j-1},\ j = 0,\ldots,n]\end{aligned}$$

where $y(t_{-1}) = \xi_{-1} = 0$.

Define $(y_n)(t)$ by

$$(y_n)(t) = e^{-t}\left[e^{t_{j-1}}y(t_{j-1}) + \frac{e^{2t} - e^{2t_{j-1}}}{e^{2t_j} - e^{2t_{j-1}}}(e^{t_j}y(t_j) - e^{t_{j-1}}y(t_{j-1}))\right]$$

for $t_{j-1} \le t \le t_j$, $j = 1, \dots, n$.

Similarly, define $(\vec{\xi_n})(t)$ by

$$(\vec{\xi_n})(t) = e^{-t}\left[e^{t_{j-1}}\xi_{j-1} + \frac{e^{2t} - e^{2t_{j-1}}}{e^{2t_j} - e^{2t_{j-1}}}(e^{t_j}\xi_j - e^{t_{j-1}}\xi_{j-1})\right]$$

for $t_{j-1} \le t \le t_j$, $j = 1, \dots, n$.

Then, $(y_n)(t_j) = y(t_j)$ and $(\vec{\xi_n})(t_j) = \xi_j$ at each $t_j \in \tau$. Furthermore, (y_n) and $y - (y_n)$ are independent processes as one can easily check using the covariance function of y. Thus, we conclude that

$$E[F(y)|y(t_j) = \xi_j,\ j = 0, 1, \dots, n] = E[F(y - (y_n) + (\vec{\xi_n}))].$$

4. Conditional expectation of functions involving stochastic integrals. Using the same notation as in section 3 above, for $h \in L_2[0, T]$ let

$$(4.1)\qquad \begin{aligned} h_{(n)}(t) &= \mathcal{P}_n h(t) = \sum_{j=1}^{n}(h, \alpha_j)\alpha_j(t) \text{ and} \\ h_{(\infty)}(t) &= \mathcal{P}_\infty h(t) = \sum_{j=1}^{\infty}(h, \alpha_j)\alpha_j(t) \end{aligned}$$

Then, we have the following:

LEMMA 1. *Let $h \in L_2[0, T]$. Then*

$$(4.2)\qquad \int_0^T h(t)h_{(n)}(t)dt = \int_0^T h_{(n)}^2(t)dt = ||h_{(n)}||^2 = \sum_{j=1}^{n}(h, \alpha_j)^2,$$

and

$$(4.3)\qquad ||h - h_{(n)}||^2 = ||h||^2 - ||h_{(n)}||^2.$$

Obviously, the above formulas hold when $n = \infty$, and $||h - h_{(\infty)}|| = 0$ if $\mathcal{H} = L_2[0, T]$.

Our next theorem gives an interesting relationship involving h, $h_{(n)}$, x and x_n that is very useful in computing conditional and ordinary expectations of functions involving the stochastic integral $\int_0^T h(t)dx_n(t)$.

THEOREM 3. *Let $h \in L_2[0,T]$. Then for each $x \in C[0,T]$*

$$\int_0^T h(t)dx_n(t) = \int_0^T h_{(n)}(t)dx(t) = \int_0^T h_{(n)}(t)dx_n(t) \tag{4.4}$$

The formula also holds for $n=\infty$ if we consider $\int_0^T h(t)dx_\infty(t) = \sum_{j=1}^{\infty} \gamma_j(x)(h,\alpha_j)$.

Proof. Using 3.1, 3.2, 4.1 and the fact that the α_j's are orthonormal, it is quite easy to show that for each $x \in C[0,T]$, each of the stochastic integrals in 4.4 equals the expression

$$\sum_{j=1}^{n}(h,\alpha_j)\int_0^T \alpha_j(t)dx(t).$$

□

COROLLARY 4. *Let $h \in L_2[0,T]$. Then*

$$E\left[\exp\left\{-\int_0^T h(t)dx_n(t)\right\}\right] = \exp\left\{\frac{1}{2}||h_{(n)}||^2\right\}. \tag{4.5}$$

Proof. By 4.4 and a well known Wiener integration formula

$$\begin{aligned} E&\left[\exp\left\{-\int_0^T h(t)dx_n(t)\right\}\right] \\ &= E\left[\exp\left\{-\int_0^T h_{(n)}(t)dx(t)\right\}\right] \\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty} \exp\left\{-||h_{(n)}||u\right\}\exp\left\{-\frac{u^2}{2}\right\}du \\ &= \exp\left\{\frac{1}{2}||h_{(n)}||^2\right\}. \end{aligned}$$

□

THEOREM 4. *Let $h \in L_2[0,T]$ and assume that*

$$F(x) = f\left[\int_0^T h(t)dx(t)\right]$$

is in $L_1(C[0,T], m_w)$.

a). *If h is a linear combination of $\{\alpha_1, \dots, \alpha_n\}$, say $h(t) = c_1\alpha_1(t) + \dots + c_n\alpha_n(t)$ on $[0,T]$, then*

$$(4.6)\quad E\left[f\left[\int_0^T h(t)dx(t)\right] | X_n(x) = \vec{\xi}\right] = f(c_1\xi_1 + \dots + c_n\xi_n).$$

b). *If $\{h, \alpha_1, \dots, \alpha_n\}$ is a linearly independent set of functions in $L_2[0,T]$, then*

$$(4.7)\qquad E\left[f\left[\int_0^T h(t)dx(t)\right] | X_n(x) = \vec{\xi}\right]$$
$$= [2\pi(||h||^2 - ||h_{(n)}||^2)]^{-1/2}$$
$$\cdot \int_{-\infty}^{\infty} f(u) \exp\left\{ -\frac{\left(u - \int_0^T h(t)d\vec{\xi}_n(t)\right)^2}{2||h - h_{(n)}||^2} \right\} du.$$

Proof. a).In this case $h_{(n)}(t) \equiv h(t)$ and so by 3.4, 4.4 and 3.2,

$$E\left[f\left[\int_0^T h(t)dx(t)\right] | X_n(x) = \vec{\xi}\right]$$
$$= E\left[f\left[\int_0^T h(t)d\{x(t) - x_n(t) + \vec{\xi}_n(t)\}\right]\right]$$
$$= E\left[f\left[\int_0^T (h(t) - h_{(n)}(t))dx(t) + \int_0^T h(t)d\vec{\xi}_n(t)\right]\right]$$
$$= E\left[f\left[\int_0^T h(t)d\vec{\xi}_n(t)\right]\right]$$
$$= f\left[\int_0^T h(t)d\vec{\xi}_n(t)\right]$$
$$= f(c_1\xi_1 + \dots + c_n\xi_n).$$

b). In this case we use 3.4, 4.4, and a well known Wiener integration formula to obtain

$$E\left[f\left[\int_0^T h(t)dx(t)\right] |X_n(x) = \vec{\xi}\right]$$
$$= E\left[f\left[\int_0^T h(t)d\{x(t) - x_n(t) + \vec{\xi}_n(t)\}\right]\right]$$
$$= E\left[f\left[\int_0^T (h(t) - h_{(n)}(t))dx(t) + \int_0^T h(t)d\vec{\xi}_n(t)\right]\right]$$
$$= (2\pi)^{-1/2}\int_{-\infty}^{\infty} f\left(||h - h_{(n)}||u + \int_0^T h(t)d\vec{\xi}_n(t)\right)\exp\{-u^2/2\}du.$$

□

In Theorem 4 above the two extreme cases occur when $h \equiv \alpha_j$ for some j or when h is orthogonal to all the α_j's.

COROLLARY 5. *Let h, F and f be as in Theorem 4. Then*

$$E\left[f\left[\int_0^T \alpha_j(t)dx(t)\right] |X_n(x) = \vec{\xi}\right] = f(\xi_j), \tag{4.8}$$

while if $\{h, \alpha_1, \dots, \alpha_n\}$ is an orthogonal set of functions in $L_2[0, T]$,

(4.9)
$$E\left[f\left[\int_0^T h(t)dx(t)\right] |X_n(x) = \vec{\xi}\right] = E\left[f\left[\int_0^T h(t)dx(t)\right]\right]$$
$$= [2\pi||h||^2]^{-1/2}\int_{-\infty}^{\infty} f(u)\exp\left\{-\frac{u^2}{2||h||^2}\right\}du.$$

Proceeding as above we obtain the following generalization of formula 4.9.

COROLLARY 6. *If $\{\phi_1, \dots, \phi_m, \alpha_1, \dots, \alpha_n\}$ is an orthonormal set of functions in $L_2[0, T]$ and if*

$$F(x) = f\left[\int_0^T \phi_1(t)dx(t), \dots, \int_0^T \phi_m(t)dx(t)\right]$$

is in $L_1(C[0,T], m_w)$, *then*

$$E\left(f\left[\int_0^T \phi_1(t)dx(t), \dots, \int_0^T \phi_m(t)dx(t)\right] | X_n(x) = \vec{\xi}\right)$$
$$= E\left[f\left[\int_0^T \phi_1(t)dx(t), \dots, \int_0^T \phi_m(t)dx(t)\right]\right]$$
$$= \left[\prod_{j=1}^m [2\pi]^{-1/2}\right] \int_{\mathbb{R}^m} f(u_1, \dots, u_m) \exp\left\{-\sum_{j=1}^m \frac{u_j^2}{2}\right\} d\vec{u}.$$

Our next corollary follows from the observations that $\int_0^T (h(t) - h_{(n)}(t))d\vec{\xi}_n(t) = 0$, and $(h - h_{(n)})_{(n)}(t) = 0$.

COROLLARY 7. *Let* h, F *and* f *be as in Theorem 4. Then*

$$E\left[f\left[\int_0^T h(t)d\{x(t) - x_n(t)\}\right] | X_n(x) = \vec{\xi}\right]$$
$$= E\left[f\left[\int_0^T \{h(t) - h_{(n)}(t)\}dx(t)\right] | X_n(x) = \vec{\xi}\right]$$
$$= E\left[f\left[\int_0^T \{h(t) - h_{(n)}(t)\}dx(t)\right]\right]$$
$$= [2\pi||h - h_{(n)}||^2]^{-1/2} \int_{-\infty}^{\infty} f(u) \exp\left\{-\frac{u^2}{2||h - h_{(n)}||^2}\right\} du.$$

Many interesting examples of conditional Wiener integrals can be obtained as special cases of the following theorem.

THEOREM 5. *Let* $g \in L_2[0,T]$. *Then*

(4.10)
$$E\left[\exp\left\{\int_0^T g(s)x(s)ds\right\} | X_n(x) = \vec{\xi}\right]$$
$$= \exp\left\{\sum_{j=1}^n \xi_j(g, \beta_j) + \frac{1}{2}\int_0^T \left[\int_s^T g(t)dt\right]^2 ds - \frac{1}{2}\sum_{j=1}^n (g, \beta_j)^2\right\}.$$

Proof. Using integration by parts it follows that

$$\int_0^T g(s)x(s)ds = \int_0^T \left[\int_s^T g(t)dt\right] dx(s)$$

and that

$$\int_0^T \left[\int_s^T g(t)dt\right] \alpha_j(s)ds = \int_0^T g(s)\beta_j(s)ds = (g, \beta_j).$$

Hence using (3.4) we obtain

$$\begin{aligned}
E&\left[\exp\left\{\int_0^T g(s)x(s)ds\right\} |X_n(x) = \vec{\xi}\right] \\
&= E\left[\exp\left\{\int_0^T \left[\int_s^T g(t)dt\right] d(x(s) - x_n(s) + \vec{\xi}_n(s))\right\}\right] \\
&= \exp\left\{\sum_{j=1}^n \xi_i \int_0^T \left[\int_s^T g(t)dt\right] \alpha_j(s)ds\right\} \\
&\quad \cdot E\left[\exp\left\{\int_0^T \left[\int_s^T g(t)dt\right] dx(s)\right.\right. \\
&\quad \left.\left. - \sum_{j=1}^n \gamma_j(x) \int_0^T \left[\int_s^T g(t)dt\right] \alpha_j(s)ds\right\}\right] \\
&= \exp\left\{\sum_{j=1}^n \xi_j(g, \beta_j)\right\} \\
&\quad \cdot E\left[\exp\left\{\int_0^T \left[\int_s^T g(t)dt - \sum_{j=1}^n (g, \beta_j)\alpha_j(s)\right] dx(s)\right\}\right] \\
&= \exp\left\{\sum_{j=1}^n \xi_j(g, \beta_j) + \frac{1}{2}\int_0^T \left[\int_s^T g(t)dt - \sum_{j=1}^n (g, \beta_j)\alpha_j(s)\right]^2 ds\right\}
\end{aligned}$$

from which 4.10 follows. □

COROLLARY 8. *Let $g(s) \equiv 1$ and let the α_j's be given by 3.6.*

Then

$$E\left[\exp\left\{\int_0^T x(s)ds\right\}|X_n(x)=\vec{\xi}\right]$$
$$=\exp\left\{\frac{T^3}{6}+\frac{1}{2}\sum_{j=1}^n(\xi_j+\xi_{j-1})(t_j-t_{j-1})\right.$$
$$\left.-\frac{1}{8}\sum_{j=1}^n(t_j-t_{j-1})(t_j+t_{j-1})^2\right\}.$$

COROLLARY 9. *Let* $n=1$ *and* $\alpha_1(s)\equiv 1/\sqrt{T}$. *Then*

$$E\left[\exp\left\{\int_0^T g(s)x(s)ds\right\}|x(T)=\xi\right]$$
$$=\exp\left\{\frac{\xi}{T}\int_0^T tg(t)dt+\frac{1}{2}\int_0^T\left[\int_s^T g(t)dt\right]^2 ds-\frac{1}{2T}\left[\int_0^T tg(t)dt\right]^2\right\},$$
$$E\left[\exp\left\{\int_0^T sx(s)ds\right\}|x(T)=\xi\right]=\exp\left\{\frac{\xi T^2}{3}+\frac{T^5}{90}\right\},$$

and

$$E\left[\exp\left\{\int_0^T x(s)ds\right\}|x(T)=\xi\right]=\exp\left\{\frac{\xi T}{2}+\frac{T^3}{24}\right\}.$$

5. Translation of generalized conditional Wiener integrals. The Cameron-Martin Theorem [**3**], [**11**] states that if $x_0(t)=\int_0^t h(s)ds$ for all $t\in[0,T]$ with $h\in L_2[0,T]$, and if T_1 is the transformation from $C[0,T]$ into itself defined by

$$T_1(x)=x+x_0 \text{ for } x\in C[0,T],$$

then for any Wiener integrable function F on $C[0,T]$ and any Wiener measurable set Γ

$$\int_\Gamma F(y)m_w(dy)=\int_{T_1^{-1}(\Gamma)}F(x+x_0)J(x_0,x)m_w(dx) \tag{5.1}$$

where

$$J(x_0,x)=\exp\left\{-\frac{1}{2}||h||^2-\int_0^T h(t)dx(t)\right\}. \tag{5.2}$$

In particular, if $\Gamma = C[0,T]$, then 5.1 becomes:

$$E[F(y)] = E[F(x+x_0)J(x_0,x)]. \tag{5.3}$$

In [**14**], Yeh gives a conditional version of 5.3 which states that

$$E[F(y)|y(T)=\xi] = E\left[F(y)|\int_0^T dy(t) = \xi\right]$$
$$= E[F(x+x_0)J(x_0,x)|x(T) = \xi - x_0(T)]\exp\left\{-\frac{x_0^2(T)}{2T} + \frac{\xi x_0(T)}{T}\right\}.$$

Our next theorem is a generalized conditional version of 5.3.

THEOREM 6. *Let $h \in L_2[0,T]$ and let $x_0(t) = \int_0^t h(s)ds$ for $t \in [0,T]$. Let $F \in L_1(C[0,T], m_w)$ and let the α_j's be as in Section 2. Then*

$$\begin{aligned} E[F(y)|X_\alpha(y) = \vec{\xi}] \\ &= E[F(x+x_0)J(x_0,x)|X_n(x+x_0) = \vec{\xi}] \\ &\cdot \exp\left\{\int_0^T h(t)d\vec{\xi}_n(t) - \frac{1}{2}||h_{(n)}||^2\right\}\end{aligned} \tag{5.4}$$

where $J(x_0,x)$ is given by 5.2 *and $h_{(n)}(t)$ is given by* 4.1. *The result holds for $n=\infty$ as well.*

Proof. By 3.4 we see that

$$E[F(y)|X_n(y) = \vec{\xi}] = E[F(y - y_n + \vec{\xi}_n)]. \tag{5.5}$$

Using 5.3 and noting that $(x+x_0)_n = x_n + (x_0)_n$, we have

$$E[F(y - y_n + \vec{\xi}_n)] = E[F(x + x_0 - x_n - (x_0)_n + \vec{\xi}_n)J(x_0,x)]. \tag{5.6}$$

Next we rewrite $J(x_0, x)$ in the form

(5.7)
$$\begin{aligned} J(x_0, x) = \exp&\left\{-\frac{1}{2}||h||^2\right\} \\ &\cdot \exp\left\{-\int_0^T h(t)d(x(t) - x_n(t) + \vec{\xi}_n(t) - (x_0)_n(t))\right\} \\ &\cdot \exp\left\{-\int_0^T h(t)dx_n(t)\right\} \\ &\cdot \exp\left\{\int_0^T h(t)d(\vec{\xi}_n(t) - (x_0)_n(t))\right\}. \end{aligned}$$

Using 4.1 we see that

(5.8)
$$\int_0^T h(t)d(x_0)_n(t) = \int_0^T h_{(n)}^2(t)dt = ||h_{(n)}||^2.$$

Since $x_n(t)$ and $x(t) - x_n(t)$ are independent processes on $[0, T]$ by Corollary 1, $\exp\left\{-\int_0^T h(t)dx_n(t)\right\}$ and

$$\begin{aligned} F(x + x_0 - x_n - (x_0)_n + \vec{\xi}_n)& \\ \cdot \exp&\left\{-\int_0^T h(t)d(x(t) - x_n(t) + \vec{\xi}_n(t) - (x_0)_n(t))\right\} \end{aligned}$$

are also independent. Thus using 5.7, 4.5 and 5.8,

(5.9)
$$\begin{aligned} &E[F(x + x_0 - x_n - (x_0)_n + \vec{\xi}_n)J(x_0, x)] \\ &= E\Bigg[F(x + x_0 - x_n - (x_0)_n + \vec{\xi}_n) \\ &\cdot \exp\left\{-\int_0^T h(t)d(x(t) - x_n(t) + \vec{\xi}_n(t) - (x_0)_n(t))\right\}\Bigg] \\ &\cdot \exp\left\{-\frac{1}{2}||h||^2 + \frac{1}{2}||h_{(n)}||^2 + \int_0^T h(t)d\vec{\xi}_n(t) - ||h_{(n)}||^2\right\}. \end{aligned}$$

Therefore, by using 5.9 and 3.4 we obtain

$$
\begin{aligned}
&E[F(x + x_0 - x_n - (x_0)_n + \vec{\xi}_n)J(x_0, x)] \\
&\quad = E\left(\left[F(x + x_0)\exp\left\{-\int_0^T h(t)d(x(t))\right\}\right] |X_\alpha(x + x_0) = \vec{\xi}\right) \\
&\quad\quad \cdot \exp\left\{-\frac{1}{2}||h||^2 + \int_0^T h(t)d\vec{\xi}_n(t) - \frac{1}{2}||h_{(n)}||^2\right\} \\
&\quad = E\left([F(x + x_0)J(x_0, x)] |X_\alpha(x + x_0) = \vec{\xi}\right) \\
&\quad\quad \cdot \exp\left\{\int_0^T h(t)d\vec{\xi}_n(t) - \frac{1}{2}||h_{(n)}||^2\right\}.
\end{aligned}
$$

This together with 5.6 and 5.5 yields 5.4. The case $n = \infty$ follows by the martingle convergence theorem. □

REMARK. By choosing the α_j's as in 3.6, we see that Theorem 4 on page 391 of [**8**] is a Corollary of Theorem 6 above.

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REFERENCES

[1] L. Breiman, *Probability,* Addison-Wesley, Reading, Mass. 1968.

[2] R.H. Cameron, *A "Simpsons Rule" for the numerical evaluation of Wiener's integrals in function space,* Duke Math. J., **18** (1951), 111-130.

[3] R.H. Cameron and W.T. Martin, *Transformation of Wiener integrals under translations,* Ann. Math., **45** (1944), 386-396.

[4] K.S. Chang and J.S. Chang, *Evaluation of some conditional Wiener integrals,* Bull. Korean Math. Soc., **21** (1984), 99-106.

[5] J.L. Doob, *Stochastic processes,* Wiley, New York, 1953.

[6] M. Kac, *Probability and Related Topics in Physical Sciences,* Interscience Publishers, New York, 1959.

[7] C. Park, *Representations of Gaussian processes by Wiener processes,* Pacific J. Math., **94** (1981), 407-415.

[8] C. Park and D.L. Skoug *A simple formula for conditional Wiener integrals with applications,* Pacific J. Math., **135** (1988), 381-394.

[9] L.A. Shepp *Radon-Nikodym derivative of Gaussian measures,* Ann. Math. Stat., **37** (1966), 321-354.

[10] H.G. Tucker, *A graduate course in probability,* Academic Press, New York, 1967.

[11] J. Yeh, *Stochastic Processes and the Wiener Integral,* Marcel Dekker, New York, 1973.

[12] ———, *Stochastic processes and the Weiner integral,* Marcel Dekker, New York, 1973.

[13] ———, *Inversion of conditional Wiener integrals,* Pacific J. Math., **59** (1975), 623-638.

[14] ———, *Transformation of conditional Wiener integrals under translation and the CAmeron-MArtin translation theorem,* Tôhoku Math. Journ., **30** (1978), 505-515.

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