PACIFIC JOURNAL OF MATHEMATICS
Vol. 167, No. 2, 1995

L^p ESTIMATES FOR OPERATORS ASSOCIATED TO FLAT CURVES WITHOUT THE FOURIER TRANSFORM

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The purpose of this paper is to provide new proofs to known theorems on the L^p boundedness of the maximal function and Hilbert transform corresponding to curves in R^n which are "infinitely flat" at the origin. The old proofs use the Fourier transform in a crucial way. The present proofs avoid the Fourier transform and hence at least have the potential of being used in more general situations.

1. Introduction. For each $x \in R^n$ let $\Gamma(x,t) = \Gamma_x(t)$ be a smooth curve in R^n with $\Gamma(x,0) = x$. For $f \in C_0^\infty(R^n)$ we define

$$M_\Gamma f(x) = \sup_{0<r\leq 1} \frac{1}{r}\int_0^r |f(\Gamma(x,t))|\, dt,$$

and

$$H_\Gamma f(x) = \int_{-1}^{1} f(\Gamma(x,t))\frac{dt}{t}.$$

In recent years there has been much attention devoted to the study of L^p bounds for M_Γ and H_Γ. In particular positive results have been obtained under a hypothesis that a certain type of curvature does not vanish to infinite order. See [**C1**] and [**CNSW**]. In the case that $\Gamma(x,t) = x + \gamma(t)$, the condition means that the vectors $\gamma'(0), \gamma''(0), \ldots,$ span R^n. More recently, a great deal of effort has been directed towards obtaining L^p bounds in the special case that $\Gamma(x,t) = x + \gamma(t)$, but where the curvature condition is not satisfied. The proofs of these results depend heavily on the Fourier transform.

The use of the Fourier transform is not viable in the general setting i.e. where $\Gamma(x,t)$ is not of the special form $x + \gamma(t)$. Thus it seems that the first step in obtaining results for general $\Gamma(x,t)$ in the case that the curvature condition is violated is to find proofs of positive results in the setting that $\Gamma(x,t) = x + \partial t$ without using the Fourier transform.

In this paper we give a new proof of Theorem 5.2 in [**CVWW**] which does not use the Fourier transform. Our present proof uses the "TT^*" method. This method depends upon the fact that if $d\mu$ is a measure supported on a piece of a curve, then $d\mu * \ldots * d\mu$ might have an L^1 density with some smoothness. This idea was used by Stein and Fefferman in studying the restriction problem. See [**F**]. In the context of Hilbert transforms and maximal functions related to curves, this idea first appears in [**NSW1**].

This "smoothing" principle was first shown to be applicable in great generality by Christ [**C2**]. See also [**RS**], [**C1**], and [**CNSW**]. Our proof here follows the general ideas of [**C2**] in combination with the dilations introduced in [**CCVWW**] and [**CVWW**].

The setting of Theorem 5.2 in [**CVWW**] is as follows. Let $\Gamma(t) = (t, \gamma_2(t), \ldots, \gamma_n(t)) : \mathbb{R} \to \mathbb{R}^n$ be an odd curve of class C^n and $\Gamma(0) = 0$. The condition imposed on $\Gamma(t)$ are expressed in terms of $D_j(t)$ and $N_j(t)$, $1 \le j \le n$, where for $t > 0$,

$$D_j(t) = \det \begin{pmatrix} 1 & \gamma_2'(t) & \cdots & \gamma_j'(t) \\ 0 & \gamma_2''(t) & \cdots & \gamma_j''(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_2^{(j)}(t) & \cdots & \gamma_j^{(j)}(t) \end{pmatrix}$$

and

$$N_j(t) = \det \begin{pmatrix} t & \gamma_2(t) & \cdots & \gamma_j(t) \\ 1 & \gamma_2'(t) & \cdots & \gamma_j'(t) \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_2^{(j-1)}(t) & \cdots & \gamma_j^{(j-1)}(t) \end{pmatrix}$$

Set

$$h_j(t) = \frac{N_j(t)}{D_{j-1}(t)},\ 1 \le j \le n$$

where $D_0(t) = 1$.

THEOREM [**CVWW**]. *Suppose*

$$D_j(t) > 0,\ 1 \le j \le n,\ t > 0,$$

and

$$h_j'(t) \geq \epsilon \frac{h_j(t)}{t},\ 2 \leq j \leq n,\ t > 0$$

for some $\epsilon > 0$. *Then*

$$||M_\Gamma f||_p \leq C_p ||f||_p,\ 1 < p \leq \infty$$

and

$$||H_\Gamma f||_p \leq C_p ||f||_p,\ 1 < p < \infty.$$

We will prove the theorem for the maximal function first. The proof of the theorem for the Hilbert transform is similar. After the study of the maximal function, we will indicate the modification needed for the Hilbert transform. As in [**CVWW**], our proof will use certain dilations and a Littlewood- Paley decomposition. We need to recall from [**NVWW**], Lemma 2, that $D_j(t) > 0$ implies that $h_j(t) > 0$, $1 \leq j \leq n$, and hence $h_j'(t)$ is positive. The dilations that we need are defined in terms of the following differential operators. For $1 \leq j \leq n$, set

$$R_j f(t) = \left(\frac{f(t)}{h_j(t)}\right)' \frac{h_j^2(t)}{h_j'(t)}.$$

Note that these operators are well-defined by the above remarks. The dilations are given by

$$\delta(t) = (\Gamma(t), R_1\Gamma(t), \ldots, R_{n-1}R_{n-2}\ldots R_1\Gamma(t)).$$

$\delta(t)$ are lower triangular matrices and the j^{th} diagonal entry is $h_j(t)$. Furthermore for $s \geq t$, we have

$$||\delta^{-1}(s)\delta(t)|| \leq C \left(\frac{t}{s}\right)^\epsilon \tag{1}$$

for some $\epsilon > 0$. This is the content of Proposition 4.2 in [**CVWW**].

The Littlewood-Paley decomposition is defined in terms of the family of invertible matrices

$$A_j = \delta(\lambda^{-j}),\ j \in \mathbf{Z}$$

where $\lambda > 1$ is chosen so large that for some α independent of $j \in \mathbf{Z}$. This easily follows from (1). Next choose $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi = 1$ and $\psi(x) = \psi(-x)$. Set

$$\Psi_j(x) = \frac{1}{\det A_{j+1}}\psi(A_{j+1}^{-1}x) - \frac{1}{\det A_j}\psi(A_j^{-1}x).$$

The following Littlewood-Paley inequalities can be found in [**CVWW**], Theorem 2.1.

$$\sum_{j\in\mathbf{Z}} \Psi_j * f = f, \tag{3}$$

and

$$\left\|\left(\sum_{j\in\mathbf{Z}} |\Psi_j * f|^2\right)^{1/2}\right\|_p \leq C_p\|f\|_p,\ 1 < p < \infty. \tag{4}$$

The sum in (3) converges in L^p, $1 < p < \infty$.

The proof of (3) in [**CVWW**] uses the Fourier transform, but it is easy to modify the argument so as not to use the Fourier transform. The proof of (4) in [**CVWW**] uses the Fourier transform only to show that (4) is valid for $p = 2$. However this argument can be replaced by applying the Cotlar-Stein lemma. In fact the estimate

$$\|\Psi_k * \Psi_j * f\|_2 \leq C2^{-\epsilon|k-j|}\|f\|_2 \text{ for some } \epsilon > 0 \tag{5}$$

is a special case of Lemma 5 below. (5) together with the Cotlar-Stein lemma implies (4) for $p = 2$.

2. The main estimate. The heart of our argument for the L^p boundedness of M_Γ is the following L^∞ estimate.

LEMMA 1. *Let*

$$Q = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n | 1 \leq y_k \leq \lambda,\ 1 \leq k \leq n\}$$

and $\Gamma_j(t) = A_j^{-1}\Gamma(\lambda^{-j}t)$. *Also let*

$$\varphi_j(y) = \Gamma_j(y_1) - \Gamma_j(y_2) + \dots + (-1)^{n+1}\Gamma_j(y_n)$$
$$= \sum^{n} (-1)^{k+1}\Gamma_j(y_k)$$

Set

$$I_h(f)(x) = \int_Q [f(x+h+\varphi_j(y)) - f(x+\varphi_j(y))]dy.$$

Then

$$||I_h(f)||_\infty \leq C|h|^\epsilon ||f||_\infty \tag{6}$$

for some $\epsilon > 0$.

Proof. Let $Z = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n | y_j = y_k \text{ for some } j \neq k\}$ and consider a Whitney decomposition of

$$Q \setminus Z = \bigcup_{\substack{l>0 \\ 1\leq m \leq C2^{(n-1)l}}} Q_{lm}$$

where Q_{lm} is a cube such that

$$\text{diameter}(Q_{lm}) = \sqrt{n}2^{-l} \approx \text{distance}(Q_{lm}, Z).$$

Let $\{\psi_{lm}\}$ be a partition of unity with respect to $\{Q_{lm}\}$ such that

$$||\partial^\alpha \psi_{lm}||_\infty \leq C_\alpha 2^{|\alpha| l}, \ \forall \alpha. \tag{7}$$

See [**S**, pp. 167-170]. Thus

$$I_h(f)(x) = \sum_{\substack{l\geq 0 \\ 1\leq m \leq C2^{(n-1)l}}} \int_{Q_{lm}} \psi_{lm}(y)[f(x+h+\varphi_j(y)) - f(x+\varphi_j(y))]dy.$$

We will need an estimate on the Jacobian of φ_j,

$$J_{\varphi_j} = \det(\Gamma_j'(y_1), -\Gamma_j'(y_2), \ldots, (-1)^{n+1}\Gamma_j'(y_n)).$$

Set

$$\Omega_k(y) = ((-1)^{k+1}\omega_{1k}(y), (-1)^{k+2}\omega_{2k}(y), \ldots, (-1)^{n+k}\omega_{nk}(y))$$

where $\omega_{rk}(y)$ denotes the (r,k) minor of the matrix and note that for each $k = 1, 2, \ldots, n$, $J_{\varphi_j} = (-1)^{k+1}\Gamma_j'(y_k) \cdot \Omega_k(y)$. The following estimate is a corollary of results obtained in [**CVWW**]. There is an $\epsilon > 0$ such that for each $k = 1, 2, \ldots, n$ and $y \in Q_{lm}$,

$$(8) \quad |J_{\varphi_j}(y)| \geq \epsilon 2^{-(n-1)l}|\Omega_k(y)| = \epsilon 2^{-(n-1)l}\left(\sum_{i=1}^{n} |\omega_{ik}(y)|^2\right)^{1/2}$$

In fact suppose that

$$Q_{lm} = \{(y_1, \ldots, y_n) \in \mathbb{R}^n | a_p \leq y_p \leq b_p,\ p = 1, \ldots, n\}.$$

For a fixed $\bar{y} = (y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$ where $a_p \leq y_p \leq b_p$, $p \neq k$, set $f(t) = \Gamma_j(t) \cdot \omega$ where $\omega = (-1)^{k+1}\Omega_k(\bar{y})$ and note that (8) can be written as

$$(9) \qquad |f'(s)| \geq \epsilon 2^{-(n-1)l}|\omega|$$

for $a_k \leq s \leq b_k$. The interval $[1, \lambda]$ can be divided into a bounded number of subintervals such that on each subinterval, we have the estimate

$$(10) \qquad |f'(s) - f'(t)| \geq \epsilon |s - t|^{n-1}|\omega|$$

for some $\epsilon > 0$. (10) is implicitly contained in the proof of Proposition 3.1 of [**CVWW**].

Furthermore $f'(t)$ has at most $n-1$ zeros and $f''(t)$ has at most $n-2$ zeros. This is the content of Lemma 3 of [**NVWW**]. However we know that $f'(s) = 0$ when $s = y_p$, $p \neq k$ and so f' has exactly $n-1$ zeros. Therefore f'' has exactly $n-2$ zeros. Since Q_{lm} is a Whitney cube, the zeros of f' are at least a distance of 2^{-l} from the interval $[a_k, b_k]$. Also the monotonicity of f' changes exactly once between two consecutive zeros of f'. Thus for a fixed s, $a_k \leq s \leq b_k$, there is a closest zero y_p of f' to s such that the monotonicity of f' does not change between y_p and s. Therefore we may apply (10) on certain subintervals $[c, d]$ (there are only a bounded number of such subintervals on which (10) applies) between y_p and s to obtain

$$\begin{aligned} |f'(s)| &= |f'(s) - f'(y_p)| \geq |f'(c) - f'(d)| \\ &\geq \epsilon |c - d|^{n-1}|\omega|. \end{aligned}$$

Since $|s - y_p| \geq 2^{-l}$, there is at least one such subinterval $[c, d]$ such that $|d - c| \geq \epsilon 2^{-l}$ for some $\epsilon > 0$. This implies (9) and thus (8).

From the fact that J_{φ_j} never vanishes on Q_{lm}, we see that φ_j is $1 - 1$ on Q_{lm}. In fact note that for any two distinct points $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in Q_{lm},

$$\varphi_j(x) - \varphi_j(y) = \sum_{k=1}^{n} (-1)^{k+1}(\Gamma_j(x_k) - \Gamma_j(y_k)) \stackrel{def}{=} \sum_{k=1}^{n} v_k.$$

Suppose that $x_k \neq y_k$ for all $k = 1, 2, \dots, n$. The general case will then follow from an induction argument. Note that

$$\begin{aligned}\det(v_1, \dots, v_n) &= \int_{y_n}^{x_n} \cdots \int_{y_1}^{x_1} \det(\Gamma_j'(t_1), \\ &\quad - \Gamma_j'(t_2), \dots, (-1)^{n+1}\Gamma_j'(t_n))\, dt_1 \cdots dt_n \\ &= \int_{y_n}^{x_n} \cdots \int_{y_1}^{x_1} J_{\varphi_j}(t_1, \dots, t_n)\, dt_1 \cdots dt_n \\ &=^{+}_{-} \int_{\tilde{Q}} J_{\varphi_j}(t)\, dt\end{aligned}$$

where $\tilde{Q}$ is some subset of Q_{lm} of positive measure and the choice of $\pm$ depends on the number of changes of sign of $\{x_k - y_k\}_{k=1}^{n}$. Thus $\det(v_1, \dots, v_n) \neq 0$ and so

$$\varphi_j(x) - \varphi_j(y) = \sum_{k=1}^{n} v_k \neq 0.$$

For a fixed (l, m), $l \geq 0$ and $1 \leq m \leq C2^{(n-1)l}$, consider

$$\begin{aligned}I_{lm}(f)(x) &= \int_{Q_{lm}} \psi_{lm}(y)[f(x + h + \varphi_j(y)) - f(x + \varphi_j(y))]\, dy \\ &= \int_{Q_{lm}} \psi_{lm}(y) f(x + h + \varphi_j(y))\, dy \\ &\quad - \int_{Q_{lm}} \psi_{lm}(y) f(x + \varphi_j(y))\, dy\end{aligned}$$

and note that

$$I_h(f)(x) = \sum_{\substack{l \geq 0 \\ 1 \leq m \leq C2^{(n-1)l}}} I_{lm}(f)(x).$$

We will make the change of variables $\bar{x} = h + \varphi_j(y)$ in the first integral and $\bar{x} = \varphi_j(y)$ in the second integral. The change of variables is justified since φ_j is $1-1$ on Q_{lm}. Thus

$$I_{lm}(f)(x) = \int_{\mathbb{R}^n} f(\bar{x}+x)\left[\frac{\psi_{lm}\circ\varphi_j^{-1}(\bar{x}+h)}{|J_{\varphi_j}\circ\varphi_j^{-1}(\bar{x}+h)|} - \frac{\psi_{lm}\circ\varphi_j^{-1}(\bar{x})}{|J_{\varphi_j}\circ\varphi_j^{-1}(\bar{x})|}\right] d\bar{x}$$

and so

$$(11) \qquad ||I_{lm}(f)||_\infty \le \int_{\mathbb{R}^n} |k(x+h) - k(x)|\, dx||f||_\infty$$

where $k(x) = \frac{\psi_{lm}\circ\varphi^{-1}(x)}{|J_{\varphi_j}\circ\varphi_j^{-1}(x)|}$. We will show that the following two estimates:

$$(12) \qquad \int_{\mathbb{R}^n} |k(x+h) - k(x)|\, dx \le C2^{-nl}$$

and

$$(13) \qquad \int |k(x+h) - k(x)|\, dx \le C|h|.$$

(12) is clear from the definition of $k(x)$ and the fact that $|\operatorname{supp}(\psi_{lm})| \le C2^{-nl}$. For (13), consider $II = \int_{\mathbb{R}^n} |\nabla k(x)| dx$. We will again make the change of variables $x = \varphi_j(y)$ in II but first let us observe that $\nabla_x k = (\varphi_j'^*)^{-1}\nabla_y k$ where $\nabla_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ and $x = (x_1, \dots, x_n)$. Note that

$$(\varphi_j'^*)^{-1}(y) = \frac{1}{J_{\varphi_j}(y)} C(y)$$

where $C(y)$ is the matrix of cofactors of $\varphi_j'(y)$, $C(y) = ((-1)^{j+k} \cdot \omega_{jk}(y))$. Therefore

$$\begin{aligned} II &= \int_{\mathbb{R}^n} |\nabla_x k(x)| dx = \int_{\mathbb{R}^n} |C(y)\nabla_y k(y)| dy \\ &\le \sum_{r,s=1}^{n} \int_{\mathbb{R}^n} \left|\omega_{rs}(y)\frac{\partial k}{\partial y_s}(y)\right| dy. \end{aligned}$$

For a fixed r and s, note that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \omega_{rs}(y) \frac{\partial k}{\partial y_s}(y) \right| dy \\
\leq &\int_{\mathbb{R}^n} \left| \frac{\partial \psi_{lm}}{\partial y_s}(y) \right| \left| \frac{\omega_{rs}(y)}{J_{\varphi_j}(y)} \right| dy \\
&+ \int_{\mathbb{R}^n} \psi_{lm}(y) \left| \omega_{rs}(y) \frac{\partial}{\partial y_s} \left(\frac{1}{J_{\varphi_j}} \right) \right| dy \\
= &A + B.
\end{aligned}$$

From (7) and (8), we see that

$$A \leq C 2^{(n-1)l} 2^l |\operatorname{supp}(\psi_{lm})| \leq C. \tag{14}$$

For B, we will assume without loss of generality that $s = 1$. Again suppose that

$$Q_{lm} = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n | a_k \leq y_k \leq b_k,\ k = 1, 2, \dots, n\}$$

and let

$$\tilde{Q}_{lm} = \{y = (y_2, \dots, y_n) \in \mathbb{R}^{n-1} | a_k \leq y_k \leq b_k,\ k = 2, \dots, n\}.$$

In B integrate with respect to y_1 first, dividing up the y_1 integral where $\frac{\partial J_{\varphi_j}}{\partial y_1}$ vanishes.

Recall that for a fixed $(y_1, \dots, y_n)$, $\frac{\partial J_{\varphi_j}}{\partial y_1}$ has exactly $n-2$ zeros. Since for each $r = 1, 2, \dots, n$, $\omega_{r1}(y)$ is independent of y_1,

$$\begin{aligned}
B &\leq \int_{\tilde{Q}_{lm}} |\omega_{r1}(y)| dy_2 \dots dy_n \int_{a_1}^{b_1} \left| \frac{\partial}{\partial y_1} \left(\frac{1}{J_{\varphi_j}} \right) \right| dy_1 \\
&\leq \sum_{k=1}^{N} \int_{\tilde{Q}_{lm}} |\omega_{r1}(y)| dy_2 \dots dy_n \left| \int_{x_k}^{x_{k+1}} \frac{\partial}{\partial y_1} \left(\frac{1}{J_{\varphi_j}} \right) dy_1 \right| \\
&\leq C \int_{\tilde{Q}_{lm}} \left| \frac{\omega_{r1}(y)}{J_{\varphi_j}(\bar{y}_1, y_2, \dots, y_n)} \right| dy_2 \dots dy_n
\end{aligned}$$

where $N \leq n-2$ and $a_1 \leq \bar{y}_1 = \bar{y}_1(y_2, \dots, y_n) \leq b_1$ is a point where $|J_{\varphi_j}(y_1, y_2, \dots, y_n)|$ takes on its minimum value in $[a_1, b_1]$ as a function of y_1 ($(y_2, \dots, y_n)$ being fixed). Therefore by (8), we have

$$B \leq C 2^{(n-1)l} |\tilde{Q}_{lm}| \leq C. \tag{15}$$

Hence from (14) and (15), we have

$$II = \int_{\mathbb{R}^n} |\nabla k(x)|\, dx \leq C$$

and so

$$\int_{\mathbb{R}^n} |k(x+h) - k(x)|\, dx \leq C|h| \int_{\mathbb{R}^n} |\nabla k(x)|\, dx \leq C|h|$$

which gives (13). Together, (12) and (13) imply that for $0 < \epsilon \leq 1$,

$$\sup_{h\in\mathbb{R}^n} \frac{1}{|h|^\epsilon} \int_{\mathbb{R}^n} |k(x+h) - k(x)|\, dx \leq C2^{-nl(1-\epsilon)}.$$

From (11), we then have

$$||I_{lm}f||_\infty \leq C2^{-nl(1-\epsilon)}|h|^\epsilon ||f||_\infty$$

for $0 < \epsilon \leq 1$. Since

$$I_h f(x) = \sum_{\substack{l\geq 0 \\ 1\leq m\leq C2^{(n-1)l}}} I_{lm}f(x),$$

we see that

$$||I_h f||_\infty \leq \sum_{l\geq 0} 2^{-l(1-n\epsilon)}|h|^\epsilon ||f||_\infty.$$

So for $\epsilon > 0$ small enough, we have (6) and this completes the proof of the lemma. $\square$

3. Boundedness of M_Γ. For the L^p boundedness of M_Γ, note that it is sufficient to estimate the maximal operator

$$\mathcal{M}f(x) = \sup_{k\in\mathbf{Z}} |M_k f(x)|$$

where

$$M_k f(x) = \frac{\lambda^k}{\lambda - 1} \int_{\lambda^{-k}}^{\lambda^{-k+1}} f(x - \Gamma(t))\, dt.$$

(Recall that $\lambda > 1$ was introduced in (2).) In fact we have the pointwise estimate

$$M_\Gamma f(x) \leq C\mathcal{M}f(x)$$

for $f \geq 0$. The L^p boundedness of $\mathcal{M}$ will follow from a well- known bootstrap argument contained in the following three lemmas, see [**NSW2**].

LEMMA 2. *$\mathcal{M}$ is bounded in L^2.*

LEMMA 3. *Suppose that*

$$\left\|\left(\sum_{k\in\mathbf{Z}}|M_kf_k|^2\right)^{1/2}\right\|_{p_0} \leq C_{p_0}\left\|\left(\sum_{k\in\mathbf{Z}}|f_k|^2\right)^{1/2}\right\|_{p_0}$$

for some $p_0 < 2$. Then

$$||\mathcal{M}f||_p \leq C_p||f||_p, \ p_0 < p \leq 2.$$

LEMMA 4. *If $||\mathcal{M}f||_{p_0} \leq C_{p_0}||f||_{p_0}$ for some $p_0 \leq 2$, then*

$$\left\|\left(\sum_{k\in\mathbf{Z}}|M_kf_k|^2\right)^{1/2}\right\|_{p} \leq C_p\left\|\left(\sum_{k\in\mathbf{Z}}|f_k|^2\right)^{1/2}\right\|_{p}, \ \frac{1}{p} \leq \frac{1}{2}\left(\frac{1}{p_0}+1\right).$$

The proof of Lemma 4 follows from a standard interpolation argument since the operators M_k are positive and uniformly bounded in L^p, $1 \leq p \leq \infty$. See [**NSW2**]. To prove Lemmas 2 and 3, let $\varphi \geq 0 \in C_0^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\varphi = 1$ and set

$$\Phi_j(x) = \frac{1}{\det A_j}\varphi(A_j^{-1}x).$$

Then as shown in [**CCVWW**], $\sup_{j\in\mathbf{Z}}|\Phi_j * f|$ is bounded in $L^p(\mathbb{R}^n)$, $1 < p \leq \infty$. Therefore if

$$N_kf(x) = M_kf(x) - \Phi_k * f(x),$$

then

$$\mathcal{M}f(x) \leq \sup_{k\in\mathbf{Z}}|\Phi_k * f(x)| + \sup_{k\in\mathbf{Z}}|N_kf(x)|$$

and so to prove the L^p estimates for $\mathcal{M}$ in Lemmas 2 and 3, it suffices to prove the L^p estimates for

$$\sup_{k\in\mathbf{Z}}|N_kf(x)|.$$

For these L^p estimates, note that from (3) we may write

$$N_k f(x) = \sum_{l \in \mathbf{Z}} \Psi_{k+l} * N_k f(x)$$

and so

$$\begin{aligned}\sup_{k \in \mathbf{Z}} |N_k f(x)| &\leq \sum_{l \in \mathbf{Z}} \left(\sum_{k \in \mathbf{Z}} |\Psi_{k+l} * N_k f(x)|^2 \right)^{1/2} \\ &= \sum_{l \in \mathbf{Z}} G_l f(x).\end{aligned}$$

We will momentarily prove

$$||G_l f||_2 \leq C 2^{-\delta |l|} ||f||_2 \tag{16}$$

for some $\delta > 0$. (16) implies the conclusion of Lemma 2.

For the proof of Lemma 3, note that

$$\left\| \left(\sum_{k \in \mathbf{Z}} |\Phi_k * f_k|^2 \right)^{1/2} \right\|_p \leq C_p \left\| \left(\sum_{k \in \mathbf{Z}} |f_k|^2 \right)^{1/2} \right\|_p, \ p > 1.$$

In fact one can easily see that Lemmas 2, 3, and 4 are true when the operators M_k are replaced by convolution with Φ_k. Therefore under the assumption of Lemma 3, we see that

$$\left\| \left(\sum_{k \in \mathbf{Z}} |N_k f_k|^2 \right)^{1/2} \right\| \leq C_{p_0} \left\| \left(\sum_{k \in \mathbf{Z}} |f_k|^2 \right)^{1/2} \right\|_{p_0}. \tag{17}$$

Thus

$$\begin{aligned}||G_l f||_{p_0} &= \left\| \left(\sum_{k \in \mathbf{Z}} |N_k (\Psi_{k+l} * f)|^2 \right)^{1/2} \right\|_{p_0} \\ &\leq C_{p_0} \left\| \left(\sum_{k \in \mathbf{Z}} |\Psi_{k+l} * f|^2 \right)^{1/2} \right\|_{p_0} \\ &\leq C_{p_0} ||f||_{p_0}.\end{aligned} \tag{18}$$

The last inequality follows from (4). Interpolating the estimates (16) and (18) gives us

$$||G_l f||_p \leq C2^{-\epsilon_p |l|}||f||_p, \ \epsilon_p > 0$$

for $p_0 < p \leq 2$. Therefore $\sup |N_k f|$ and hence $\mathcal{M}$ is bounded in L^p, $p_0 < p \leq 2$ and this will then complete the proof of Lemma 3.

It remains to establish (16). Write

$$T_k^l f(x) =^+_- \Psi_{k+l} * N_k f(x)$$

and set $S_l f(x) = \sum_{k \in \mathbf{Z}} T_k^l f(x)$. We will show that

$$(19) \qquad ||T_k^l (T_j^l)^* f||_2 \leq C2^{-\delta|l|}2^{-\epsilon|j-k|}||f||_2$$

for some δ, $\epsilon > 0$. Since T_k^l and $(T_j^l)^*$ commute, we will also have

$$(20) \qquad ||(T_k^l)^* T_j^l f||_2 \leq C2^{-\delta|l|}2^{-\epsilon|j-k|}||f||_2.$$

The Cotlar-Stein lemma then implies

$$||S_l f||_2 \leq C2^{-\delta|l|}||f||_2,$$

which in turn implies (16) by a standard Rademacher function argument. Note that it suffices to prove (19) when $\delta = 0$ and then when $\epsilon = 0$.

We will first establish (19) when $\delta = 0$ by proving the following lemma.

LEMMA 5. $||\Psi_{k+l} * \Psi_{j+l} * f||_2 \leq C2^{-\epsilon|j-k|}||f||_2$ *for some* $\epsilon > 0$.

Proof. It suffices to show

$$||\Psi_{k+l} * \Psi_{j+l}||_1 \leq C2^{-\epsilon|j-k|}.$$

Without loss of generality, assume $j > k$. Since Ψ_{k+l} has mean value zero, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\Psi_{k+l} * \Psi_{j+l}(x)|dx \\
&= \int_{\mathbb{R}^n} dx \left| \int_{\mathbb{R}^n} [\Psi_{k+l}(x-y) - \Psi_{k+l}(x)]\Psi_{j+l}(y)\, dy \right| \\
&\leq \int_{\mathbb{R}^n} |\Psi_{j+l}(y)|\, dy \int_{\mathbb{R}^n} \frac{1}{\det A_{k+l+1}} |\psi(A_{k+l+1}^{-1}(x-y)) \\
&\quad - \psi(A_{k+l+1}^{-1}(x))|\, dx \\
&\quad + \int_{\mathbb{R}^n} |\Psi_{j+l}(y)|\, dy \int_{\mathbb{R}^n} \frac{1}{\det A_{k+l}} |\psi(A_{k+l}^{-1}(x-y)) \\
&\quad - \psi(A_{k+l}^{-1}(x))|\, dx \\
&= I + II.
\end{aligned}$$

Changing variables in I, we see that

$$\begin{aligned}
I = &\int_{\mathbb{R}^n} |\Psi_{j+l}(y)|\, dy \int_{\mathbb{R}^n} |\psi(x - A_{k+l+1}^{-1}y) - \psi(x)|\, dx \\
\leq &C||\nabla\psi||_1 \int_{\mathbb{R}^n} |\Psi_{j+l}(y)|\, |A_{k+l+1}^{-1}y|\, dy \\
\leq &C\left[\int_{\mathbb{R}^n} |\psi(y)|\, |A_{k+l+1}^{-1}A_{j+l+1}y|\, dy \right. \\
&\left. + \int_{\mathbb{R}^n} |\psi(y)|\, |A_{k+l+1}^{-1}A_{j+l}y|\, dy\right] \\
\leq &C[||A_{k+l+1}^{-1}A_{j+l+1}|| + ||A_{k+l+1}^{-1}A_{j+l}||] \leq C\lambda^{-\epsilon(j-k)}
\end{aligned}$$

for some $\epsilon > 0$. The last inequality follows from (1). The same estimate for II follows similarly. Therefore

$$||\Psi_{k+l} * \Psi_{j+l}||_1 \leq C2^{-\epsilon(j-k)}$$

for some $\epsilon > 0$ and this completes the proof of the lemma.

Next we will prove (19) when $\epsilon = 0$. We will divide the argument into two cases.

Case 1. $l \geq 0$. Write

$$T_k^l f(x) = R_k^l f(x) - Q_k^l f(x)$$

where

$$R_k^l f(x) = \Psi_{k+l} * (\pm M_k f(x))$$

and

$$Q_k^l f(x) = \Psi_{k+l} * (\pm \Phi_k * f(x)).$$

We will prove the estimates for R_k^l and Q_k^l separately. In fact for R_k^l, we will show the stronger estimate

$$(21) \qquad ||R_k^l f||_2 \leq C 2^{-\delta l} ||f||_2$$

for some $\delta > 0$. Since $||R_k^l||_{op}^2 = ||R_k^l (R_k^l)^*||_{op}$ ($||R||_{op}$ denotes the operator norm of R) and convolution with Ψ_{k+l} is uniformly bounded in L^2, it suffices to estimate $||\Psi_{k+l} * (M_k M_k^*)||_{op}$. Iterating this observation n times, we see that (21) follows from the estimate

$$(22) \qquad ||\Psi_{k+l} * Tf||_2 \leq C 2^{-\delta l} ||f||_2$$

where

$$\begin{aligned} Tf(x) =& \underbrace{M_k M_k^* M_k \cdots M_k^*}_{n \text{ times}} f(x) \\ =& \frac{\lambda^{nk}}{(\lambda-1)^n} \int_{\substack{\lambda^{-k} \leq y_j \leq \lambda^{-k+1} \\ j=1,2,\dots,n}} f(x - \Gamma(y_1) \\ & + \Gamma(y_2) - \dots + (-1)^n \Gamma(y_n)) \, dy \\ =& \int_Q f(A_k[A_k^{-1}x - \varphi_k(y)]) \, dy. \end{aligned}$$

(22) in turn follows from the L^∞ estimate

$$(23) \qquad ||\Psi_{k+l} * Tf||_\infty \leq C 2^{-\delta l} ||f||_\infty.$$

In fact since $\{\Psi_{k+l} * T\}$ is uniformly bounded in L^1, then interpolation with (23) gives (22). Note that

$$Tf(x-y) - Tf(x) = I_{A_k^{-1}y}(f_k)(A_k^{-1}x)$$

where $f_k(x) = f(A_k x)$. By Lemma 1, we see that

$$|Tf(x-y) - Tf(x)| \leq C|A_k^{-1}y|^\epsilon ||f||_\infty$$

for some $\epsilon > 0$ and so since Ψ_{k+l} has mean value zero,

$$\begin{aligned}|\Psi_{k+l} * Tf(x)| =&\left|\int_{\mathbb{R}^n} \Psi_{k+l}(y)[Tf(x-y) - Tf(x)]\,dy\right| \\ \leq& C||f||_\infty \int_{\mathbb{R}^n} |\Psi_{k+l}(y)|\,|A_k^{-1}y|^\epsilon\,dy \\ \leq& C||f||_\infty \left[\int_{\mathbb{R}^n} |A_k^{-1}A_{k+l+1}y|^\epsilon|\psi(y)|\,dy\right. \\ &\left.+\int_{\mathbb{R}^n} |A_k^{-1}A_{k+l}y|^\epsilon|\psi(y)|\,dy\right] \\ \leq& C\lambda^{-\delta l}||f||_\infty\end{aligned}$$

for some $\delta > 0$. The last inequality follows from (1) since $l \geq 0$. This gives us (23). A similar but easier argument gives us

$$||Q_k^l f||_2 \leq C2^{-\delta l}||f||_2$$

for some $\delta > 0$. Thus (19) holds when $\epsilon = 0$ and $l \geq 0$.
Case 2. $l \leq 0$. To prove (19) when $\epsilon = 0$, it is sufficient to show that

$$||N_k(\Psi_{k+l})(x)||_1 \leq C2^{\delta l} \tag{24}$$

for some $\delta > 0$. Note that

$$\begin{aligned}N_k(\Psi_{k+l})(x) =&\frac{\lambda^k}{\lambda-1}\int_{\lambda^{-k}}^{\lambda^{-k+1}} \Psi_{k+l}(x-\Gamma(t))\,dt \\ &-\int_{\mathbb{R}^n} \Phi_k(y)\Psi_{k+l}(x-y)\,dy \\ =&\frac{\lambda}{1-\lambda}\int_{\frac{1}{\lambda}}^{1}[\Psi_{k+l}(x-\Gamma(\lambda^{-k+1}t)) - \Psi_{k+l}(x)]\,dt \\ &-\int_{\mathbb{R}^n} \Phi_k(y)[\Psi_{k+l}(x-y) - \Psi_{k+l}(x)]\,dy \\ =&I(x) + II(x).\end{aligned}$$

We will prove (24) for $I(x)$. The L^1 estimate for $II(x)$ is somewhat easier. For $I(x)$, write

$$\begin{aligned}&\Psi_{k+l}(x-\Gamma(\lambda^{-k+1}t)) - \Psi_{k+l}(x) \\ &\quad= \int_0^1 \nabla\Psi_{k+l}(x - s\Gamma(\lambda^{-k+1}t))\cdot\Gamma(\lambda^{-k+1}t)\,ds \\ &\quad= \int_0^1 A_{k-1}^*\nabla\Psi_{k+l}(x - s\Gamma(\lambda^{-k+1}t))\cdot\Gamma_{k-1}(t)\,ds.\end{aligned}$$

Therefore

$$
\begin{aligned}
(25)\quad & \int_{\mathbb{R}^n} |\Psi_{k+l}(x - \Gamma(\lambda^{-k+1}t)) - \Psi_{k+l}(x)|\, dx \\
&\leq |\Gamma_{k-1}(t)| \int_0^1 ds \int_{\mathbb{R}^n} |A^*_{k-1} \nabla \Psi_{k+l}(x - s\Gamma(\lambda^{-k+1}t))|\, dx \\
&= |\Gamma_{k-1}(t)| \int_{\mathbb{R}^n} |A^*_{k-1} \nabla \Psi_{k+l}(x)|\, dx.
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{\partial \Psi_{k+l}}{\partial x_r}(x) = & \frac{1}{\det A_{k+l+1}} \sum_{s=1}^{n} \frac{\partial \psi}{\partial x_s}(A^{-1}_{k+l+1}x) b_{sr} \\
& - \frac{1}{\det A_{k+l}} \sum_{s=1}^{n} \frac{\partial \psi}{\partial x_s}(A^{-1}_{k+l}x) c_{sr}
\end{aligned}
$$

where

$$(b_{sr}) = A^{-1}_{k+l+1} \text{ and } (c_{sr}) = A^{-1}_{k+l}.$$

Thus the p^{th} component of $A^*_{k-1}\nabla\Psi_{k+l}(x)$ is

$$
\begin{aligned}
& \frac{1}{\det A_{k+l+1}} \sum_{s=1}^{n} \frac{\partial \psi}{\partial x_s}(A^{-1}_{k+l+1}x) \sum_{r-1}^{n} b_{sr} a_{rp} \\
& - \frac{1}{\det A_{k+l}} \sum_{s=1}^{n} \frac{\partial \psi}{\partial x_s}(A^{-1}_{k+l}x) \sum_{r=1}^{n} c_{sr} a_{rp}
\end{aligned}
$$

where $(a_{rp}) = A_{k-1}$. And so

$$
\begin{aligned}
& \int_{\mathbb{R}^n} |A^*_{k-1} \nabla \Psi_{k+l}(x)|\, dx \\
& \quad \leq C \int_{\mathbb{R}^n} |\nabla \psi(x)|\, dx [||A^{-1}_{k+l+1} A_k|| + ||A^{-1}_{k+l} A_k||] \\
& \quad \leq C \lambda^{\delta l}
\end{aligned}
$$

for some $\delta > 0$. The last inequality follows from (1) since $l \leq 0$. Therefore from (25), we have

$$
\begin{aligned}
||I(x)||_1 &\leq C \int_{1/\lambda}^{1} |\Gamma_{k-1}(t)|\, dt \cdot \int_{\mathbb{R}^n} |A^*_{k-1} \nabla \Psi_{k+l}(x)|\, dx \\
&\leq C\lambda^{\delta l} \int_{1/\lambda}^{1} |\Gamma_{k-1}(t)|\, dt \leq C\lambda^{\delta l}
\end{aligned}
$$

for some $\delta > 0$. The last inequality follows from (1). In fact for $t > 0$, $\Gamma(t) = \delta(t)e$ where $e = (1, 0, \dots, 0)$ and so by (1),

$$\begin{aligned}|\Gamma_{k-1}(t)| &= |\delta^{-1}(\lambda^{-k+1})\Gamma(\lambda^{-k+1}t)| \\ &= |\delta^{-1}(\lambda^{-k+1})\delta(\lambda^{-k+1}t)e| \leq ||\delta^{-1}(\lambda^{-k+1})\delta(\lambda^{-k+1}t)|| \leq C\end{aligned}$$

when $t \leq 1$. This establishes (24) and thus finishes the proof of (19) and hence (20) when $\epsilon = 0$. This completes the proof of the L^p boundedness of $\mathcal{M}$ and hence M_Γ. □

The proof of the theorem for the Hilbert transform, H_Γ, is similar. Write

$$H_j f(x) = \int_{\lambda^{-j}}^{\lambda^{-j+1}} f(x - \Gamma(t)) \frac{dt}{t} = f * d\sigma_j(x)$$

(recall that $\lambda > 1$ was introduced in (2)) and note that

$$\begin{aligned}H_\Gamma f &= \sum_j H_j f = \sum_j \sum_l \Psi_{j+l} * H_j f \\ &= \sum_j \sum_l \sum_m \Psi_{j+m} * \Psi_{j+l} * H_j f \\ &= \sum_{l,m} \sum_j \Psi_{j+m} * \Psi_{j+l} * H_j f \\ &= \sum_{l,m} g_{l,m} f\end{aligned}$$

where

$$g_{l,m} f = \sum_j \Psi_{j+m} * \Psi_{j+l} * H_j f.$$

As in (16) and (18), it suffices to show

$$||g_{l,m} f||_p \leq C ||f||_p, \; 1 < p < \infty \tag{26}$$

and

$$||g_{l,m} f||_2 \leq C 2^{-\epsilon|l|} 2^{-\epsilon|m|} ||f||_2 \tag{27}$$

for some $\epsilon > 0$. To show (26), take $w \in L^{p'}$ and note that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \sum_j (\Psi_{j+m} * \Psi_{j+l} * H_j f)(x) w(x)\, dx\right| \\
&= \left|\sum_j \int_{\mathbb{R}^n} \Psi_{j+l} * H_j f(x) \Psi_{j+m} * w(x)\, dx\right| \\
&\leq \left\|\left(\sum_j |\Psi_{j+l} * H_j f|^2\right)^{1/2}\right\|_p \left\|\left(\sum_j |\Psi_{j+m} * w|^2\right)^2\right\|_{p'} \\
&\leq C_p \left\|\left(\sum_j [M_j(|\Psi_{j+l} * f|)]^2\right)^{1/2}\right\|_p ||w||_{p'} \\
&\leq C_p \left\|\left(\sum_j |\Psi_{j+l} * f|^2\right)^{1/2}\right\|_p ||w||_{p'} \\
&\leq C_p ||f||_p\, ||w||_{p'}.
\end{aligned}$$

We will show (27) by the Cotlar-Stein lemma. We have to show

$$\begin{aligned}
&||\Psi_{j+m} * \Psi_{j+l} * d\sigma_j * \Psi_{k+m} * \Psi_{k+l} * d\tilde{\sigma}_k * f||_2 \\
&\leq 2^{-\epsilon(|l|+|m|+|k-j|)} ||f||_2.
\end{aligned}$$

Suppose for example $|l| \geq |m|$. Then

$$\begin{aligned}
&||\Psi_{j+m} * \Psi_{j+l} * d\sigma_j * \Psi_{k+m} * \Psi_{k+l} * d\tilde{\sigma}_k * f||_2 \\
&\leq C ||\Psi_{j+l} * d\sigma_j * \Psi_{k+l} * d\tilde{\sigma}_k * f||_2 \\
&\leq C 2^{-\epsilon(|l|+|j-k|)} ||f||_2
\end{aligned}$$

as in (19) and (20), where we have used the fact that $\Psi_{j+m} * \Psi_{k+m}$ has uniformly bounded L^1 norm.

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Received October 20, 1992 and in revised form November 27, 1992. The first and fifth authors were supported in part by an SERC grant. The third author was supported in part by an NSF grant at the University of Wisconsin.

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