RATIO TESTS FOR CONVERGENCE OF SERIES

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1. Introduction. The following theorem was proved and used by Jehlke [2] to obtain elegant improvements of the classic tests of Gauss and Weierstrass for convergence of series of real and of complex terms.

THEOREM 1. *If the terms of two series* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are such that*

$$\frac{b_{n+1}}{b_n} = \frac{a_{n+1}}{a_n}(1 + c_n) \qquad (n = 0, 1, \cdots), \tag{1}$$

where $\sum_{n=0}^{\infty} c_n$ *is absolutely convergent, then the two series* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *are both convergent or both divergent.*

It is the main object of this note to prove that Theorem 1 is a best possible theorem in that no hypothesis weaker than the hypothesis that $\sum_{n=0}^{\infty} |c_n| < \infty$ is sufficient to imply the conclusion of the theorem. The final result, Theorem 4, is obtained from two preliminary theorems, Theorems 2 and 3, which seem to have independent interest.

2. Preliminary theorems. We first establish the following result.

THEOREM 2. *Let* $c_n \neq -1$, $n = 0, 1, 2, \cdots$. *In order that the sequence* $\{c_n\}$ *be such that* $\sum_{n=0}^{\infty} b_n$ *converges whenever* (1) *holds and* $\sum_{n=0}^{\infty} a_n$ *converges, it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} |(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})c_n| < \infty. \tag{2}$$

Proof. To prove Theorem 2, let (1) hold. Then

$$\frac{b_{n+1}}{a_{n+1}} = \frac{b_n}{a_n}(1 + c_n) \qquad (n = 0, 1, 2, \cdots), \tag{3}$$

and hence

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$$\frac{b_n}{a_n} = \frac{b_0}{a_0}(1+c_0)(1+c_1)\cdots(1+c_{n-1}) \qquad (n = 1, 2, \cdots). \tag{4}$$

Let

$$p_n = \frac{b_0}{a_0}(1+c_0)(1+c_1)\cdots(1+c_{n-1}) \qquad (n = 1, 2, \cdots). \tag{5}$$

Then $b_n = p_n a_n$. But by a well-known theorem of Hadamard [1], $\sum_{n=0}^{\infty} p_n a_n$ converges whenever $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} |p_{n+1} - p_n| < \infty$. But (5) implies that

$$p_{n+1} - p_n = \frac{b_0}{a_0}(1+c_0)(1+c_1)\cdots(1+c_{n-1})c_n, \tag{6}$$

and the conclusion of Theorem 2 follows.

THEOREM 3. *Let $c_n \neq -1$, $n = 0, 1, 2, \cdots$. In order that the sequence $\{c_n\}$ be such that $\sum_{n=0}^{\infty} a_n$ converges whenever* (1) *holds and $\sum_{n=0}^{\infty} b_n$ converges, it is necessary and sufficient that*

$$\sum_{n=1}^{\infty} \left| \frac{1}{1+c_0}\,\frac{1}{1+c_1}\cdots\frac{1}{1+c_{n-1}}\,\frac{c_n}{1+c_n} \right| < \infty. \tag{7}$$

Proof. Theorem 3 may be proved by revising the proof of Theorem 2 to use the relations

$$\frac{a_{n+1}}{a_n} = \frac{b_{n+1}}{b_n}\,\frac{1}{1+c_n} \qquad (n = 0, 1, 2, \cdots) \tag{8}$$

instead of (1) or, which amounts to the same thing, replacing $1 + c_k$ by $1/(1 + c'_k)$ in (2) and then removing the primes.

3. Theorem. Our main result is the following.

THEOREM 4. *Let $c_n \neq -1$, $n = 0, 1, 2, \cdots$. In order that this sequence be such that the two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent or both divergent whenever* (1) *holds, it is necessary and sufficient that $\sum_{n=0}^{\infty} |c_n| < \infty$.*

Proof. To prove necessity, suppose $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both convergent or both divergent whenever (1) holds. Then, by Theorems 2 and 3, both (2) and (7) hold. Denoting the nth terms of the series in (2) and (7) by u_n and v_n, we see that, as $n \longrightarrow \infty$, we have $u_n \longrightarrow 0$ and $v_n \longrightarrow 0$ and hence

$$u_n v_n = \frac{c_n^2}{1 + c_n} \longrightarrow 0 \ . \tag{9}$$

This implies that $c_n \longrightarrow 0$ and hence that $|1/(1 + c_n)| > 1/2$ for n sufficiently great. This and (7) imply that

$$\sum_{n=1}^{\infty} \left| \frac{1}{1 + c_0} \frac{1}{1 + c_1} \cdots \frac{1}{1 + c_{n-1}} c_n \right| < \infty \ . \tag{10}$$

If we let $x_n = |(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})|$, then (2) and (10) imply that

$$\sum_{n=1}^{\infty} (x_n + x_n^{-1}) \ |c_n| < \infty \ . \tag{11}$$

But the mere fact that $x_n > 0$ implies that $(x_n + x_n^{-1}) \geq 2$, and it follows that $\Sigma_{n=0}^{\infty} |c_n| < \infty$. This proves necessity. To prove sufficiency, suppose that $\Sigma_{n=0}^{\infty} |c_n| < \infty$. Then the infinite product $\prod(1 + c_k)$ converges to a number not zero, and this means that each of $(1 + c_0)(1 + c_1) \cdots (1 + c_{n-1})$ and $[(1 + c_0)(1 + c_1) \cdots (1 + c_n)]^{-1}$ converges to a number not zero. This and $\Sigma_{n=0}^{\infty} |c_n| < \infty$ imply (2) and (7). Therefore Theorems 2 and 3 imply that $\Sigma_{n=0}^{\infty} a_n$ and $\Sigma_{n=0}^{\infty} b_n$ are both convergent or both divergent. This completes the proof of Theorem 4.

References

1. J. Hadamard, *Deux théorèmes d'Abel sur la convergence des series*, Acta. Math., 27 (1903), 177-183.

2. H. Jehlke, *Eine Bemerkung zum Konvergenzkriterium von Weierstrass*, Mathematische Zeitschrift, 52 (1949), 60-61.

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